Identification of firms' beliefs in structural models of market competition

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Abstract. Firms make decisions under uncertainty and differ in their ability to collect and process information. As a result, in changing environments, firms have heterogeneous beliefs on the behaviour of other firms. This heterogeneity in beliefs can have important implications on market outcomes, efficiency and welfare. This paper studies the identification of firms' beliefs using their observed actions—a *revealed preference and beliefs approach*. I consider a general structural model of market competition where firms have incomplete information and their beliefs and profits are nonparametric functions of decisions and state variables. Beliefs may be out of equilibrium. The framework applies both to continuous and discrete choice games and includes as particular cases models of competition in prices or quantities, auction models, entry games and dynamic games of investment decisions. I focus on identification results that exploit an exclusion restriction that naturally appears in models of competition: an observable variable that affects a firm's cost (or revenue) but does not have a direct effect on other firms' profits. I present identification results under three scenarios—common in empirical industrial organization—on the data available to the researcher.

Résumé. Identification des convictions des entreprises dans des modèles structurels de concurrence commerciale. En matière de décision, les entreprises agissent dans l'incertitude et diffèrent dans leurs capacités à recueillir et traiter l'information. Par conséquent, dans des environnements en constante évolution, les « convictions » des entreprises sur le comportement de leurs concurrents sont hétérogènes. Cette hétérogénéité peut avoir des conséquences importantes sur les effets du marché, le rendement et le bien-être. Grâce à une approche fondée sur les préférences et les croyances révélées, cet article s'intéresse à l'identification des convictions des entreprises en fonction de leurs actions observées. Nous considérons donc un modèle structurel général de concurrence commerciale au sein duquel les entreprises disposent de données incomplètes et où leurs convictions, de même que leurs profits, sont des fonctions non paramétriques des décisions et des variables d'état. Les convictions peuvent se situer hors équilibre. Ce cadre s'applique à la fois aux jeux à choix continus et aux jeux à choix discrets et comprend, à titre de cas particuliers, des modèles de concurrence de prix ou de quantité, des modèles d'enchères, des jeux d'entrée ainsi que des jeux dynamiques de décisions d'investissement. L'attention est portée sur les résultats d'identification qui exploitent une restriction d'exclusion apparaissant naturellement dans les modèles de concurrence : une variable observable ayant une incidence sur les coûts (ou les recettes) des entreprises mais n'entraînant aucun effet direct sur les profits de leurs concurrents. En matière de

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données mises à la disposition des chercheurs, les résultats d'identification sont présentés sous trois scénarios, ce qui est habituel dans les organisations industrielles empiriques.

JEL classification: C57, D81, D83, D84, L13

1. Introduction

A FIRM'S BEHAVIOUR DEPENDS on its beliefs about the actions of other firms in the same market. In a retail market, the choice of price depends on beliefs about competitors' prices. In a procurement auction, a firm's bid depends on its expectations about other firms' bids. Managers form their beliefs under uncertainty on demand, costs and competitors' decisions. These managers and their firms have different abilities to collect and process information and, as a result, they are heterogeneous in their expectations. This heterogeneity in beliefs can have important implications on firms' performance and survival in the market. Nevertheless, firms with different accuracy in beliefs can survive in the same industry, for the same reasons as they can coexist with different productivities.

The importance of firms' heterogeneity in their ability to form expectations—and the possibility of biased or non-equilibrium beliefs—has been long recognized in economics, at least since the work of Simon (1958, 1959). However, in most fields in economics, the status quo has been to assume rational expectations. In empirical industrial organization (IO), some of the most commonly used structural models of oligopoly competition assume complete information, perfect certainty and Nash equilibrium. For instance, this is the case in models of price competition with differentiated product (Berry et al. 1995, Berry and Haile 2014) and in empirical games of market entry (Bresnahan and Reiss 1991, Ciliberto and Tamer 2009).[1] Though there is a substantial literature on structural models of incomplete information in empirical IO, it is mostly concentrated in auctions (Guerre et al. 2000, Athey and Haile 2002) and in discrete choice games, both static (Seim 2006, Sweeting 2009, Bajari et al. 2010) and dynamic (Aguirregabiria and Mira 2007, Igami 2017). Empirical applications to models of quantity or price competition are not so common, though Armantier and Richard (2003) and Aryal and Zincenko (2019) are good exceptions.

1 Berry and Haile (2014) assume complete information and perfect certainty, but the supply side of their model allows for a general form of competition, not necessarily Nash equilibrium. In their section 6, in the spirit of Bresnahan (1982), they study the joint identification of marginal costs and the *nature of competition*. I will comment on this in section 4.3.2 and example 6, when I discuss the relationship between the identification of beliefs in my model of incomplete information and the identification of the *nature of competition* in a model of complete information.

Most empirical applications of games of incomplete information assume that firms have homogeneous beliefs that correspond to a Bayesian Nash equilibrium. Nevertheless, recent papers in structural IO relax equilibrium assumptions and present evidence of substantial heterogeneity and biases in firms' beliefs. As one would expect, biased beliefs are more likely in new markets and after regulatory changes: for instance, after deregulation of the US telecommunication industry (Goldfarb and Xiao 2011), the UK electricity market (Doraszelski et al. 2018), the Texas electricity spot market (Hortaçsu and Puller 2008 and Hortaçsu et al. 2019) or the Washington State liquor market (Huang et al. 2018) and in the early years of the fast-food restaurant industry in UK (Aguirregabiria and Magesan 2020) or China (Xie 2018). All these papers use a *revealed preference and beliefs approach* to identify the structural parameters in costs and demand together with firms' subjective beliefs. That is, a firm's observed actions reveal information about its preferences (i.e., the structure of its profit function) and beliefs.[2]

Identification of beliefs using a revealed preference and beliefs *(RP&B)* approach requires restrictions either on profit or on beliefs functions. The papers mentioned above use different restrictions to identify subjective beliefs. In this paper, I present a systematic analysis of the joint identification of firms' beliefs and structural parameters in a general class of empirical games of market competition. The analysis introduces minimum restrictions on preferences (profits) and beliefs which are nonparametric functions of firms' actions and state variables. I investigate the identification of beliefs under very weak restrictions.

I present a framework where firms have incomplete information and their beliefs on competitors' behaviour are unrestricted (nonparametric) probability functions on the space of competitors' actions and conditional on observable state variables. Beliefs may be out of equilibrium. Revenue and cost functions are also nonparametrically specified. The framework applies both to continuous and discrete choice games and includes as particular cases models of competition in prices or quantities, auction models, entry games and dynamic investment games. I focus on identification results that exploit an exclusion restriction that naturally appears in models of competition: an observable variable that affects a firm's cost (or revenue) but does not have a direct effect on other firms' profits. Examples of this type of variable are firm-specific input prices, total factor productivity and predetermined variables such as

2 Using observed behaviour to identify agents' subjective beliefs was already proposed by Frank Ramsey in his article titled "Truth and Probability" (Ramsey 1926). Ramsey argues that probability is related to the knowledge that each individual possesses, leading to the notion of subjective probability or beliefs. Then, he explains how subjective beliefs can be inferred using observed behaviour. On pages 174 and 175 in his article, Ramsey uses a simple example to illustrate the identification of beliefs from observed actions.

the firm's previous period capital stock or incumbency status. I show the identification power of this exclusion restriction under several scenarios on the data available to the researcher: data only on firms' choices and state variables, as well as applications where the researcher has price and quantity data to identify the revenue function or the cost function. Identification conditions vary substantially when the model is static or dynamic, so I study separately these two cases.

This paper relates to several literatures in empirical IO and econometrics. As mentioned above, it belongs to a growing literature on structural models of market competition where firms have biased beliefs (see the references above).

In the econometrics of games, several papers have studied identification of models that allow for biased beliefs but impose other restrictions, such as *level-k rationality* (An 2017), *cognitive hierarchy* (Brown et al. 2013) or *rationalizability* (Aradillas-Lopez and Tamer 2008, Kline 2018). In this paper, I do not impose these restrictions and show that they are testable. An et al. (2018) show the identification subjective expectations in single-agent dynamic structural models using a RP&B approach. The exclusion restriction that these authors use is quite different from the one that I present in this paper.

In experimental economics, a good number of papers have estimated agents' beliefs in games played in laboratory experiments and have investigated *strategic uncertainty* (Van Huyck et al. 1990, Heinemann et al. 2009). In Aguirregabiria and Xie (2020), we propose a simple to implement experimental design that generates an exclusion restriction that identifies agents' beliefs in a nonparametric model where agents can have *other-regarding preferences* (i.e., altruism or concerns for inequality).

The identification of subjective beliefs in games also relates to the identification of structural games with multiple equilibria in the data. In both cases, we are interested in identifying the variation in players' behaviour that comes from variation in beliefs, keeping preferences constant. Recent contributions include De Paula and Tang (2012), Otsu et al. (2016) and Aguirregabiria and Mira (2019).

Finally, as I describe in section 3, the identification of firms' beliefs on competitors' strategies relates to the traditional IO literature on identification of the *nature of competition*, pioneered by Bresnahan (1982). In contrast to the traditional approach, the model in this paper acknowledges that firms' beliefs are endogenous objects that depend on all the state variables affecting demand and costs. I show that the exclusion restrictions typically used to identify the form of competition (or the so-called *conjectural variation parameters*) cannot identify firms' beliefs.

I have organized the rest of the paper as follows. Sections 2 and 3 present general frameworks—static and dynamic, respectively—that include as particular cases most models of competition in empirical applications in IO. Section 4 presents our main results on the identification of firms' beliefs using a RP&B approach. I summarize and conclude in section 5.

2. Static games

2.1. Basic framework

Consider N firms competing in a market. Firms are indexed by i. The profit function of firm i is $\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$, where $a_i \in \mathcal{A}$ is the action of firm i and $\boldsymbol{a}_{-i} = (a_j : j \neq i) \in \mathcal{A}^{N-1}$ is the vector with the actions of the other firms. The decision variable, a_i, can be either discrete or continuous, and it can represent—among other possibilities—a firm's level of output, its price, the binary indicator of entry in a market, the firm's number of stores or its investment in R&D. The vector $\mathbf{x} \in \mathcal{X}$ represents variables that are known by all the firms. The term $\varepsilon_i \in \mathbb{R}$ is private information of firm i. For instance, this private information can be a component of the firm's cost, or a private signal about the state of the demand. I denote ε_i as the "type" of firm i. The vector of firms' types $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ is drawn from a distribution with cumulative distribution function $F(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N | \mathbf{x})$. I use $F_i(\varepsilon_i | \mathbf{x})$ to represent the marginal distribution function of ε_i. The primitives of the game are Π's, $\mathcal{A}$, $\mathcal{X}$ and F.

