SEQUENTIAL ESTIMATION OF DYNAMIC DISCRETE GAMES

Victor Aguirregabiria (Boston University)

and

Pedro Mira (CEMFI)

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CONTEXT AND MOTIVATION

• Many interesting questions in economics involve **dynamic strategic interactions** among economic agents.

- Market entry/exit in oligopoly industries/markets.

-Adoption of new technologies

- R&D and creation of new products

- Monetary policy.

• Dynamic games are useful tools to study these phenomena.

• Despite its interest, there have been very few empirical applications that estimate structurally dynamic games.

• Three main issues that have limited the range of applications of empirical discrete games:

(1) Dimension of state space: Computational burden

(2) Multiple equilibria

(3) Permanent unobserved heterogeneity.

• Contribution of this paper:

(1) Proposes an estimation method that deals with these three issues.

(2) Applies the method to estimate a model of entry/exit in oligopoly markets.

OUTLINE

1. MODEL AND ASSUMPTIONS.

2. ESTIMATION METHODS

3. MONTE CARLO EXPERIMENTS.

4. EMPIRICAL APPLICATION

1. MODEL AND ASSUMPTIONS.

• We consider a general class of dynamic discrete **games of incomplete information**.

• For the sake of presentation, it is useful to think in a particular application: **entry and exit in local retail markets**.

* Retail industry: banks, supermarkets, hotels
* M **independent** (isolated) local retail markets, indexed by m.
* N_m potential entrants in market m, indexed by i.
* The set of potential entrants can change across markets.

• **A firm decision problem**

Every period t firms decide simultaneously to be active or not in the market.

$a_{it} \in \{0, 1\}$ is the decision of firm i at period t.

• **State variables**: At the beginning of period t a firm is characterized by two vectors of state variables, x_{it} and ε_{it}, which affect its profitability.

> x_{it} is common knowledge; e.g., exogenous market characteristics; incumbent status at previous period, etc.
>
> ε_{it} is private information of firm i; e.g., a component of fixed costs.

- Current profits of firm i:

$$\boxed{\Pi_{it} = \tilde{\Pi}_i(a_t, x_t, \varepsilon_{it})}$$

where $x_t \equiv (x_{1t}, x_{2t}, ..., x_{Nt})$ and $a_t \equiv (a_{1t}, a_{2t}, ..., a_{Nt})$.

- For instance,

$$\Pi_{it} = \begin{cases} R_i(S_t, a_t) - \theta_{FC,i} - \theta_{EC} \ (1 - a_{i,t-1}) - \omega - \varepsilon_{it} & if \quad a_{it} = 1 \\ \\ \theta_{SV} \ a_{i,t-1} & if \quad a_{it} = 0 \end{cases}$$

$R_i(S_t, a_t)$ is an "indirect" variable profit function (e.g., from Cournot or Betrand static competition)

$S_t =$ Market size; $\theta_{FC,i} =$ Fixed cost;

$\theta_{EC} =$ Entry cost; $\theta_{SV} =$ Exit value.

• In this example:

$$x_t = \Big(S_t \ , \ a_{1,t-1} \ , \ a_{2,t-1} \ , \ ... \ , \ a_{N,t-1}\Big)$$

ASSUMPTION: $\{\varepsilon_{it}\}$ are $i.i.d.$ across firms, across markets and over time.

ASSUMPTION: $\{x_t\}$ follows a **controlled Markov process** with transition probability $f(x_{t+1}|\ a_t, x_t)$

• In this example:

a_{t-1} follows a trivial transition

S_t follows an exogenous Markov process.

- **MARKOV PERFECT EQUILIBRIA**

- Firms' strategies depend only on payoff relevant state variables (x_t, ε_{it})

- Let $\alpha = \{\alpha_i(x_t, \varepsilon_{it})\}$ be a set of **strategy functions**.

