Identification of Structural Parameters in Dynamic Discrete Choice Games with Fixed Effects Unobserved Heterogeneity

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CONTEXT & MOTIVATION

- Dynamic Discrete Choice Games are useful tools for the empirical analysis of **dynamic interactions** (social, strategic) among agents.
- Two types of **structural parameters** that play a crucial role in the predictions of these models:
  - Those capturing dynamics or **state dependence**: adjustment costs and investment costs.
  - Those capturing **interactions between players**: competition effects, peer effects, and spillovers.
- The identification of these two sets of parameters critically relies on controlling for **player and market heterogeneity** which is observable to the players but unobservable to the researcher.

CONTEXT & MOTIVATION [2/2]

- Misspecification of the distribution of **Unobserved Heterogeneity (UH)** can lead to two types of biases in key parameters of interest.
- Neglecting **UH that is persistent over time** introduces biases in the estimation of parameters that capture dynamics.
  - **Spurious evidence of dynamics** or state dependence.
- Neglecting **UH that is correlated across players** introduces biases in estimation of parameters capturing strategic or social interactions.
  - **Spurious evidence of positive spillover effects** between players.

THIS PAPER

- We integrate & extend two strands of literature:

1. Identification of **Dynamic Discrete Games with UH:**
   - This literature has considered only Random Effects / Finite Mixture models: Kasahara & Shimotsu (2009), Arcidiacono & Miller (2011), Aguirregabiria & Mira (2019).
   - In contrast, we adopt a **Fixed Effects** model without restrictions on the distribution or support of the UH.
2. Identification of **Fixed Effects Panel Data DC Models:**
   - This literature examines 'reduced form' models: Bonhomme (2012), Honoré, & Weidner (2020), Dobronyi, Gu, Kim, & Russell (2021)
   - We extend it to models with **Multiple Equilibria** and **Forward-Looking** agents.

THIS PAPER [2/2]

- We apply and extend several approaches for the identification of Fixed Effects models, with their relative advantages and limitations:

1. For versions of the model with Equilibrium Uniqueness:
    - 'Traditional' **Conditional Likelihood/Sufficient Statistics** approach.
        - Pros: Simple and guarantees uniqueness.
        - Cons: It is sufficient but not necessary.
    - New **Functional Differencing** approach.
        - Pros: Necessary & Sufficient.
        - More complex, subject to problem of Finite Underidentification.
2. For versions of the model with Multiple Equilibria:
    - We extend the 'Traditional' Conditional Likelihood/Sufficient Statistics approach to deal with **Partial Identification**.

THE REST OF THIS PRESENTATION

1. MODEL

2. IDENTIFICATION RESULTS

   2.1. Myopic Players

      a. Sequential moves – Equiibrium Uniqueness.

      b. Simultaneous moves – Multiple Equiibria.

   2.2. Forward-Looking Players

      a. Sequential moves – Equiibrium Uniqueness.

      b. Simultaneous moves – Multiple Equiibria.

3. EMPIRICAL APPLICATION
   - Dynamic Pricing Game with Hi-Lo Pricing.

1. MODEL

MODEL: BASIC FEATURES

- **Panel data**: N players (indexed by i), M markets or games (indexed by m), T periods (indexed by t). T is small.

- For today's talk, I will focus on a dynamic game with:

  1. Two players: $i, j \in \{1, 2\}$
  2. Binary choice: $y_{imt} \in \{0, 1\}$
  3. The only endogenous state variables are $(y_{1m,t-1}, y_{2m,t-1})$
  4. No exogenous observable state variables
  5. Complete information

MODEL: PREFERENCES & STATE VARIABLES

- Every period t, each player chooses action $y_{it} \in \{0,1\}$ to maximize:

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \delta_i^s \; U_{i,t+s} \right]$$

where δ_i is the discount factor, and **utility function**:

$$U_{it} \;=\; u_i \left(y_{it}, \; y_{jt}, \; y_{i,t-1}, \; y_{j,t-1} \right) + \varepsilon_{it} \left(y_{it} \right).$$

- The **state variables** of the DG are $(\mathrm{y}_{t-1}, \boldsymbol{\varepsilon}_t)$, where:
    - $\mathrm{y}_{t-1} \equiv (y_{1,t-1}, \; y_{2,t-1})$.
    - $\boldsymbol{\varepsilon}_t \equiv (\varepsilon_{1t}(0), \; \varepsilon_{1t}(1), \; \varepsilon_{2t}(0), \; \varepsilon_{2t}(1))$ i.i.d. Extreme Value Type I.