Firms simultaneously decide their actions to maximize their respective expected profits. Under the standard solution concept of Bayesian Nash equilibrium (BNE), the primitives of the model are assumed common knowledge—that is, every firm knows that every firm knows these primitives. The model that I consider here does not impose this restriction. This model assumes that each firm knows only its own profit function, the vector of variables $\mathbf{x}$ and its private information ε_i. For instance, some firms may not know the distribution function F, or the profit functions of other firms. Furthermore, the fact that $\mathbf{x}$ is known to all the firms might not be common knowledge.

A firm does not know the private information of its competitors and therefore it does not know their actions. Firms form probabilistic beliefs about the actions of competitors. Let $b_i(\boldsymbol{a}_{-i} | \varepsilon_i, \mathbf{x})$ be a probability density function that represents firm i's beliefs. This is a probability function in the space of the actions of firms other than i and conditional on firm $i's$ information. We use $B_i(\boldsymbol{a}_{-i} | \varepsilon_i, \mathbf{x})$ to denote the distribution function associated with the density $b_i(\boldsymbol{a}_{-i} | \varepsilon_i, \mathbf{x})$. Given its beliefs, a firm's expected profit is

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}, b_i) = \int_{\boldsymbol{a}_{-i}} \Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) b_i(\boldsymbol{a}_{-i} | \varepsilon_i, \mathbf{x}) d\boldsymbol{a}_{-i}. \tag{1}$$

The integral is over the Lebesgue measure on $\mathcal{A}^{N-1}$, which can be either continuous or discrete.

A key condition in this model is that firms maximize their (subjective) expected profits. The following assumption establishes this condition.

ASSUMPTION 1. *A firm—given its subjective beliefs—chooses its strategy function $\sigma_i(\varepsilon_i, x, b_i)$ to maximize its expected profit. That is,*

$$\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = \underset{a_i \in \mathcal{A}}{arg\ max} \Pi_i^e(a_i, \varepsilon_i, \mathbf{x}, b_i). \tag{2}$$

In other words, a firm's strategy is a best response to its beliefs. ■

The belief-based best response model that I consider in this paper is general and has a long tradition in economics. However, it rules out some forms of strategic behaviour such as *reinforcement learning* (RL). Models of RL do not incorporate agents' beliefs or the maximization of expected payoff. Instead, they formalize the following principle: actions that have yielded payoffs above (below) the agent's aspiration level are more (less) likely to be chosen in the future. RL is common in computer science, it has also a tradition in economics (Erev and Roth 1998, Börgers and Sarin 2000).

It is convenient to represent a firm's strategy as a cumulative distribution function. Let $p_i\left(a_i|\mathbf{x}\right)$ be the probability density function of choice variable a_i conditional on $\mathbf{x}$, that is, the so called *conditional choice probability (CCP)* function. We use $P_i\left(a_i|\mathbf{x}\right)$ to denote the cumulative distribution function associated to p_i. According to the model, this distribution comes from firm i's best response and satisfies the following equation. For any value $a^0 \in \mathcal{A}$,

$$P_i\left(a^0|\mathbf{x}\right) \equiv \Pr\left(\sigma_i(\varepsilon_i,\mathbf{x},b_i) \leq a^0|\mathbf{x};b_i\right) = \int 1\left\{\sigma_i(\varepsilon_i,\mathbf{x},b_i) \leq a^0\right\} dF_i(\varepsilon_i|\mathbf{x}). \quad (3)$$

In this framework, the payoff functions $\{\Pi_i\}$ and the distribution of private signals $F(.|\mathbf{x})$ are primitives of the model. The predictions of the model are the choice probabilities. The belief functions $\{b_i\}$ are endogenous outcomes of the model. However, the model is incomplete in the sense that it does not specify how these beliefs are determined. Instead, it specifies a general framework that includes as particular cases many different models for the determination of beliefs such as Bayesian Nash equilibrium, cognitive hierarchy models and many others.

2.2. Additional assumptions

I focus on models where a firm's action a_i is a single variable that can be either continuous or discrete. If the decision is continuous, then $\mathcal{A} = \mathbb{R}$. If the decision is discrete, then it is ordered and $\mathcal{A} = \{0, 1, \ldots, J\}$, e.g., number of products, stores, etc.[3] The framework imposes some restrictions on the marginal profit function to guarantee that a firm's best response function is strictly monotonic in ε_i. For the rest of this subsection, I present assumptions such that: (i) these marginal conditions of optimality are necessary and sufficient such that they fully characterize a firm's optimal best response function and (ii) we can obtain a simple characterization of the optimal decision rule using the *cumulative choice probability function* $P_i\left(a_i|\mathbf{x}\right)$.

Let $\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$ be the *marginal profit function.* Here the concept of *marginal profit* is broad and depends on the decision variables a_i. If the decision variable is continuous, such as output or price, the marginal profit is at the *intensive margin* and it corresponds to the mathematical concept of partial derivative: $\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) \equiv \partial\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})/\partial a_i$. If the decision variable

3 Note that any binary choice (e.g., a market entry decision) is a particular case of ordered discrete choice variable.

is discrete—such as entry decision, number of stores or number of products—the marginal profit is at the *extensive margin* and corresponds to a difference function: $\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) \equiv \Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) - \Pi_i(a_i - 1, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$.

The following assumption provides sufficient conditions for the monotonicity of a firm's best response function with respect to its type ε_i.

Assumption 2. *For any value of $(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$, the marginal profit function $\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$ is (A) additively separable in the private information ε_i such that*

$$\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) = \Delta\pi_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) - \varepsilon_i \text{ and} \tag{4}$$

(B) $\Delta\pi_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x})$ is strictly decreasing in the own action a_i. For discrete choice models, and without loss of generality, I adopt the notational convention of $\Delta\pi_i(0, \boldsymbol{a}_{-i}, \mathbf{x}) = +\infty$ and $\Delta\pi_i(J+1, \boldsymbol{a}_{-i}, \mathbf{x}) = -\infty$. ■

The strict monotonicity of the marginal profit with respect to the own action—assumption 2(B)—holds in most models of market competition. For models of competition in price or quantity, a downward sloping demand curve and a non-decreasing marginal costs are sufficient conditions for the monotonicity of the marginal profit with respect to the own action.

The additivity of the private information in assumption 2(A) is not without loss of generality. The following example presents a simple model to illustrate some restrictions imposed by this assumption.

Example 1. Consider a model of Cournot competition in an homogeneous product industry. Variable $a_i \in \mathbb{R}_+$ represents firm i's amount of output. The inverse demand function is linear: $p = \alpha - \beta \sum_{j=1}^{N} a_j$. Parameters α and β are the true demand parameters. Firms do not know these parameters. Instead, each firm receives a private and independent signal (α_i, β_i) about the value of these parameters. The cost function of firm i is: $\gamma_i a_i + \delta_i a_i^2$, where γ_i and δ_i are private information of this firm. Therefore, the profit function is $\Pi_i = (\alpha_i - \beta_i \sum_{j=1}^{N} a_j)\ a_i - \gamma_i a_i - \delta_i a_i^2$ and the marginal profit function is

$$\Delta\Pi_i = \Pi_i(a_i) - \Pi_i(a_i - 1) = \alpha_i - \beta_i \sum_{j=1}^{N} a_j - \beta_i a_i - \gamma_i - \delta_i a_i. \tag{5}$$

This expression shows that the private information variables α_i and γ_i enter additively in the marginal profit. However, this is not the case for the private information variables β_i and δ_i which interact with the own output and with the output of other firms. Therefore, for this model, assumption 2(A) restricts private information to enter in the intercept of the demand curve (α_i) or/and in the linear term of cost function (γ_i). It does not allow private information in the slope of the demand (β_i) or in the quadratic term of the cost function (δ_i). ■

For all the identification results in this paper, assumption 2(A) can be replaced with the weaker condition that $\Delta\Pi_i = \Delta\pi_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) - g_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x})\varepsilon_i$, where function $g_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x})$ is positive valued and known to the researcher.

For notational simplicity, I use assumption 2(A) instead of this weaker version.[4] See Athey (2001) or Athey and Bagwell (2001) for more general results that require monotonicity but not additivity in ε_i.

Assumption 3 establishes the restriction that firms' types are independently distributed—*independent private values* (IPV)—and that a firm's beliefs about other firms' actions do not depend on the firm's own type.

ASSUMPTION 3. *(A) The private information variables* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ *are independently distributed conditional on* $\mathbf{x}$*:* $F(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N|\mathbf{x}) = \prod_{i=1}^{N} F_i(\varepsilon_i|\mathbf{x})$. *(B) Every firm i knows that other firms' private information is independent of its own private information such that* $\boldsymbol{a}_{-i}$ *and* ε_i *are independent conditional on* $\mathbf{x}$. *(C) The beliefs function* $b_i(\boldsymbol{a}_{-i}|\varepsilon_i, \mathbf{x})$ *does not depend on the firm's own type,* ε_i. *I represent a beliefs function as* $b_i(\boldsymbol{a}_{-i}|\mathbf{x})$. ■

The identification results in this paper—in propositions 1 to 5—still hold when we allow for two types of unobservables for the researcher: the firm-specific private information unobservables, ε_i—with an IPV structure—and unobservables, ω, which vary across markets but not over firms, can be known by every firm and can be payoff-relevant or not. For notational simplicity, I focus on a model without market unobserved heterogeneity ω.[5]

ASSUMPTION 4. *The private information* ε_i *is a continuous random variable with the real line as support and with a cumulative distribution function conditional on* $\mathbf{x}$*—*$F_i(\varepsilon_i|\mathbf{x})$*—that is strictly increasing over the whole real line.* ■

2.3. Characterization of firms' best response functions

Define the expected marginal profit (without including the private component ε_i) as

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) \equiv \int_{\boldsymbol{a}_{-i}} \Delta\pi_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) b_i(\boldsymbol{a}_{-i}|\mathbf{x}) d\boldsymbol{a}_{-i}. \tag{6}$$

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4 In a recent paper, Allen and Rehbeck (2019) show that the assumption of additive separability of unobservables provides identification of preferences and counterfactual behaviour in a very general class of single-agent static decision models.