- Given α we can define **choice probabilities** $P^\alpha = \{P_i^\alpha(x_t)\}$

$$P_i^\alpha(x_t) = \int I\{\alpha_i(x_t, \varepsilon_{it}) = 1\}\ dG_i(\varepsilon_{it})$$

- We represent a **MPE in the space of players' choice probabilities.** Let α^* be a MPE, and let P^* be the set probabilities associated with α^*. Then, P^* solves a mapping:

$$P^* = \Lambda(P^*)$$

• **AN ALTERNATIVE EQUILIBRIUM MAPPING**

• We consider **an alternative mapping that is much simpler to evaluate than** $\Lambda(P)$.**for different values of** θ **and fixed** P**.**

• A MPE associated with θ, say P_θ^*, also solves the mapping

$$P_\theta^* = \mathbf{\Psi}_{\boldsymbol{\theta}}(P_\theta^*)$$

where (in the entry/exit example):

$$\mathbf{\Psi}_{\boldsymbol{\theta}}(P)(i, x_t) = \mathbf{\Phi}\left(\ Z_i(x_t, P)\ \frac{\theta}{\sigma} + \lambda_i(x_t, P)\right)$$

and $Z_i(x_t, P)$ and $\lambda_i(x_t, P)$ vectors which depend on P and transition probabilities.

2. ESTIMATION

2.1. Data Generating Process

• A researcher observes players' actions and common knowledge state variables across M geographically separate markets over T periods, where M is large and T is small:

$$Data = \{a_{mt}, x_{mt} : m = 1, 2, ..., M;\ t = 1, 2, ..., T\}$$

ASSUMPTION 5: There is a unique $\theta^0 \in \Theta$ such that $P^0 = \Psi(P^0; \theta^0)$ and $P^0 \neq \Psi(P^0; \theta)$ for any $\theta \neq \theta^0$.

2.2. Maximum Likelihood Estimation

• Let $\Upsilon = \{1, 2, 3...\}$ be the set of equilibrium types. An equilibrium type is a probability function $P^{\tau}(\theta)$ where $\tau \in \Upsilon$ is the index that represents the type.

• Under Assumption 5 the population probabilities P^0 belong to one and only one equilibrium type. There is a $\tau_0 \in \Upsilon$ and $\theta^0 \in \Theta$ such that $P^0 = P^{\tau_0}(\theta^0)$.

• The MLE of θ^0 is:

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in \Theta} \left\{ \ \sup_{\tau \in \Upsilon} \ \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{N} \log P_i^{\tau}(a_{imt} | x_{mt}; \theta) \ \right\}$$

2.3. Pseudo Maximum Likelihood Estimation

• PML estimators try to minimize the number of evaluations of Ψ for different vectors of players' probabilities P.

• We define first the *pseudo likelihood function*:

$$Q_M(\theta, P) = \frac{1}{M} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{N} \ln \Psi_i(a_{imt} | x_{mt}; P, \theta)$$

• Suppose that we knew the population probabilities P^0, and consider the following PML estimator:

$$\hat{\theta}_U \equiv \arg\max_{\theta \in \Theta} \; Q_M(\theta, P^0)$$

• This PML estimator is unfeasible because P^0 is unknown.

• Suppose that we can obtain a $\sqrt{M}-$consistent nonparametric estimator of P^0. The feasible two-step PML estimator:

$$\hat{\theta}_{2S} \equiv \arg\max_{\theta \in \Theta} Q_M(\theta, \hat{P}^0).$$

• **Limitations of this PML:**

(1) Asymptotically inefficient.

(2) Seriously biased in small samples.

(3) Does not deal with permanent unobserved het.

2.4. Nested PML

• NPL generates a sequence of estimators $\{\hat{\theta}_K : K \geq 1\}$ where the $K-stage$ estimator is defined as:

$$\hat{\theta}_K = \arg\max_{\theta \in \Theta} Q_M(\theta, \hat{P}_{K-1})$$

and the probabilities $\{\hat{P}_K : K \geq 1\}$ are obtained recursively as:

$$\hat{P}_K = \Psi(\hat{\theta}_K, \hat{P}_{K-1})$$

ESTIMATION WITH UNOBSERVED MARKET HETEROGENEITY

1. Assumptions on Permanent Unobserved Heterogeneity

• Let x_{mt} be the observable state variables, and suppose that there is also a time invariant common knowledge unobservable ω_m.

$$\tilde{\Pi}_{imt}(1) = \theta_R \, S_{mt} \, \left(2 + \sum_{j \neq i} a_{imt}\right)^{-2} - \theta_{FC} - \theta_{EC}(1 - a_{im,t-1}) + \boldsymbol{\omega}_{\mathbf{m}} + \varepsilon_{imt}$$

ASSUMPTION: The unobservable variable ω_m is such that:

(A) it has a discrete and finite support $\Omega = \left\{\omega^1, \omega^2, \ldots, \omega^B\right\}$;

(B) it is independently and identically distributed over markets with probability mass function $\varphi(\omega) \equiv \Pr(\omega_m = \omega)$;

(C) ω_m does not enter into the conditional transition probability of x_{mt}, i.e., $\Pr(x_{m,t+1} | a_{mt}, x_{mt}, \omega_m) = f(x_{m,t+1} | a_{mt}, x_{mt})$.