MARKOV PERFECT EQUILIBRIA

- A MPE consists of strategy functions of the state variables:

$$y_{1t} = \sigma_1(\mathrm{y}_{t-1}, \varepsilon_t) \quad \text{and} \quad y_{2t} = \sigma_2(\mathrm{y}_{t-1}, \varepsilon_t)$$

- Satisfying the **best response equations**:

$$y_{1t} \;=\; 1\left\{\widetilde{u}_1\left(y_{2t}, \mathrm{y}_{t-1}\right) + \widetilde{V}_1^{\sigma}\left(y_{2t}\right) - \varepsilon_{1t} \geq 0\right\}$$

$$y_{2t} \;=\; 1\left\{\widetilde{u}_2\left(y_{1t}, \mathrm{y}_{t-1}\right) + \widetilde{V}_2^{\sigma}\left(y_{1t}\right) - \varepsilon_{2t} \geq 0\right\}$$

$\varepsilon_{it} \equiv \varepsilon_{it}(0) - \varepsilon_{it}(1)$; $\widetilde{u}_{it} \equiv u_{it}(1) - u_{it}(0)$; and

$$\widetilde{V}_i^{\sigma}\left(y_{jt}\right) \equiv \delta_i \int \left(V_i^{\sigma}(1, y_{jt}, \varepsilon_{t+1}) - V_i^{\sigma}(0, y_{jt}, \varepsilon_{t+1})\right) g\left(\varepsilon_{t+1}\right)\, d\varepsilon_{t+1}$$

STRUCTURAL & INCIDENTAL PARAMETERS

- The utility difference $\widetilde{u}_{imt}$ has the following structure:

$$\begin{aligned}\widetilde{u}_{1mt} &= \alpha_{1m} + \gamma_1 \; y_{2mt} + \beta_1 \; y_{1m,t-1} + \lambda_1 \; y_{2m,t-1} \\ \widetilde{u}_{2mt} &= \alpha_{2m} + \gamma_2 \; y_{1mt} + \beta_2 \; y_{2m,t-1} + \lambda_2 \; y_{1m,t-1}\end{aligned}$$

- **Structural Parameters**: $\boldsymbol{\theta} = (\gamma_1, \; \gamma_2, \; \beta_1, \; \beta_2, \; \lambda_1, \; \lambda_2)$.

- **Incidental Parameters**: $\boldsymbol{\alpha} = (\alpha_{1m}, \; \alpha_{2m}, \; \delta_{1m}, \; \delta_{2m} : m = 1, 2, ..., M)$

- **Econometric Model**:

$$\begin{aligned}y_{1mt} &= 1\left\{\alpha_{1m} + \beta_1 \; y_{1m,t-1} + \gamma_1 \; y_{2mt} + \widetilde{V}_{1m}\left(y_{2mt}\right) - \varepsilon_{1mt} \geq 0\right\} \\ y_{2mt} &= 1\left\{\alpha_{2m} + \beta_2 \; y_{2m,t-1} + \gamma_2 \; y_{1mt} + \widetilde{V}_{2m}\left(y_{1mt}\right) - \varepsilon_{2mt} \geq 0\right\}\end{aligned}$$

2. IDENTIFICATION

DATA

- The researcher observes panel data for the players playing the game at M markets and T time periods:

$$\text{Data} = \{\ y_{1mt},\ y_{2mt} : m = 1, 2, ..., M\ ;\ t = 0, 1, ..., T\}$$

  M is large and T is small.

- The researcher is interested in the estimation of the structural parameters β_1, β_2, γ_1, γ_2, λ_1, λ_2.