5 To extend the identification results in this paper to a model with unobserved market heterogeneity, we can apply results from Aguirregabiria and Mira (2019). In that paper, the authors assume that unobserved market heterogeneity ω is independent of ε_i—but not independent of $\mathbf{x}$—and has a distribution with finite support. Using results from the literature on nonparametric finite mixtures, they establish the identification of the probability distribution of a firm's action conditional on $\mathbf{x}$ and ω (propositions 1 and 2 in Aguirregabiria and Mira 2019). Given the identification of this distribution, it is straightforward to extend propositions 1 to 5 to a model with this type of market unobserved heterogeneity such that a firm's beliefs function is $b_i(\boldsymbol{a}_{-i}|\mathbf{x}, \omega)$.

Consider the marginal condition of optimality. If a_i is firm i's optimal choice, then: (i) for models with continuous decision variable, $\Delta\pi_i^e(a_i, x; b_i) - \varepsilon_i = 0$ and (ii) for models with discrete decision variable, $\Delta\pi_i^e(a_i + 1, x; b_i) < \varepsilon_i \leq \Delta\pi_i^e(a_i, x; b_i)$, where—remember—I have adopted the convention that $\Delta\pi_i(0, \boldsymbol{a}_{-i}, \mathbf{x}) = +\infty$ and $\Delta\pi_i(J+1, \boldsymbol{a}_{-i}, \mathbf{x}) = -\infty$. Let $\sigma_i(\varepsilon_i, x, b_i)$ be firm i's best response function.

Proposition 1 establishes that the marginal condition is necessary and sufficient. This proposition also characterizes a firm's best response function in terms of the cumulative choice probability function.

Proposition 1. *(A) Under assumptions 1 to 3, the marginal condition of optimality is a necessary and sufficient condition for firm i's best response, i.e.,* $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = a_i$ *if and only if the marginal condition holds for value* a_i. *(B) Under assumptions 1 to 4, for any value* a^0 *in the choice set* $\mathcal{A}$, *we have the following relationship between the cumulative choice probability function* $P_i(a^0|\mathbf{x})$ *and the expected marginal profit:*

$$P_i(a^0|\mathbf{x}) = F_i\left[\Delta\pi_i^e(a^0, \mathbf{x}; b_i)|\mathbf{x}\right]. \tag{7}$$

This relationship is invertible such that we have

$$Q_i(a^0|\mathbf{x}) = \Delta\pi_i^e(a^0, \mathbf{x}; b_i), \tag{8}$$

where $Q_i(a^0|\mathbf{x}) \equiv F_i^{-1}\left[P_i(a^0|\mathbf{x})\right]$ *and* F_i^{-1} *is the inverse of* F_i *(quantile function).* ■

The following examples illustrate this characterization of best response functions using four standard models of competition: Cournot, auctions, market entry and number of stores, respectively.

Example 2 (Cournot competition). Consider a Cournot game of quantity choice in an homogeneous product industry. Let $a_i \in \mathbb{R}_+$ be firm i's amount of output. The market demand function is $d(Q, \mathbf{x})$, where $Q = \sum_{i=1}^N a_i$ and a firm's marginal cost function is $c_i(a_i, \mathbf{x}) + \varepsilon_i$. The expected marginal profit function is $\Delta\pi_i^e(a_i, \mathbf{x}; b_i) - \varepsilon_i$, where

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) = -c_i(a_i, \mathbf{x}) + \int_{\boldsymbol{a}_{-i}} a_i d\left(\sum_{j \neq i} a_j, \mathbf{x}\right) b_i(\boldsymbol{a}_{-i}|\mathbf{x}) d\boldsymbol{a}_{-i}. \qquad ■(9)$$

Example 3 (market entry game). Consider a game of market entry. Let $a_i \in \{0, 1\}$ be the indicator of the event "firm i is active in the market." A firm's profit if not active in the market is zero, $\Pi_i(0, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) = 0$. If active in the market, a firm's profit is equal to the variable profit $v_i(\boldsymbol{a}_{-i}, \mathbf{x})$ (i.e., revenue minus variable cost) minus the entry cost $ec_i(\mathbf{x}) + \varepsilon_i$. The marginal profit is the profit if active minus the profit if not active, which, in this case, is zero. Then, the expected marginal profit function is

$$\Delta\pi_i^e(1, \mathbf{x}; b_i) - \varepsilon_i = -ec_i(\mathbf{x}) - \varepsilon_i + \sum_{\boldsymbol{a}_{-i}} v_i(\boldsymbol{a}_{-i}, \mathbf{x}) b_i(\boldsymbol{a}_{-i}|\mathbf{x}). \qquad ■(10)$$

Example 4 (competition in number of stores or products). Consider a game where $a_i \in \{0,1,\dots,J\}$ represents the number of products that the firm has in the market. Similarly as in the market entry game, a firm's profit is equal to the variable profit $v_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x})$ minus the fixed cost $fc_i(a_i, \mathbf{x}) + a_i\varepsilon_i$. In this case, the private information $\varepsilon_i$is associated to the increase in the fixed cost from managing one more product. Then, the expected marginal profit function is

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) - \varepsilon_i = -\Delta fc_i(a_i, \mathbf{x}) - \varepsilon_i + \sum_{\boldsymbol{a}_{-i}} \Delta v_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) b_i(\boldsymbol{a}_{-i}|\mathbf{x}), \tag{11}$$

with $\Delta fc_i(a_i, \mathbf{x}) \equiv fc_i(a_i, \mathbf{x}) - fc_i(a_i - 1, \mathbf{x})$ and $\Delta v_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) \equiv v_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) - v_i(a_i - 1, \boldsymbol{a}_{-i}, \mathbf{x})$. ■

Example 5 (procurement auction). This example is slightly different from the previous ones because the private information variable ε_i is not additive in the marginal condition of optimality. I use this example to illustrate how this condition is not necessary to obtain the characterization of the best response function that I use for the identification results in this paper. Consider a procurement auction where $a_i \in \mathbb{R}$ represents firm i's bid. The profit function is $\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) = (a_i - c_i(\mathbf{x}) - \varepsilon_i)\ 1\{a_j > a_i \forall j \neq i\}$, where $c_i(\mathbf{x}) + \varepsilon_i$ is the cost and $1\{.\}$ is the indicator function such that $1\{a_j > a_i \forall j \neq i\}$ is the indicator of the event "firm i has the lowest bid and wins the auction."

The expected profit function is

$$\begin{aligned} \Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) &= (a_i - c_i(\mathbf{x}) - \varepsilon_i) \int_{\boldsymbol{a}_{-i}} 1\{a_j > a_i \forall j \neq i\} b_i(\boldsymbol{a}_{-i}|\mathbf{x}) d\boldsymbol{a}_{-i} \\ &= (a_i - c_i(\mathbf{x}) - \varepsilon_i)\, W(a_i, \mathbf{x}, b_i), \end{aligned} \tag{12}$$

where $W(a_i, \mathbf{x}, b_i) \equiv \int_{\boldsymbol{a}_{-i}} 1\{a_j > a_i \forall j \neq i\} b_i(\boldsymbol{a}_{-i}|\mathbf{x})\ d\boldsymbol{a}_{-i}$ is firm i's subjective probability of wining the auction given its beliefs. Therefore, the expected marginal profit function is

$$\Delta\pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) = W(a_i, \mathbf{x}, b_i) + (a_i - c_i(\mathbf{x}) - \varepsilon_i)\, \Delta W(a_i, \mathbf{x}, b_i), \tag{13}$$

where $\Delta W(a_i, \mathbf{x}, b_i)$ represents the derivative of the subjective probability of winning with respect to the own bid a_i.

Note that this expected marginal profit function is not additively separable in ε_i. However, $\Delta\pi_i^e$ is strictly monotonic in a_i and ε_i, and this implies that the best response function is strictly monotonic in ε_i. More specifically, we have that $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = a^0$ if and only if $W(a^0, \mathbf{x}, b_i) + (a^0 - c_i(\mathbf{x}) - \varepsilon_i)\Delta W(a^0, \mathbf{x}, b_i) = 0$, and this is equivalent to $\varepsilon_i = a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$. This implies that $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) \leq a^0$ if and only if $\varepsilon_i \leq a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$ such that we can represent the best response function using the following formula for the conditional quantile function:

$$Q_i(a^0|\mathbf{x}) = a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}. \quad ■ \tag{14}$$

2.4. Common equilibrium restrictions in empirical applications

The framework presented above includes as particular cases most games of competition with incomplete information in empirical IO applications. A main difference is that most empirical applications have assumed that firms' beliefs satisfy some equilibrium restrictions. Different equilibrium concepts have been used in the literature. I present here the equilibrium concepts that have received more attention in empirical applications in IO.

All these equilibrium concepts assume that firms choose their best response strategies given their beliefs: that is, they impose the best response conditions described above. In addition, they restrict beliefs to satisfy some additional equilibrium restrictions. I describe below these additional restrictions.

(a) *Bayesian Nash equilibrium (BNE) with independent private values.* This is the most commonly used solution concept in games of incomplete information in IO. It has received particular attention in auction games (e.g., Guerre et al. 2000, Athey and Haile 2002) and in discrete choice games (e.g., Seim 2006, Sweeting 2009). It has been used also in empirical applications of Cournot competition models with incomplete information (Armantier and Richard 2003, Aryal and 2019).

A BNE can be described as N cumulative choice probability functions—$\{P_i(a_i|\mathbf{x}) : i = 1, 2, \ldots, N\}$—satisfying the following conditions: (i) [best responses] $P_i(a_i|\mathbf{x})$ satisfies the best response equation (7) given beliefs b_i and (ii) [rational beliefs] the cumulative beliefs function B_i is equal to the actual distribution function of the choices of the other firms conditional on $\mathbf{x}$:

$$B_i(\boldsymbol{a}_{-i}|\mathbf{x}) = \prod_{j \neq i} P_j \left(a_j|\mathbf{x}\right). \tag{15}$$

(b) *Cognitive hierarchy (CH) and level-K models.* These models propose equilibrium concepts where firms have biased beliefs, that is, $B_i(\boldsymbol{a}_{-i}|\mathbf{x}) \neq \prod_{j \neq i} P_j \left(a_j|\mathbf{x}\right)$. They are based on the following ideas. Firms are heterogeneous in their beliefs and there is a finite number of belief types. That is, $B_i(\boldsymbol{a}_{-i}|\mathbf{x})$ belongs to a finite family of K belief functions, $\{B^{(k)}(\boldsymbol{a}_{-i}|\mathbf{x}) : k = 1, 2, \ldots, K\}$. Each member of this family is a "belief type." Belief types correspond to different levels of strategic sophistication and are determined by a hierarchical structure.