• Assumption 6C states that all markets are homogenous with respect to transitions, and it implies that the transition probability functions f can still be estimated from transition data without solving the model.

• Now the vector of structural parameters θ includes the parameters in the distribution of the unobservables ω. The vector P now stacks the distributions of players' actions conditional on all values of observable and unobservable common knowledge state variables.

• Now $P = \{P_b : b = 1, 2, ..., B\}$ where P_b is the vector with players' choice probabilities when the "market type" is $\omega_m = \omega^b$.

PML Estimation with Permanent Unobserved Heterogeneity

- Let $P = \{P_b : b = 1, 2, ..., B\}$. The pseudo likelihood function now is:

$$\begin{aligned}\log \Pr(Data|\theta, P) &= \textstyle\sum_{m=1}^{M} \log \Pr(\tilde{a}_m, \tilde{x}_m|\theta, P) \\ &= \textstyle\sum_{m=1}^{M} \log \left(\sum_{b=1}^{B} \varphi(\omega^b) \; \Pr(\tilde{a}_m, \tilde{x}_m|\omega^b, \theta, P) \right)\end{aligned}$$

where $\tilde{a}_m = \{a_{mt} : t = 1, 2, ..., T\}$ and $\tilde{x}_m = \{x_{mt} : t = 1, 2, ..., T\}$.

- Applying the Markov structure of the model, and assumption 6C, we get:

$$\begin{aligned}\Pr(\tilde{a}_m, \tilde{x}_m|\omega^b; \theta, P) &= \left(\prod_{t=1}^{T} \Pr(a_{mt}|x_{mt}, \omega^b, \theta, P) \right) \\ &\quad \left(\prod_{t=2}^{T} \Pr(x_{mt}|a_{m,t-1}, x_{m,t-1}, \omega^b) \right) \\ &\quad \Pr(x_{m1}|\omega^b, \theta, P)\end{aligned}$$

- And:

$$\Pr(\tilde{a}_m, \tilde{x}_m | \omega^b; \theta, P) = \left(\prod_{t=1}^{T} \prod_{i=1}^{N} \Psi_i(a_{imt} | x_{mt}, \theta, P_b) \right) \left(\prod_{t=2}^{T} f(x_{mt} | a_{m,t-1}, x_{m,t-1}) \right) \Pr(x_{m1} | \omega^b, \theta, P)$$

- Solving this expression into the log likelihood, we have that:

$$\log \Pr(Data | \theta, P) = \sum_{m=1}^{M} \log \left(\sum_{b=1}^{B} \varphi(\omega^b) \left(\prod_{t=1}^{T} \prod_{i=1}^{N} \Psi_i(a_{imt} | x_{mt}, \omega^b, P_b, \theta) \right) \Pr(x_{m1} | \omega^b, \theta, P) \right) + \sum_{m=1}^{M} \sum_{t=2}^{T} \ln f(x_{mt} | a_{m,t-1}, x_{m,t-1}) \quad (1)$$

- The first component in the right hand side is the pseudo likelihood function $Q_M(\theta, P)$.

• **Initial Conditions Problem:** The observed state vector at the first observation for each market x_{m1} is not exogenous with respect to unobserved market type: $\Pr(x_{m1}|\omega_m) \neq \Pr(x_{m1})$. This is the, so called, *initial conditions problem* in the estimation of dynamic discrete models with autocorrelated unobservables (Heckman, 1981).

• Under the assumption that x_{m1} is drawn from the stationary distribution induced by the Markov perfect equilibrium, we can implement a computationally tractable solution of this problem.

• Let $p^*(x_{mt}|f, P_b)$ be the steady-state distribution of the vector of state variables x_{mt} in a market where the vector of firms' choice probabilities is P_b and the conditional transition probability function of x is f.

$$Q_M(\theta, P) = \sum_{m=1}^{M} \log \left(\begin{array}{c} \sum_{b=1}^{B} \varphi(\omega^b) \left(\prod_{t=1}^{T} \prod_{i=1}^{N} \Psi_i(a_{imt}|x_{mt}, \omega^b, P_b, \theta) \right) \\ p^*(x_{mt}|f, P_b) \end{array} \right)$$

• Given this pseudo likelihood function, the NPL estimator is defined as follow a pair $(\hat{\theta}, \hat{P})$, with $\hat{P} = \{\hat{P}_b : b = 1, 2, ..., B\}$ such that the two following conditions hold:

$$(1) \qquad \hat{\theta} = \arg\max_{\theta \in \Theta} Q_M(\theta, \hat{P})$$

$$(2) \qquad \hat{P}_b = \Psi(\hat{\theta}, \hat{P}_b, \omega^b) \text{ for every } b = 1, 2, ..., B$$

where we include ω^b as an argument in Ψ to emphasize that we have a different equilibrium mapping for every value of ω^b.