- We denote these structural parameters using the vector $\boldsymbol{\theta}$.

SUMMARY OF IDENTIFICATION RESULTS

No Contemp. Interactions $\gamma_1 = \gamma_2 = 0$	One-Direction Interactions $\gamma_1 = 0$	Two-Direction Interactions Sequential Move	Two-Direction Interactions Simul. Move
MYOPIC PLAYERS: $\widetilde{V}_{imt} = 0$			
Point identification β_1, β_2, λ_1, λ_2	Point identification β_1, β_2, γ_2	Point iden. β_1, β_2 Partial iden. γ_1, γ_2	Partial identification β_1, β_2, γ_1, γ_2
FORWARD-LOOKING PLAYERS: $\widetilde{V}_{imt} \neq 0$			
Point iden. β_1, β_2	Point iden. β_1, β_2 Partial iden. γ_2	Point iden. β_1, β_2	Partial iden. β_1, β_2

Equilibrium Uniqueness – Conditional Likelihood Approach

- Let $\widetilde{y}_m \equiv (y_{1mt}, y_{2mt} : t = 0, 1, .., T)$ be a *market history*.
- Log-likelihood function: $\ell(\boldsymbol{\alpha}, \boldsymbol{\theta}) = \sum_{m=1}^{M} \ln \mathbb{P}(\widetilde{y}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta})$

$$\ln \mathbb{P}(\widetilde{y}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta}) = \ln p(y_{1m0}, y_{2m0} | \boldsymbol{\alpha}_m)$$

$$\sum_{t=1}^{T} \ln \left(\frac{\exp\{\ y_{1mt} [\alpha_{1m} + \beta_1\ y_{1mt-1} + v_1(\boldsymbol{\alpha}_m)]\ \}}{1 + \exp\{\ y_{1mt} [\alpha_{1m} + \beta_1\ y_{1mt-1} + v_1(\boldsymbol{\alpha}_m)]\ \}} \right)$$

$$\sum_{t=1}^{T} \ln \left(\frac{\exp\{\ y_{2mt} [\alpha_{2m} + \gamma_2\ y_{1mt} + \beta_2\ y_{2mt-1} + v_2(y_{1mt}, \boldsymbol{\alpha}_m)]\ \}}{1 + \exp [\alpha_{2m} + \gamma_2\ y_{1mt} + \beta_2\ y_{2mt-1} + v_2(y_{1mt}, \boldsymbol{\alpha}_m)]} \right)$$

Equilibrium Uniqueness – Conditional Likelihood Approach [2]

- The log-prob $\ln \mathbb{P}\left(\widetilde{y}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta}\right)$ has the following structure:

$$\ln \mathbb{P}\left(\widetilde{y}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta}\right) = s(\widetilde{y}_m)' \, g(\boldsymbol{\alpha}_m) + c(\widetilde{y}_m)' \, \boldsymbol{\theta}$$

where $s(\widetilde{y}_m)$ and $c(\widetilde{y}_m)$ are vectors of statistics.

- This structure implies that $s(\widetilde{y}_m)$ is a sufficient statistics for $\boldsymbol{\alpha}_m$.

- If the elements in the vector $\left[s(\widetilde{y}_m)', c(\widetilde{y}_m)'\right]$ are linearly independent, then the maximization of the Conditional log-likelihood function implies the point identification of the vector of parameters $\boldsymbol{\theta}$.

Equilibrium Uniqueness – Conditional Likelihood Approach [3]

- More intuitively, let A and B be two market histories such that:

$$s(A) - s(B) = 0 \qquad \textit{and} \qquad c(A) - c(B) = (1, 0, 0, 0, 0, 0)$$

- This implies that:

$$\ln \mathbb{P}(A) - \ln \mathbb{P}(B) = \beta_1$$

- More generally, let A and B be two market histories such that:

$$s(A) - s(B) = 0 \qquad \textit{and} \qquad c(A) - c(B) \neq (0, 0, 0, 0, 0, 0)$$

- Then:

$$\ln \mathbb{P}(A) - \ln \mathbb{P}(B) = [c(A) - c(B)]' \boldsymbol{\theta}$$

Equilibrium Uniqueness – Conditional Likelihood Approach [4]

- For instance, in the model with sequential move, parameters (β_1, β_2) are point identified with $T \geq 3$.