A type-0 firm has arbitrary believes $B^{(0)}$. In the level-k model, a type-k firm believes that all the other firms are type (k-1). Therefore, a type-k firm has beliefs:

$$B^{(k)}(\boldsymbol{a}_{-i}|\mathbf{x}) = \prod_{j \neq i} F_j \left(\Delta\pi_j^e(a_j, \mathbf{x}; B^{(k-1)})|\mathbf{x}\right). \tag{16}$$

This recursive equation defines the belief functions for every type k between 1 and K. Note that the only unrestricted function is the beliefs function for type-0: the rest of the belief functions are known functions of $B^{(0)}$ and the primitives of the model.

The CH model is more flexible than the level-k model. In the CH model, a type-k firm believes that the other firms come from a probability distribution over types 0 to (k−1). This model has been estimated in IO applications in Goldfarb and Xiao (2011), Brown et al. (2013) and Hortaçsu et al. (2019).

These models allow for some flexibility in beliefs. However, they still impose important restrictions. More specifically, they do not include BNE or rational beliefs as a particular case, and there is a small number of belief types. In applications, K is always smaller than N and typically equal to 2 or 3.

(c) *Rationalizability (*Bernheim 1984, Pearce 1984*).* The concept of rationalizability imposes two simple restrictions on firms' beliefs and behaviour. First, every firm maximizes its own expected profit given beliefs. And second, this rationality is common knowledge, i.e., every firms knows that every firm knows...that all the firms are rational. The set of outcomes of the game that satisfy these conditions—the set of rationalizable outcomes—includes all the Bayesian Nash equilibria (BNE) of the game but also outcomes where players have biased beliefs. More specifically, in a game with multiple equilibria, each firm has beliefs that are consistent with a BNE, but these beliefs may not correspond to the same BNE, that is, a firm believes that they are playing equilibrium A and other firm believes that the equilibrium played is B.

In general, the set of rationalizable beliefs is larger than the set of BNE but substantially smaller than the set of all the possible beliefs. Therefore, the condition of common knowledge rationality imposes non trivial restrictions with respect to the model that I consider in this paper. I show that these additional restrictions are testable.

(d) *Correlated Bayesian Nash equilibrium.* In recent work, Bergemann and Morris (2013, 2016) have introduced the solution concept of Bayesian correlated equilibrium (BCE). This solution is more robust than BNE, in the sense that it delivers all predictions compatible with BNE for any information structure within a wide class. Magnolfi and Roncoroni (2017) study inference based on the BCE solution concept. Their goal is to identify payoff parameters and they do not study the identification of beliefs. Their work illustrates a trade-off between robustness to assumptions about information structures and the ability to achieve point identification.

3. Dynamic games

In this section, I extend the previous framework to a dynamic game. Time is discrete and indexed by t. Now, $\Pi_{it}(a_{it}, \boldsymbol{a}_{-it}, \varepsilon_{it}, \mathbf{x}_t)$ represents the profit function of firm i at period t. The arguments of this function have the same interpretation as in section 2. Firms choose their actions at every period t to maximize their expected and discounted profits $\mathbb{E}_t(\sum_{s=0}^{T-t} \beta_i^s \pi_{it+s})$, where β_i is the firm's discount factor and T is the time horizon that can be finite or infinite.

I introduce an additional assumption that, in dynamic structural models, is typically described as the *conditional independence assumption.*

Assumption 5. *(A) The vector of state variables* $\mathbf{x}_t$ *follows a controlled Markov process with transition probability density function* $f_{xt}(\mathbf{x}_{t+1}|a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$. *(B) The private information variable* ε_{it} *is independently and identically distributed over time.* ■

Assumption 5(B) implies that the private information variables are not serially correlated. It rules out the possibility of firms using the history of other firms' decisions to learn about these firms' types. This type of learning is the focus of the Experience-Based equilibrium concept in Fershtman and Pakes (2012).

Every period, firms choose simultaneously their actions to maximize their intertemporal values. A firm's value at period t depends on the actions of other firms at that period and in the future. Firms form probabilistic beliefs about the actions of competitors, now and in the future. Let $b_{it+s}^{(t)}(\boldsymbol{a}_{-i,t+s}|\mathbf{x}_{t+s})$ be a probability density function, in the space of the actions of firms other than i, that represents the beliefs of firm i at period t about the behaviour of competitors at period $t+s$. This representation of beliefs allows for general forms of beliefs updating. According to this notation, $b_{it+s}^{(t+1)} - b_{it+s}^{(t)}$ represents the updating from period t to period $t+1$ in the beliefs that firm i has about the behaviour of competitors at period $t+s$.

Given a firm's beliefs at period t, $b_i^{(t)} \equiv \{b_{i,t+s}^{(t)} : s \geq 0\}$, its best response at period t is the solution of a single-agent dynamic programming (DP) problem. We can represent this DP problem using Bellman's principle. Let $V_{it}^{\mathbf{b}_i^{(t)}}(\mathbf{x}_t, \varepsilon_{it})$ be the value function. Then,

$$\begin{aligned} V_{it}^{\mathbf{b}_i^{(t)}}(\mathbf{x}_t, \varepsilon_{it}) = \max_{a_{it} \in \mathcal{A}} \Bigg\{ \int_{\boldsymbol{a}_{-it}} \Bigg[&\Pi_{it}(a_{it}, \boldsymbol{a}_{-it}, \varepsilon_{it}, \mathbf{x}_t) \\ &+ v_{it}^{\mathbf{b}_i^{(t)}}(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) \Bigg] b_{it}^{(t)}(\boldsymbol{a}_{-it}|\mathbf{x}_t) d\boldsymbol{a}_{-it} \Bigg\}, \end{aligned} \tag{17}$$

where $v_{it}^{\mathbf{b}_i^{(t)}}(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$ is the continuation value that has the following expression:

$$\beta \int V_{it+1}^{\mathbf{b}_i^{(t)}}(\mathbf{x}_{t+1}, \varepsilon_{it+1}) f_{xt}(\mathbf{x}_{t+1}|a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) d\mathbf{x}_{t+1} dF_i(\varepsilon_{it+1}|\mathbf{x}_t). \tag{18}$$

Given its beliefs, a firm chooses its strategy to maximize its value. That is, the best response strategy function, $\sigma_{it}(\varepsilon_{it}, \mathbf{x}_t, b_i^{(t)})$, is the maximand of the term in brackets $\{.\}$ in equation (17). As in the static game, it is convenient to represent a firm's strategy as a cumulative distribution function, or as a quantile function.

Proposition 2. *In the dynamic game, under assumptions 1 to 5, for any value* a^0 *in the choice set* $\mathcal{A}$, *we have the following relationship between the cumulative choice probability function* $P_{it}(a^0|\mathbf{x}_t)$ *and the marginal expected intertemporal profit:*

$$P_{it}(a^0|\mathbf{x}_t) = F_i\left[\Delta\pi_{it}^e(a^0, \mathbf{x}_t; b_{it}^{(t)}) + \Delta v_{it}^{e,\mathbf{b}_i^{(t)}}(a^0, \mathbf{x}_t; b_{it}^{(t)})|\mathbf{x}_t\right], \tag{19}$$

where $\Delta\pi^{e}_{it}(a^0,\mathbf{x}_t;b^{(t)}_{it}) \equiv \int_{\boldsymbol{a}_{-it}} \Delta\pi_{it}(a_{it},\boldsymbol{a}_{-it},\mathbf{x}_t) \quad b^{(t)}_{it}(\boldsymbol{a}_{-it}|\mathbf{x}_t) \quad d\boldsymbol{a}_{-it}$ *and* $\Delta v^{e,\mathbf{b}^{(t)}_i}_{it}(a^0,\mathbf{x}_t;b^{(t)}_{it}) \equiv \int_{\boldsymbol{a}_{-it}} \Delta v^{\mathbf{b}^{(t)}_i}_{it}(a_{it},\boldsymbol{a}_{-it},\mathbf{x}_t)\ b^{(t)}_{it}(\boldsymbol{a}_{-it}|\mathbf{x}_t)\ d\boldsymbol{a}_{-it}$. *This relationship is invertible such that we have*

$$Q_{it}(a^0|\mathbf{x}_t) = \Delta\pi^{e}_{it}(a^0,\mathbf{x}_t;b^{(t)}_{it}) + \Delta v^{e,\mathbf{b}^{(t)}_i}_{it}(a^0,\mathbf{x}_t;b^{(t)}_{it}), \tag{20}$$

where $Q_{it}(a^0|\mathbf{x}_t) \equiv F_i^{-1}\left[P_{it}(a^0|\mathbf{x}_t)\right]$. ■

In this framework, the sequence of beliefs $\mathbf{b}^{(t)}_i = \left\{b^{(t)}_{i,t+s} : s \geq 0\right\}$ is completely unrestricted. This framework contains as particular cases most solution concepts in dynamics games of competition with incomplete information. I present here some common cases.

(a) *Markov perfect equilibrium (MPE).* This is the most commonly used solution concept in applications of dynamic games in empirical IO (Maskin and Tirole 1988, Ericson and Pakes 1995). Here we consider a version of MPE that allows for non-stationarity due to finite time horizon T or primitive functions π_{it} and f_{xt} that vary over time.

An MPE can be described as N sequences of cumulative choice probability functions—$\{P_{it}(a_{it}|\mathbf{x}_t) : i = 1,2,\ldots,N;\ t \geq 1\}$—satisfying the following conditions: (i) [best responses] $P_{it}(a_{it}|\mathbf{x}_t)$ satisfies the best response condition in equation (19) and (ii) [rational beliefs] beliefs $\mathbf{b}^{(t)}_i$ are equal to the actual probability distribution of the choices of the other firms: for any $t \geq 1$, $s \geq 0$, $\boldsymbol{a}_{-i} \in \mathcal{A}^{N-1}$ and $\mathbf{x} \in \mathcal{X}$,

$$b^{(t)}_{i,t+s}(\boldsymbol{a}_{-i}|\mathbf{x}) = \prod_{j \neq i} p_{j,t+s}\left(a_j|\mathbf{x}\right). \tag{21}$$

(b) *Dynamic equilibrium with belief-based learning*, such as Bayesian learning, fictitious play and other forms of adaptive learning. These equilibrium concepts impose restrictions on the evolution of beliefs over time. They are restricted versions of the general model presented above. However, as mentioned in section 2.1, games with reinforcement learning are not a particular case of our model because they are not belief-based.