• We obtain this NPL estimator using an iterative procedure that is similar to the one without unobserved heterogeneity. The main difference is that now we have to calculate the steady-state distributions $p^*(.|f, P_b)$ to deal with the initial conditions problem.

• However, the pseudo likelihood approach also reduces very significantly the cost of dealing with the initial conditions problem. The reason is that given the probabilities (f, P_b) the steady-state probabilities $p^*(.|f, P_b)$ do not depend on the structural parameters in θ. Therefore, the probabilities $p^*(.|f, P_b)$ remain constant during any pseudo maximum likelihood estimation and they are updated only between two pseudo maximum likelihood estimations when we obtain new choice probabilities P_b.

ALGORITHM

At iteration 1, start with B vectors of players' choice probabilities, one for each market type: $\hat{P}^0 = \{\hat{P}_b^0 : b = 1, 2, ..., B\}$. Then, perform the following steps.

STEP 1: For every market type $b \in \{1, 2, ..., B\}$, obtain its steady-state distribution of x_{mt} as the unique solution to the system of linear equations (see Amemiya, chapter 11):

$$p^*(x|f, \hat{P}_b^0) = \sum_{x_0 \in X} f^{\hat{P}_b^0}(x|x_0)\ p^*(x_0|f, \hat{P}_b^0) \qquad \text{for any } x \in X$$

where $f^{\hat{P}_b^0}(.|.)$ is the transition probability for x induced by the conditional transition probability $f(.|.,.)$ and the choice probabilities in $\hat{P}_b^0$. That is:

$$f^{\hat{P}_b^0}(x|x_0) = \sum_{a \in A} \left(\prod_{i=1}^{N} \hat{P}_{b,i}^0(a_i|x_0) \right)\ f(x|x_0, a)$$

STEP 2: Given the probabilities $\{p^*(.|f, \hat{P}_b^0) : b = 1, 2, ...B\}$, construct the pseudo likelihood function $Q_M(\theta, \hat{P}^0)$ and obtain the pseudo maximum likelihood estimator of θ as:

$$\hat{\theta}^1 = \arg\max_{\theta \in \Theta}\ Q_M(\theta, \hat{P}^0)$$

STEP 3: For every market type b, update the vector of players' choice probabilities using the best response probability mapping associated with market type b. That is,

$$\hat{P}_b^1 = \mathbf{\Psi}(\hat{\theta}^1, \hat{P}_b^0, \omega^b)$$

STEP 4: If $||\hat{P}^1 - \hat{P}^0||$ is smaller than a fixed constant, then stop the iterative procedure and choose $(\hat{\theta}^1, \hat{P}^1)$ as the NPL estimator. Otherwise, replace $\hat{P}^0$ by $\hat{P}^1$ and repeat steps 1 to 4.

3. MONTE CARLO EXPERIMENT

- Profit function:

$$\tilde{\Pi}_{imt} = \theta_{RS} \ \ln(S_{mt}) - \theta_{RN} \ \ln\left(1 + \sum_{j \neq i} a_{imt}\right) - \theta_{FC,i} - \theta_{EC}(1 - a_{im,t-1}) + \varepsilon_{imt}$$

Remark 1: The NPL algorithm always converged to the same estimates regardless of the value of $\hat{P}_0$ (true, nonparametric, logit or random) that we used to initialize the procedure.

Remark 3: The two-freq estimator has a very large bias in all the experiments, though its variance is similar to, and sometimes even smaller than, the variances of NPL and two-true estimators.

Remark 4: The NPL estimator performs very well relative to the two-true estimator both in terms of variance and bias.

Remark 5: The two-logit performs very well for this simple model.

Remark 6: In all the experiments, the most important gains associated with the NPL estimator occur for the entry cost parameter, α_2

4. APPLICATION

• Data: Census of Chilean firms collected by the Chilean *Servicio de Impuestos Internos* (Internal Revenue Service).