- The vectors of sufficient statistics is:

$$\text{s}(\widetilde{\text{y}}) = \left[y_{10},\ y_{20},\ y_{1T},\ y_{2T},\ T_{(1)},\ T_{(2)},\ T_{(1,2)} \right]$$

with: $T_{(1)} \equiv \sum_{t=1}^{T} y_{1t};\ T_{(2)} \equiv \sum_{t=1}^{T} y_{2t};\ T_{(1,2)} \equiv \sum_{t=1}^{T} y_{1t} y_{2t}$

- And:

$$\text{c}(\widetilde{\text{y}})'\ \boldsymbol{\theta} = \beta_1\ C_{(1,1)} + \beta_2\ C_{(2,2)}$$

$$C_{(1,1)} \equiv \sum_{t=1}^{T} y_{1t}\ y_{1t-1};\ C_{(2,2)} \equiv \sum_{t=1}^{T} y_{2t}\ y_{2t-1}$$

EXAMPLE OF IDENTIFYING PAIR OF HISTORIES

Examples of histories and identified parameters with T=3

$$A = \{\boldsymbol{y}_0, \ \boldsymbol{a}, \ \boldsymbol{b}, \ \boldsymbol{y}_3\}; \quad B = \{\boldsymbol{y}_0, \ \boldsymbol{b}, \ \boldsymbol{a}, \ \boldsymbol{y}_3\}$$

	$\boldsymbol{y}_0$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{y}_3$	$\ln \mathbb{P}(A) - \ln \mathbb{P}(B)$
Case 1:	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	β_1
Case 2:	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	β_2

EQUILIBRIUM UNIQUENESS – FUNCTIONAL DIFFERENCING

- The CML-SS approach does not impose all the restrictions on the parameters: Honore & Weidner (2021).
- The model implies additional moment equalities and inequalities.
- We follow the Functional DIfference approach in Dobronyi, Gu, Kim, & Russell (2021) to derive additional moment conditions.
- These additional moment conditions are particularly helpful for partial identification of strategic interactions parameters γ.

Multiple Equilibria – Sufficient Statistics for Bounds

- The model does **not have unique predictions** for all the outcomes $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$,
- For instance, with $\gamma_1 \leq 0$ and $\gamma_2 \leq 0$, the model has unique predictions for $\mathbb{P}(0,0)$ and $\mathbb{P}(1,1)$ but not for $\mathbb{P}(0,1)$ and $\mathbb{P}(1,0)$.
- However, the model implies bounds for these probabilities:

$$L(0,1 \mid y_{mt-1}, \boldsymbol{\alpha}_m) \quad \leq \mathbb{P}(0,1 \mid y_{mt-1}; \boldsymbol{\alpha}_m) \leq \quad U(0,1 \mid y_{mt-1}; \boldsymbol{\alpha}_m)$$

$$L(1,0 \mid y_{mt-1}; \boldsymbol{\alpha}_m) \quad \leq \mathbb{P}(1,0 \mid y_{mt-1}; \boldsymbol{\alpha}_m) \leq \quad U(1,0 \mid y_{mt-1}; \boldsymbol{\alpha}_m)$$

Multiple Equilibria – Sufficient Statistics for Bounds [2]

- Using the bounds for players' choice probabilities we can construct **bounds for the probability of a market history** $\mathbb{P}\left(\widetilde{\mathrm{y}}_m|\boldsymbol{\alpha}_m\right)$.

$$L(\widetilde{\mathrm{y}}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta}) \ \leq \ \mathbb{P}\left(\widetilde{\mathrm{y}}_m|\boldsymbol{\alpha}_m\right) \leq U(\widetilde{\mathrm{y}}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta})$$