4. Identification

4.1. Data

This section presents results on the identification of beliefs in the previous general framework. I distinguish three scenarios for the data available to the researcher which are common in empirical IO applications.

(a) *Only firms' choice data.* The researcher has a sample of M local markets, indexed by m, where she observes firms' actions and state variables: $\{a_{imt}, \mathbf{x}_{mt} : i = 1,2,\ldots,N;\ t = 1,2,\ldots,T^{data}\}$. In empirical applications of market entry models, it is often the case that the researcher has only choice data, e.g., firms' entry/exit decision, and there is no direct information on firms' revenues or costs.

(b) *Choice data + revenue function.* In addition to data on firms' choices, the researcher may have data on some components of the profit function. In many IO applications, the researcher observes prices and quantities and can estimate the demand system. Given the demand system, the researcher knows the revenue function.

(c) *Choice data + revenue function + cost function.* Data on firms' marginal costs is rare but it is sometimes available (Hortaçsu and Puller 2008, Hortaçsu et al. 2019). Marginal costs can be also obtained from the estimation of a production function if the dataset contains information on firms' output and input quantities, and input prices.

To incorporate in our framework the data that the researcher has on the revenue or cost functions, we distinguish these two components in the profit function. A firm's profit is equal to revenue minus cost: $\pi_i = r_i - c_i$. Accordingly, the marginal profit is equal to the marginal revenue minus the marginal cost: $\Delta\pi_i = \Delta r_i - \Delta c_i$. The economic interpretation of this marginal revenue and marginal cost depends on the particular decision variable of the model that can be continuous (e.g., quantity, price, investment) or discrete (e.g., entry, number of products).

For the identification analysis below, I consider that the researcher has a random sample with infinite markets: $M \to \infty$. This population level approach is standard in the literature on identification. Given this infinite sample, the cumulative choice probability function $P_i(a^0|\mathbf{x}^0)$ is identified for every firm i and at every point $(a^0, \mathbf{x}^0)$ in the support $\mathcal{A} \times \mathcal{X}$. More precisely, by definition, we have that $P_i(a^0|\mathbf{x}^0) = \mathbb{E}(1\{a_{im} \leq a^0\}|\mathbf{x}_m = \mathbf{x}^0)$, and the expectation $\mathbb{E}(1\{a_{im} \leq a^0\}|\mathbf{x}_m = \mathbf{x}^0)$ is identified from our sample. For the rest of this section, I treat P_i as a known function.

4.2. The identification problem

Consider the static game. By proposition 1, we have that

$$P_i(a^0|\mathbf{x}) = F_i\left(\Delta r_i^e(a^0, \mathbf{x}, b_i) - \Delta c_i^e(a^0, \mathbf{x}, b_i)|\mathbf{x}\right), \tag{22}$$

where $\Delta r_i^e(a^0, \mathbf{x}, b_i) \equiv \int \Delta r_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) b_i(\boldsymbol{a}_{-i}|\mathbf{x}) d\boldsymbol{a}_{-i}$ and $\Delta c_i^e(a^0, \mathbf{x}, b_i) \equiv \int \Delta c_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) b_i(\boldsymbol{a}_{-i}|\mathbf{x}) d\boldsymbol{a}_{-i}$ are the (subjective) expected marginal revenue and expected marginal cost, respectively. Equation (22) summarizes all the restrictions that the model imposes on the distribution function P_i. The left-hand side of this equation—the distribution P_i—is known to the researcher. The right-hand side depends on the model primitives—the structural functions F_i, Δr_i and Δc_i—and on beliefs b_i. The model is fully (point) identified if the system of equations in (22) implies unique values for structural functions and beliefs.

It is clear that the model is strongly under-identified. While the number of restrictions (the dimension of the distribution P_i) is $(|\mathcal{A}| - 1)|\mathcal{X}|$, we have that only the dimension of the beliefs function b_i is $(|\mathcal{A}|^{N-1} - 1)|\mathcal{X}|$, which is obviously larger than the number of restrictions. When F_i, Δr_i and Δc_i

are unknown to the researcher, the under-identification is stronger. Despite the under-identification of the model, I show below that it possible to identify a function that only depends on beliefs. The identification of this "beliefs object" can be used to test the validity of different equilibrium concepts and restrictions on beliefs.

For the sake of simplicity, I first illustrate the identification results in a simple model of competition: a binary choice game with two firms. Furthermore, I assume that the probability distribution F_i is known to the researcher. In section 4.4, I show that this identification result extends to models with: (i) multinomial and continuous choices, (ii) nonparametric specification F_i, (iii) more than two players and (iv) dynamic games.

4.3. Two-firms binary choice game

Consider a binary choice game of price competition between two firms: $a_i = 0$ represents the choice of the low price (promotion price) and $a_i = 1$ means the choice of high price (regular price). Let $q_i = d_i(a_i, a_{-i}, \mathbf{x})$ be the demand function for the product of firm i. and let $C_i(q_i, \mathbf{x})$ be the cost as a function of output. Therefore, using our notation, the revenue function is $r_i(a_i, a_{-i}, \mathbf{x}) \equiv a_i d_i(a_i, a_{-i}, \mathbf{x})$ and the cost function is $c_i(a_i, a_{-i}, \mathbf{x}) = C_i(d_i(a_i, a_{-i}, \mathbf{x}), \mathbf{x})$.[6]

Let $P_i(0|\mathbf{x})$—or in short $P_i(\mathbf{x})$—be the probability that firm i chooses the low price. And let $b_i(0|\mathbf{x})$—or in short $b_i(\mathbf{x})$—be this firm's belief about the probability that the competitor chooses the low price. The marginal profit function is $\Delta\pi_i(a_{-i}, \mathbf{x}) \equiv \pi_i(1, a_{-i}, \mathbf{x}) - \pi_i(0, a_{-i}, \mathbf{x})$, that is, the difference between the profit with high price and with low price. Marginal profit is equal to marginal revenue minus marginal cost: $\Delta\pi_i(a_{-i}, \mathbf{x}) = \Delta r_i(a_{-i}, \mathbf{x}) - \Delta c_i(a_{-i}, \mathbf{x})$.

As established in proposition 1, the best response equation can be represented as

$$Q_i(\mathbf{x}) = \Delta\pi_i(0, \mathbf{x}) + b_i(\mathbf{x}) \left[\Delta\pi_i(1, \mathbf{x}) - \Delta\pi_i(0, \mathbf{x})\right], \tag{23}$$

with $Q_i(\mathbf{x}) \equiv F_i^{-1}(P_i(\mathbf{x}))$. Given this system of equations—for every value of $\mathbf{x}$—we are interested in the identification of marginal profits $\Delta\pi_i(0, \mathbf{x})$ and $\Delta\pi_i(1, \mathbf{x})$ as well as the beliefs function $b_i(\mathbf{x})$. We are particularly interested in the identification of the beliefs function $b_i(\mathbf{x})$, or at least on the identification of an object or parameter that only depends on this function.

Without further restrictions, the model is under-identified. More specifically, the order condition for identification does not hold: for each value of $\mathbf{x}$, there is only one restriction, i.e., one value of $Q_i(\mathbf{x})$, but three unknowns,

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6 Even if a firm's cost depends only on its own output, the cost as a function of prices depends both on the own price and competitors' prices. This is simply because the quantity produced and sold by a firm depends on all the prices. In contrast, in a Cournot game, where the decision variable a_i represents a firm's output, the cost function $c_i(a_i, a_{-i}, \mathbf{x})$ does not depend on a_{-i}.

$\Delta\pi_i(0,\mathbf{x})$, $\Delta\pi_i(1,\mathbf{x})$ and $b_i(\mathbf{x})$. Section 4.3.1 describes the identification of beliefs when the researcher has revenue and cost data. In sections 4.3.2 and 4.3.3, I present an exclusion restriction that identifies firms' beliefs even when only choice data is available.

4.3.1 Identification with revenue and cost data

Suppose that the researcher knows the revenue function and the cost function: the dataset includes information on prices and quantities of inputs and outputs that can be used to identify demand and cost functions. This implies that the marginal profits $\Delta\pi_i(0,\mathbf{x})$ and $\Delta\pi_i(1,\mathbf{x})$ are known to the researcher. Therefore, under the condition that $\Delta\pi_i(1,\mathbf{x})-\Delta\pi_i(0,\mathbf{x})\neq 0$, equation (23) implies that the beliefs function is fully identified:

$$b_i(\mathbf{x})=\frac{Q_i(\mathbf{x})-\Delta\pi_i(0,\mathbf{x})}{\Delta\pi_i(1,\mathbf{x})-\Delta\pi_i(0,\mathbf{x})}. \tag{24}$$

The identification condition $\Delta\pi_i(a_{-i}=1,\mathbf{x})-\Delta\pi_i(a_{-i}=0,\mathbf{x})\neq 0$ is quite intuitive. A firm's observed behaviour reveals information about the firm's beliefs only if beliefs have an effect on behaviour, and this is the case only if other firms' actions affect the firm's profit, i.e., only if $\Delta\pi_i(a_{-i}=1,\mathbf{x})-\Delta\pi_i(a_{-i}=0,\mathbf{x})\neq 0$.

Given the identification of firms' beliefs, the researcher can test the validity of restrictions on beliefs that are commonly imposed in applications: (a) *Testing for unbiased beliefs.* We say that firm i has unbiased beliefs about the behaviour of the other firm if $b_i(\mathbf{x})-p_{-i}(\mathbf{x})=0$ for every value of $\mathbf{x}$. Given the identification of $b_i(\mathbf{x})$, and that $p_{-i}(\mathbf{x})$ is known to the researcher, we can test this null hypothesis.

(b) *Testing for BNE.* The concept of BNE imposes the restrictions that all the firms play best responses and have unbiased beliefs. Therefore, testing the null hypothesis of BNE is equivalent to test the joint restrictions $b_1(\mathbf{x})-p_2(\mathbf{x})=0$ and $b_2(\mathbf{x})-p_1(\mathbf{x})=0$ for every value of $\mathbf{x}$.

As explained in footnote 5, all the identification results in this paper still hold when we allow for unobservables that vary across markets but not over firms and have finite support. These unobservables can be payoff-relevant or not. Multiple equilibria in the data can be represented using an unobservable that affects firms' behaviour and is not payoff-relevant. Therefore, this test for BNE applies also to a model and dataset with multiple equilibria in the data.