• Includes all the firms, all the establishments that a firm has, and the geographical location of each establishment. Crucial to identify the local market where a establishment operates and all its competitors in that market.

• It is a panel and therefore I observe exits and new entries.

• Definition of market: Comuna (census tract) excluding metropolitan areas. 189 comunas in the working sample.

• Sample period 1994-1999

Table 5a
Descriptive Statistics
189 markets. Years 1994-1999

	Restaurants	Gas stations	Bookstores	Shoe shops	Fish shops
# firms per 10,000 people	14.6	1.0	1.9	0.9	0.7
Markets with 0 firms	32.2 %	58.6 %	49.5 %	67.1 %	74.1 %
Markets with 1 firm	1.3 %	15.3 %	15.8 %	10.8 %	9.6 %
Markets with 2 firms	1.2 %	7.8 %	8.0 %	6.7 %	5.0 %
Markets with 3 firms	0.5 %	5.2 %	6.9 %	3.8 %	3.4 %
Markets with 4 firms	1.2 %	4.0 %	3.6 %	2.7 %	2.0 %
Markets with $>$ 4 firms	63.5 %	9.2 %	16.2 %	8.9 %	5.9 %
Herfindahl Index (median)	0.169	0.738	0.663	0.702	0.725
Firm size	17.6	67.7	23.3	67.2	124.8
log(firms) on log(mark size)	0.383	0.133	0.127	0.073	0.062
	(0.043)	(0.019)	(0.024)	(0.020)	(0.018)

Table 5b
Descriptive Statistics
189 markets. Years 1994-1999

	Restaurants	Gas stations	Bookstores	Shoe shops	Fish shops
log(firm size) on log(mark size)	-0.019	0.153	-0.066	0.223	0.097
	(0.034)	(0.082)	(0.050)	(0.081)	(0.111)
Entry rate (%)	9.8	14.6	19.7	12.8	21.3
Exit rate (%)	9.9	7.4	13.5	10.4	14.5
Survival rate : 1 year (%)	86.2 (13.8)	89.5 (10.5)	84.0 (16.0)	86.8 (13.2)	79.7 (20.3)
Survival rate: 2 years (%)	69.5 (19.5)	88.5 (1.1)	70.0 (16.6)	71.1 (18.2)	58.1 (27.2)
Survival rate: 3 years (%)	60.1 (14.9)	84.6 (4.3)	60.0 (14.3)	52.6 (25.1)	44.6 (23.3)

Table 8

NPL estimation of Entry-Exit model

Parameters	Rest	Gas	Book	Shoe	Fish
Variable profit: $\frac{\theta_{RS}}{\sigma_{\varepsilon}}$	1.743	1.929	2.029	2.030	0.914
	(0.045)	(0.127)	(0.076)	(0.121)	(0.125)
Variable profit: $\frac{\theta_{RN}}{\sigma_{\varepsilon}}$	1.643	2.818	1.606	2.724	1.395
	(0.176)	(0.325)	(0.201)	(0.316)	(0.234)
Fixed Operating Cost: $\frac{\theta_{FC}}{\sigma_{\varepsilon}}$	9.519	12.769	15.997	14.497	6.270
	(0.478)	(1.251)	(0.141)	(1.206)	(1.233)
Entry cost: $\frac{\theta_{EC}}{\sigma_{\varepsilon}}$	5.756	10.441	5.620	5.839	4.586
	(0.030)	(0.150)	(0.081)	(0.145)	(0.121)
$\frac{\sigma_{\omega}}{\sigma_{\varepsilon}}$	1.322	2.028	1.335	2.060	1.880
	(0.471)	(1.247)	[illegible]	[illegible]	[illegible]

Table 9
Normalized Parameters

	Parameters	Rest	Gas	Book	Shoe	Fish
(1)	$\frac{\theta_{FC}}{\theta_{RS}\ \ln(S_{Med})}$	0.590	0.716	0.852	0.772	0.742
(3)	$\frac{\theta_{EC}}{\theta_{RS}\ \ln(S_{Med})}$	0.357	0.585	0.299	0.311	0.542
(3)	$100\frac{\theta_{RN}\ \ln(2)}{\theta_{RS}\ \ln(S_{Med})}$	7.1 %	10.9 %	5.9 %	10.1 %	11.4 %
(4)	$\frac{\sigma_{\omega}^2}{\theta_{RS}^2\ var\left(\ln(S)\right)+\sigma_{\omega}^2}$	0.33	0.49	0.27	0.47	0.78