- Using bounds on $\mathbb{P}(0,1)$ and $\mathbb{P}(1,0)$ which are **products of logit probabilities**, we have that the bounds on probabilities of market histories have the following form:

$$\ln L(\widetilde{\mathrm{y}}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta}) \quad = \quad \mathrm{s}_L(\widetilde{\mathrm{y}}_m)' \, g(\boldsymbol{\alpha}_m) + \mathrm{c}_L(\widetilde{\mathrm{y}}_m)' \, \boldsymbol{\theta}$$

$$\ln U(\widetilde{\mathrm{y}}_m \mid \boldsymbol{\alpha}_m, \boldsymbol{\theta}) \quad = \quad \mathrm{s}_U(\widetilde{\mathrm{y}}_m)' \, g(\boldsymbol{\alpha}_m) + \mathrm{c}_U(\widetilde{\mathrm{y}}_m)' \, \boldsymbol{\theta}$$

Multiple Equilibria – Sufficient Statistics for Bounds [3]

- Using this structure we can establish partial identification / bounds on the structural parameters.

  Let A and B be two market histories such that:

$$\mathsf{s}_L(A) = \mathsf{s}_U(B) \quad \text{AND} \quad \mathsf{c}_L(A) \neq \mathsf{c}_U(B)$$

- Then, we have that:

$$\left[\mathsf{c}_L(A) - \mathsf{c}_U(B)\right]' \boldsymbol{\theta} \leq \ln\left(\frac{\mathbb{P}(A)}{\mathbb{P}(B)}\right)$$

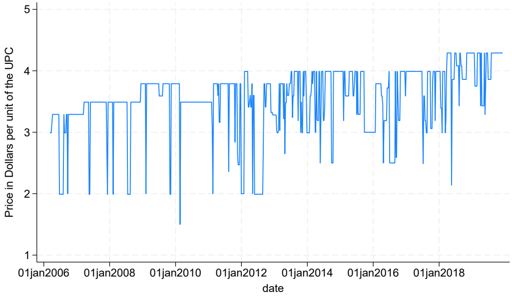
- These inequalities provide set identification of the structural parameters.

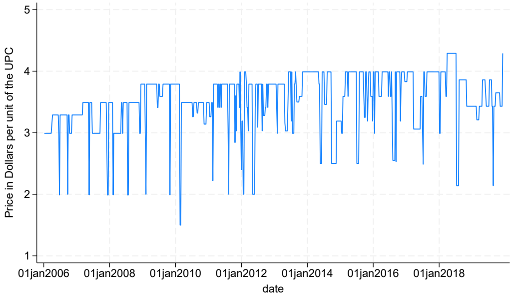
3. EMPIRICAL APPLICATION

DYNAMIC PRICING GAME – NIELSEN DATA

- Dynamic game of price competition with menu costs.
- Two possible prices: High or Low.
- $\beta_i =$ Menu cost of changing price.
- $\gamma_i =$ Competition effect.
- $\alpha_m =$ Unobserved heterogeneity in market size.

TWO LEADING DEODORANT PRODUCTS – ONE STORE





WORKING SAMPLE

- $N = 2$. Two leader female deodorant products.
- $M = 6,917$ geographic markets. Zip codes.
- Weekly prices. Original dataset has 730 weeks..
- We estimate the model using $T = 4$ spells of price histories.
- This results into a working sample with $M^* \approx 1,210,000$ with $T = 4$.

MENU COST PARAMETERS – Forward-Looking & Simultaneous

- 95% Confidence Intervals. Very Preliminary. We plan using Bootstrap inference method in Cox & Shi (REStud, 2023)

	MLE without Market UH	FE - SS-CMLE
Parameter	**95% C.I.**	**95% C.I.**
β_1	[1.7185 , 1.7306]	[0.4612 , 0.4918]
β_2	[1.7741 , 1.7900]	[0.4899 , 0.5131]

SUMMARY

- We study the identification of DDCG with a FE specification of UH and using short panels.
- Relative to reduced form panel data models, the identification should control for the continuation values and deal with multiple equilibria.
- We extend the Sufficient Statistics approach to deal with bounds on the probabilities of market histories.
- We find positive identification results on dynamic parameters. Identification of strategic interactions is quite limited.
- Complete Empirical application coming soon.