(c) *Testing for rationalizability.* Given that the researcher knows firms' profit functions, she can construct the set of rationalizable beliefs and then test if the identified beliefs—$b_1(\mathbf{x})$ and $b_2(\mathbf{x})$—belong to this set. To construct the set of rationalizable beliefs, we can use a simple iterative procedure as in Aradillas-Lopez and Tamer (2008). This iterative procedure exploits the property that the best response probability function $F_i(\Delta\pi_i(0,\mathbf{x})+b_i(\mathbf{x})[\Delta\pi_i(1,\mathbf{x})-\Delta\pi_i(0,\mathbf{x})])$ is strictly monotonic in the beliefs function $b_i(\mathbf{x})$. Suppose, without loss of generality, that $\Delta\pi_i(1,\mathbf{x})-\Delta\pi_i(0,\mathbf{x})>0$. At iteration k, the set of level-k rationalizable beliefs for firm 1 is $[L_1^{(k)}(\mathbf{x}), U_1^{(k)}(\mathbf{x})]$, with

$$\begin{cases} L_1^{(k)}(\mathbf{x}) = F_2(\Delta\pi_2(0,\mathbf{x}) + L_2^{(k-1)}(\mathbf{x})[\Delta\pi_2(1,\mathbf{x}) - \Delta\pi_2(0,\mathbf{x})]), \\ U_1^{(k)}(\mathbf{x}) = F_2(\Delta\pi_2(0,\mathbf{x}) + U_2^{(k-1)}(\mathbf{x})[\Delta\pi_2(1,\mathbf{x}) - \Delta\pi_2(0,\mathbf{x})]). \end{cases} \tag{25}$$

And we have a similar expression for the set of level-k rationalizable beliefs for firm 2. To test rationalizability of level k, we test the null hypothesis implied the inequalities $L_1^{(k)}(\mathbf{x}) \leq b_1(\mathbf{x}) \leq U_1^{(k)}(\mathbf{x})$ and $L_2^{(k)}(\mathbf{x}) \leq b_2(\mathbf{x}) \leq U_2^{(k)}(\mathbf{x})$.

4.3.2 Identification with revenue but not cost data

Suppose that the researcher has data that identifies the demand system and therefore the revenue function $r_i(a_i, a_{-i}, \mathbf{x})$. The cost function is unknown. The best response equation is

$$\begin{aligned} Q_i(\mathbf{x}) = \Delta r_i(0,\mathbf{x}) - \Delta c_i(0,\mathbf{x}) + b_i(\mathbf{x})[\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x}) \\ + \Delta c_i(1,\mathbf{x}) - \Delta c_i(0,\mathbf{x})], \end{aligned} \tag{26}$$

where $Q_i(\mathbf{x})$, $\Delta r_i(0,\mathbf{x})$ and $\Delta r_i(1,\mathbf{x})$ are known and $b_i(\mathbf{x})$, $\Delta c_i(1,\mathbf{x})$ and $\Delta c_i(0,\mathbf{x})$ are unknown. This restriction cannot identify beliefs and cost functions. For any possible value of $b_i(\mathbf{x})$, there exist values of the marginal costs $\Delta c_i(1,\mathbf{x})$ and $\Delta c_i(0,\mathbf{x})$ such that the best response equation holds. Therefore, there exist infinite combinations of $(b_i(\mathbf{x}), \Delta c_i(0,\mathbf{x}), \Delta c_i(1,\mathbf{x}))$ that satisfy the best response equation (26).

This identification problem is closely related to the traditional identification problem of the *nature of competition*, or the identification of collusive behaviour (Bresnahan 1982). Almost any observed behaviour can be justified as one with "non-collusive beliefs" if we select the appropriate marginal cost function. The following example describes connection in detail.

Example 6 (firms' beliefs and conjectural variations). In an influential paper, Bresnahan (1982) studies the identification of the *form* (or *nature*) *of competition* in a model with complete information. In a complete information game, the nature of competition can be described as a conjectural variation (CV) parameter. This CV parameter has similarities with our beliefs function, but there are also substantial differences between them. Our beliefs function is an endogenous object that varies with all the exogenous characteristics in the vector $\mathbf{x}$ affecting demand and costs. CV parameters are typically interpreted as exogenously given and do not vary when demand or costs change. As I explain below, this has important implications on the identification of beliefs relative to the identification of CV parameters.

The best response equation in Bresnahan (1982) is similar as equation (26) but replacing the beliefs function $b_i(\mathbf{x})$ with a parameter CV_i that is assumed invariant with $\mathbf{x}$. After the identification of the demand and marginal revenue functions, Bresnahan proposes an exclusion restriction that implies the identification of the parameter CV_i. I first describe this identification result using our notation, and then I show this assumption cannot provide identification of beliefs in our model where beliefs are endogenous.

Bresnahan's identification of the nature of competition (CV). Suppose that the vector of exogenous variables $\mathbf{x}$ has two components $(\widetilde{\mathbf{x}}, z)$, where z is a variable that satisfies two conditions: (i) (Bresnahan-*i*) z affects the marginal revenue function, or more precisely, the function $\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x})$, that is, variation in z "rotates" the demand curve such that marginal revenue changes, and (ii) (Bresnahan-*ii*) z does not enter in the marginal cost function.

Consider the best response equation (26) but where the beliefs function $b_i(\mathbf{x})$ is replaced with a constant parameter CV_i. Let z^a and z^b be two different values of the special variable that "rotates" the demand curve. Consider the best response equation evaluated at two different points, $(\widetilde{\mathbf{x}}, z^a)$ and $(\widetilde{\mathbf{x}}, z^b)$, and obtain the difference between these two equations. We get

$$\begin{aligned} &Q_i(\widetilde{\mathbf{x}}, z^a) - Q_i(\widetilde{\mathbf{x}}, z^b) = \Delta r_i(0,\widetilde{\mathbf{x}}, z^a) - \Delta r_i(0,\widetilde{\mathbf{x}}, z^b) \\ &+ CV_i \left[\Delta r_i(1,\widetilde{\mathbf{x}}, z^a) - \Delta r_i(0,\widetilde{\mathbf{x}}, z^a) - \Delta r_i(1,\widetilde{\mathbf{x}}, z^b) + \Delta r_i(0,\widetilde{\mathbf{x}}, z^b)\right]. \end{aligned} \tag{27}$$

Everything in this equation except parameter CV_i is known to the researcher. Furthermore, the identification assumption (Bresnahan-ii) above implies that $\Delta r_i(1,\widetilde{\mathbf{x}}, z^a) - \Delta r_i(0,\widetilde{\mathbf{x}}, z^a) - \Delta r_i(1,\widetilde{\mathbf{x}}, z^b) + \Delta r_i(0,\widetilde{\mathbf{x}}, z^b)$ is different to zero. Therefore, we can solve for CV_i to identify this parameter.

This exclusion restriction does not work for the identification of endogenous beliefs. In our model, the beliefs function $b_i(\mathbf{x})$ is an endogenous object that depends on all the exogenous variables affecting demand or costs. Therefore, under Bresnahan's identification assumptions (i) and (ii), we have that right-hand side of equation (27) becomes

$$\begin{aligned} \Delta r_i(0,\widetilde{\mathbf{x}}, z^a) - \Delta r_i(0,\widetilde{\mathbf{x}}, z^b) + &\left[b_i(\widetilde{\mathbf{x}}, z^a) - b_i(\widetilde{\mathbf{x}}, z^b)\right] \\ &[\Delta c_i(1,\widetilde{\mathbf{x}}) - \Delta c_i(0,\widetilde{\mathbf{x}})] + b_i(\widetilde{\mathbf{x}}, z^a)[\Delta r_i(1,\widetilde{\mathbf{x}}, z^a) \\ &- \Delta r_i(0,\widetilde{\mathbf{x}}, z^a)] - b_i(\widetilde{\mathbf{x}}, z^b)\left[\Delta r_i(1,\widetilde{\mathbf{x}}, z^b) - \Delta r_i(0,\widetilde{\mathbf{x}}, z^b)\right]. \end{aligned} \tag{28}$$

This expression depends both on beliefs and costs and it cannot be used to separately identify one from the other. ■

Berry and Haile (2014, sections 5 and 6) extend Bresnahan's analysis to a model of price competition in a differentiated product industry with complete information. They show that Bresnahan's exclusion restriction still identifies the nature of competition in their model and that there are other sources of exogenous variation that identify the nature of competition, such as a change in the firm's own marginal cost. However, this identification result in Berry and Haile (2014) does not extend to a model of competition with incomplete information and unrestricted beliefs. The argument is the same as the one presented above for the case of Bresnahan's exclusion restriction. In the model with incomplete information, the (expected) marginal revenue function depends on the firm's beliefs, that in turn depend on all the exogenous variables of the model affecting demand or marginal costs. Any exogenous variation in the firm's own marginal cost implies a variation in beliefs. Without knowledge of the beliefs function, it is not possible to identify how much of the observed change in prices or quantities is due to variation in the firm's own marginal cost and how much it is because the change in beliefs.

A useful exclusion restriction: variation in competitors' cost. The rest of this subsection presents an exclusion restriction that provides identification of firms' beliefs. Suppose that the vector $\mathbf{x}$ contains a firm-specific variable that affects the marginal cost of a firm but not the marginal cost of its competitors. For instance, input prices (e.g., wages, prices of intermediate inputs) can have firm specific variation because long-term contracts, bargaining with suppliers, internal labour markets, etc. Assumption 6 describes this condition more formally.

ASSUMPTION 6. *Vector $\mathbf{x}$ has the following elements ($\widetilde{\mathbf{x}}$, z_1, z_2,..., z_N) where $\widetilde{\mathbf{x}}$ can affect the marginal revenues and marginal costs of all the firms in an unrestricted way, and each variable z_i is firm-specific and satisfies the following conditions: (A) firm i's marginal cost (or/and marginal revenue) depends on z_i and (B) firm i's marginal cost and marginal revenue do not depend on $\mathbf{z}_{-i} \equiv \{z_j : j \neq i\}$.* ■

Under assumption 6, it is possible to identify firm i's beliefs using this firm's change in behaviour when $\mathbf{z}_{-i}$ varies. Proposition 3 establishes formally this result.

PROPOSITION 3. *Let z_{-i}^a, z_{-i}^b and z_{-i}^c be three different values for the variable z_{-i}. Then, under assumptions 1 to 4 and 6, the following equation holds:*

$$\frac{b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^c) - b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)}{b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)} = \frac{Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^c) - Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)}{Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)}, \tag{29}$$

such that an object that depends only on beliefs (left-hand side) is identified using the firm's observed behaviour (right-hand side). ■

The term $Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a)$ $-Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)$ captures the change in the behaviour of firm i when z_{-i} varies from z_{-i}^a to z_{-i}^b: that is, the change in the probability that firm i charges a low price when the competitor's wage rate changes. Since variable z_{-i} does not affect firm i's marginal revenue or marginal cost—assumption 6(B)—we can conclude that the change in firm i's pricing behaviour is the result of a change in this firm's beliefs. The difference between the best-response equation at points $(\widetilde{\mathbf{x}}, z_i, z_{-i}^a)$ and $(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)$ is

$$\begin{aligned} Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b) = &\left[b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)\right] \\ &[\Delta\pi_i(1, \widetilde{\mathbf{x}}, z_i) - \Delta\pi_i(0, \widetilde{\mathbf{x}}, z_i)]. \end{aligned} \tag{30}$$

This difference is not sufficient to identify the beliefs parameter $b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)$. The reason is that $\Delta\pi_i(1, \widetilde{\mathbf{x}}, z_i)-$ $\Delta\pi_i(0, \widetilde{\mathbf{x}}, z_i)$ depends on unknown marginal costs through the term $\Delta c_i(1, \widetilde{\mathbf{x}}, z_i)-$ $\Delta c_i(0, \widetilde{\mathbf{x}}, z_i)$. However, we can also obtain the difference between the best-response equation at points $(\widetilde{\mathbf{x}}, z_i, z_{-i}^c)$ and $(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)$ to get an equation similar to (30) but for $Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^c) - Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)$. Note that the term $\Delta\pi_i(1, \widetilde{\mathbf{x}}, z_i) - \Delta\pi_i(0, \widetilde{\mathbf{x}}, z_i)$

is common between these equations. Therefore, we can cancel this unknown term by obtaining the ratio between these two difference equations.

Equation (29) shows that the observed variation in the pricing behaviour of firm i when the competitor's input prices change reveals information about this firm's beliefs. We can separate beliefs from the primitives in the profit function.

In some models, the cost function of a firm does not depend on the action of other firms. For instance, this is the case in Cournot models of quantity competition or in the entry games because, in these models, the cost function c_i is a "pure" cost function and not the composition of the true cost function and the demand function. In these models, we have that $\Delta c_i(1, \widetilde{\mathbf{x}}, z_i) - \Delta c_i(0, \widetilde{\mathbf{x}}, z_i) = 0$ such that $\Delta \pi_i(1, \widetilde{\mathbf{x}}, z_i) - \Delta \pi_i(0, \widetilde{\mathbf{x}}, z_i) = \Delta r_i(1, \widetilde{\mathbf{x}}) - \Delta r_i(0, \widetilde{\mathbf{x}})$ that is known to the researcher. Therefore, we can identify the following beliefs parameter:

$$b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b) = \frac{Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^a) - Q_i(\widetilde{\mathbf{x}}, z_i, z_{-i}^b)}{\Delta r_i(1, \widetilde{\mathbf{x}}) - \Delta r_i(0, \widetilde{\mathbf{x}})}. \tag{31}$$

Given the identification of these beliefs objects—either in equation (29) or in (31), we can implement tests for the null hypotheses of unbiased beliefs and BNE in a similar way, as I have described at the end of section 4.3.1.

4.3.3 Identification using only firms' choice data

The previous exclusion restriction can be applied to the identification of beliefs also when the researcher has not identified the revenue function. Proposition 3 applies also to this case. Similarly, we can use the identified beliefs parameters to test the null hypotheses of unbiased beliefs and BNE.

Xie (2018) has obtained similar results on the identification beliefs without imposing the exclusion restriction in assumption 6. Instead, he shows that asymmetry in players' choice sets—together with sample variation in these choice sets—can provide identification of beliefs.

4.3.4 Full identification of the beliefs function

Given the result in proposition 3, the full identification of the beliefs function $b_i(\widetilde{\mathbf{x}}, z_i, z_{-i})$ requires two "normalization" restrictions for every value of $(\widetilde{\mathbf{x}}, z_i)$. The researcher needs to fix the value of beliefs at two different points in the support of the excluded variable, z_{-i}.

A possible way of fixing these values is to assume that beliefs are rational—that is, they are equal to the true value of the competitor's CCP—at two points in the support of z_{-i}. In this case, an important issue is how to select these two points. Aguirregabiria and Magesan (2020) describe several approaches that can help the researcher when making this modelling decision (see section 3.2.6 in that paper). A possible approach consists of applying a test of equilibrium beliefs to every possible triple of points of z_{-i} and then choosing two points from a triple where the null hypothesis is not rejected. This procedure involves multiple testing, and therefore p-values should be adjusted using, for instance, Bonferroni's correction. Furthermore, the effective application of this approach requires that, in the DGP, beliefs are unbiased at no less than three points in the support set of z_{-i}; otherwise, the test of unbiased beliefs will reject the null hypothesis at every triple. Other approach consists in testing for the monotonicity in z_{-i} of both the firm's beliefs function and the competitor's CCP function. Note that the restrictions in equation (29) imply that monotonicity of the beliefs function is a testable condition. If the two functions are strictly monotonic and z_{-i} has a large support, then the two functions converge to each other at extreme points of the support, and it seems reasonable to assume unbiased beliefs at these extreme points.

In principle, the researcher can fully identify the beliefs function by imposing functional form restrictions. However, the parametric restrictions necessary to fully identify this function are strong. The reason is that the set of restrictions in proposition 3—equation (29)—do not provide any information about the location or the scale of the beliefs function. Therefore, a necessary condition for the system of equations in (29) to fully identify a parametric beliefs function is that the parametric family is such that the location and the scale of the beliefs function are either: (i) known to the researcher or (ii) can be obtained from the second order or higher order derivatives of this function. To illustrate this point, it is useful to consider a polynomial specification of the beliefs function: that is, $b_i(\widetilde{\mathbf{x}}, z_i, z_{-i}) = \beta_{i0}(\widetilde{\mathbf{x}}, z_i) + \beta_{i1}(\widetilde{\mathbf{x}}, z_i) z_{-i} + \beta_{i2}(\widetilde{\mathbf{x}}, z_i) z_{-i}^2 + \cdots + \beta_{ip}(\widetilde{\mathbf{x}}, z_i) z_{-i}^p$, where $\beta_{i0}(\widetilde{\mathbf{x}}, z_i)$, $\beta_{i1}(\widetilde{\mathbf{x}}, z_i)$,..., $\beta_{ip}(\widetilde{\mathbf{x}}, z_i)$ are unknown parameters. Solving this function into equation (29), it is straightforward to show the identification of the parameters $\widetilde{\beta}_{ik}(\widetilde{\mathbf{x}}, z_i)$ for $k \geq 2$, where $\widetilde{\beta}_{ik}(\widetilde{\mathbf{x}}, z_i)$ is defined as $\beta_{ik}(\widetilde{\mathbf{x}}, z_i)/\beta_{i1}(\widetilde{\mathbf{x}}, z_i)$. Given this result, the full identification of the beliefs function requires either the a priori knowledge of the location parameter β_{i0} and the scale parameter β_{i1} or a parametric family where β_{ik} for $k \geq 2$ are related to β_{i0} and β_{i1} such that these location and scale parameters are identified from the knowledge of the $\widetilde{\beta}$s. Of course, these restrictions are not innocuous, and if incorrect, they imply the mis-specification of the model and the inconsistent estimation of beliefs and profit functions.

4.4. Extensions

Under assumption 6 and the condition that F_i is known to the researcher, the identification result in proposition 3 extends to models where the decision variable is (ordered) multinomial or continuous, as long as the support of the the state variable z_i has at least as many points as the decision variable a_i. The proof is a bit more technical than proposition 3 because it requires to represent the model in vector form and to show that a linear function of beliefs can be written as linear projection of the vector of quantiles. See proposition 1 in Aguirregabiria and Magesan (2020).

For the sake of simplicity, for the other extensions, I focus here on a two-player binary choice game.

4.4.1 Identification in dynamic games

We can apply proposition 2 to the two-player dynamic binary choice game to obtain the following expression for the quantile best response condition:

$$
\begin{aligned}
Q_{it}(\mathbf{x}_t) = {} & \Delta r_{it}(0,\mathbf{x}_t) - \Delta c_{it}(0,\mathbf{x}_t) + b_{it}^{(t)}(\mathbf{x}_t)[\Delta r_{it}(1,\mathbf{x}_t) - \Delta r_{it}(0,\mathbf{x}_t) \\
& - \Delta c_{it}(0,\mathbf{x}_t) + \Delta c_{it}(1,\mathbf{x}_t)] + \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0,\mathbf{x}_t) \\
& + b_{it}^{(t)}(\mathbf{x}_t)[\Delta v_{it}^{\mathbf{b}_i^{(t)}}(1,\mathbf{x}_t) - \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0,\mathbf{x}_t)]. \qquad (32)
\end{aligned}
$$

In this equation, the first line is identical to the static game and the second line contains the continuation value and therefore the dynamics of the game.

Unfortunately, the exclusion restriction in assumption 6 is not sufficient for the identification of beliefs in the dynamic game. Without further assumptions, z_{-it} is a state variable that affects the (marginal) continuation values of the two firms, even if this variable satisfies assumption 6 such that it does not have a different effect on the marginal profit of firm i at period t. For instance, suppose that z_{-it} is the wage rate paid by firm $-i$, the competitor of firm i. Though the value of the competitors' wage rate at period t does not have a direct effect on the contemporaneous marginal profit of firm i—once we account for the expected decision of the competitor—it can affect the continuation value as it affects future wages.

Assumption 6* extends assumption 6 such that we can get identification of beliefs in dynamic games.

Assumption 6∗. *Vector* $\mathbf{x}_t$ *has the following elements* $(\widetilde{\mathbf{x}}_t, z_{1t}, z_{2t}, \ldots, z_{Nt})$, *which satisfy conditions (A) and (B) in assumption 6. In addition, they also satisfy (C)—the transition probability of the state variable* z_{it} *is such that* z_{it+1} *does not depend on* $(z_{it}, \mathbf{z}_{-it})$ *once we condition on* a_{it} *and* $\widetilde{\mathbf{x}}_t$, *i.e.*,

$$Pr(z_{it+1}|a_{it}, \widetilde{\mathbf{x}}_t, z_{it}, \mathbf{z}_{-it}) = Pr(z_{it+1}|a_{it}, \widetilde{\mathbf{x}}_t)\,. \qquad \blacksquare(33)$$

Assumption 6*(C) holds in many applications of dynamic games in empirical IO. The incumbent status, capacity, capital stock or product quality of a firm at period $t-1$ are state variables that enter in the firm's payoff function at period t due to investment and adjustment costs. A firm's payoff function at period t depends also on the competitors' values of these variables at period t, but it does not depend on the competitors' values of these variables at $t-1$. Very importantly, assumption 6*(C) does not mean that firm i does not condition his behaviour on the state variables $\mathbf{z}_{-it}$. Each firm conditions his behaviour on all the state variables that affect the profit of a firm in the game, even if these variables are excluded from his own payoff.

More specifically, an important class of dynamic games that satisfies assumption 6*(C) consists of models where the transition rule for this state variable is $z_{i,t+1} = a_{it}$. It is clear that this transition rule is a particular case of equation (33). This is an important class of dynamic games that includes as particular cases games of market entry/exit, technology adoption, pricing with menu costs and some dynamic games of quality or capacity competition, among others.

Of course, assumption 6*(C) rules out many candidates for exclusion restrictions that do satisfy conditions assumption 6(A) and assumption 6(B). As I have mentioned above, this is the case of the competitor's input prices because they typically follow Markov processes where $z_{i,t+1}$ depends on z_{it}.

A key implication of assumption 6*(C) is that the marginal continuation values $\Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{-it}, \mathbf{x}_t)$, which are defined as conditional on the competitor's decision a_{-it}, do not depend on the state variable z_{-it}, that is, $\Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{-it}, \widetilde{\mathbf{x}}_t, z_{it}, z_{-it}) = \Delta v_{it}^{\mathbf{b}_i^{(t)}}(a_{-it}, \widetilde{\mathbf{x}}_t)$. This state variable affects future expected continuation values $\Delta v_{it}^{e,\mathbf{b}_i^{(t)}}(\mathbf{x}_t; b_{it}^{(t)})$ only through the firm's beliefs, that is, $\Delta v_{it}^{e,\mathbf{b}_i^{(t)}}(\mathbf{x}_t; b_{it}^{(t)}) = \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0, \widetilde{\mathbf{x}}_t) + b_{it}^{(t)}(\widetilde{\mathbf{x}}_t, z_{it}, z_{-it})[\Delta v_{it}^{\mathbf{b}_i^{(t)}}(1, \widetilde{\mathbf{x}}_t) - \Delta v_{it}^{\mathbf{b}_i^{(t)}}(0, \widetilde{\mathbf{x}}_t)]$. This property implies the identification of beliefs.

PROPOSITION 4. *Consider the two-player binary choice dynamic game under assumptions 1 to 5 and 6*. Then, the following equation holds:*

$$\frac{b_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^c_{-it}) - b_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^b_{-it})}{b_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^a_{-it}) - b_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^b_{-it})} = \frac{Q_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^c_{-it}) - Q_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^b_{-it})}{Q_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^a_{-it}) - Q_{it}(\widetilde{\mathbf{x}}_t, z_{it}, z^b_{-it})} \tag{34}$$

such that an object that depends only on beliefs (left-hand side) is identified using the firm's observed behaviour (right-hand side). ■

Note that proposition 4 does not impose any restriction on the evolution of beliefs over time. Therefore, the result is robust to very general forms of firms' belief-based learning. On the negative side, proposition 4 establishes the identification of firms' beliefs about competitors contemporaneous behaviour: that is, beliefs at period t about the opponents' behaviour at period t. We cannot identify beliefs about the opponent's behaviour in the future. However, Aguirregabiria and Magesan (2019) show that the identification of the evolution of these contemporaneous beliefs is enough for testing a very general class of models of learning and beliefs formation.

4.4.2 Identification of beliefs with unknown distribution of private information

For all the identification results presented above, we have imposed the restriction that the researcher knows the distribution function of the private information variable ε_i. This is not an innocuous assumption. Mis-specifying this distribution function can generate wrong conclusions about beliefs. Therefore, it is important to study the identification of beliefs when the distribution function F_i is unknown to the researcher.

Proposition 5 shows that, when the state variables z_i have continuous support, beliefs can be identified even when the distribution F_i is unknown to the researcher and nonparametrically specified.

Proposition 5. *Consider the static binary choice game under assumptions 1 to 4 and 6. Suppose that: (i) the distribution F_i is independent of z_i and z_{-i} but may depend on $\widetilde{\mathbf{x}}$, (ii) z_i and z_{-i} are continuous random variables, (iii) $P_i(\widetilde{\mathbf{x}}, z_i, z_{-i})$ is strictly monotonic in z_i and z_{-i} and asymmetric in these two arguments, i.e., for $z \neq z'$, generically, $P_i(\widetilde{\mathbf{x}}, z, z') \neq P_i(\widetilde{\mathbf{x}}, z', z)$, (iv) the researcher knows the revenue function and (v) firm i's marginal cost does not depend on a_{-i}. Let (z_i^A, z_{-i}^A) and (z_i^B, z_{-i}^B) be two arbitrary values of (z_i, z_{-i}). Then, the following results hold.*
(A) There exist values z_{-i}^{AB*} and z_{-i}^{BA*} that are uniquely identified and satisfy the following three properties: (1) $z_{-i}^{AB*} \neq z_{-i}^{BA*}$, (2) $P_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^A) = P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})$ and (3) $P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^B) = P_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^{BA*})$.
(B) The following condition holds:

$$\frac{b_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^A) - b_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^{BA*})}{b_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^B) - b_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})} = -\frac{\Delta r_i(1, \widetilde{\mathbf{x}}, z_i^A) - \Delta r_i(0, \widetilde{\mathbf{x}}, z_i^A)}{\Delta r_i(1, \widetilde{\mathbf{x}}, z_i^B) - \Delta r_i(0, \widetilde{\mathbf{x}}, z_i^B)} \tag{35}$$

such that an object that depends only on beliefs (left-hand side) is identified using the firm's observed behaviour and revenue function.

Proof. Condition (v) establishes that firm i's marginal cost does not depend on the other firm's actions. This condition applies to Cournot, market entry, auctions and many other models. Under this condition, the best response quantile equation becomes

$$Q_i(\mathbf{x}) = \Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x}) + b_i(\mathbf{x})\,[\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x})]. \tag{36}$$

Under conditions (ii) and (iii) in proposition 5, for any value p in the interior of the image set of function $P_i(\widetilde{\mathbf{x}}, z_i^B, .)$, we can find a unique value z_{-i}^* that solves equation $p = P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i})$ with respect to z_{-i}. Given that the function $P_i(.)$ is identified, this value z_{-i}^* is also identified. Define z_{-i}^{AB*} as the unique value of z_{-i} that solves the equation $P_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^A) = P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i})$ with respect to z_{-i}, and similarly, define z_{-i}^{BA*} as the unique value of z_{-i} that solves the equation $P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^B) = P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i})$ with respect to z_{-i}. Given the asymmetry of function P_i with respect to z_i and z_{-i}—condition (iii)—we have that z_{-i}^{AB*} and z_{-i}^{BA*} are two different values. Therefore, we have found values z_{-i}^{AB*} and z_{-i}^{BA*} that satisfy conditions (1) to (3) in proposition 5.

Since $P_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^A) = P_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})$ and the distribution function F_i is independent of (z_i, z_{-i})—condition (i)—the quantile values $Q_i(\widetilde{\mathbf{x}}, z_i^A, z_{-i}^A)$ and $Q_i(\widetilde{\mathbf{x}}, z_i^B, z_{-i}^{AB*})$ are also the same. Applying this condition to the best response quantile equation, we have that (omitting $\widetilde{\mathbf{x}}$ as an argument for notational simplicity)

$$\begin{aligned}
0 &= Q_i(z_i^A, z_{-i}^A) - Q_i(z_i^B, z_{-i}^{AB*}) \\
&= \left\{\Delta r_i(0, z_i^A) - \Delta c_i(0, z_i^A) + b_i(z_i^A, z_{-i}^A)\left[\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A)\right]\right\} \\
&\quad - \left\{\Delta r_i(0, z_i^B) - \Delta c_i(0, z_i^B) + b_i(z_i^B, z_{-i}^{AB*})\left[\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B)\right]\right\}.
\end{aligned} \tag{37}$$

Similarly, we obtain the condition

$$\begin{aligned} 0 &= Q_i(z_i^B, z_{-i}^B) - Q_i(z_i^A, z_{-i}^{BA*}) \\ &= \Big\{ \Delta r_i(0, z_i^B) - \Delta c_i(0, z_i^B) + b_i(z_i^B, z_{-i}^B) \left[\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B) \right] \Big\} \\ &\quad - \Big\{ \Delta r_i(0, z_i^A) - \Delta c_i(0, z_i^A) + b_i(z_i^A, z_{-i}^{BA*}) \left[\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A) \right] \Big\}. \end{aligned} \tag{38}$$

Adding up equations (37) and (38), we obtain the condition $0 = [b_i(z_i^B, z_{-i}^B) - b_i(z_i^B, z_{-i}^{AB*})]\ \ [\Delta r_i(1, z_i^B) - \Delta r_i(0, z_i^B)] + [b_i(z_i^A, z_{-i}^A) - b_i(z_i^A, z_{-i}^{BA*})]\ \ [\Delta r_i(1, z_i^A) - \Delta r_i(0, z_i^A)]$, and this condition implies equation (35) in proposition 5. ■

5. Conclusions

Firms face substantial uncertainty about competitors' strategies. The assumption of complete information may be convenient in some empirical applications, but it ignores sources of strategic uncertainty that can be important to understand the observed variation in behaviour between firms, across markets and over time. Furthermore, in changing economic environments—e.g., new regulations, mergers, pandemics—the assumption that firms have unbiased beliefs can be very unrealistic and affect our estimates of structural parameters. Perhaps most importantly, heterogeneity in firms' beliefs can be an important source of misallocation and inefficiency in some industries.

In this paper, I have presented a framework where firms have incomplete information, face strategic uncertainty and may have biased beliefs. I have shown that standard exclusion restrictions, which are present in most applications in empirical IO, provide identification of objects that depend only on firms' beliefs. The approach imposes minimal restrictions on beliefs and on structural functions. As such, the identified objects can be used to explore the determinants and patterns of firms' beliefs and learning over time.

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