EULER EQUATIONS FOR THE ESTIMATION OF DYNAMIC DISCRETE CHOICE STRUCTURAL MODELS

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ABSTRACT

We derive marginal conditions of optimality (i.e., Euler equations) for a general class of Dynamic Discrete Choice *(*DDC*) structural models. These conditions can be used to estimate structural parameters in these models without having to solve for approximate value functions. This result extends to discrete choice models the* GMM-Euler equation *approach proposed by Hansen and Singleton (1982) for the estimation of dynamic continuous decision models. We first show that DDC models can be represented as models of continuous choice where the decision variable is a vector of choice probabilities. We then prove that the marginal conditions of optimality and the envelope conditions required to construct Euler equations are also satisfied in* DDC *models. The GMM estimation of these Euler equations avoids the curse of dimensionality associated to the computation of value functions and the explicit integration over the space of state variables. We present an empirical application and compare*

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estimates using the GMM-Euler equations method with those from maximum likelihood and two-step methods.

Keywords: Dynamic discrete choice structural models; Euler equations; choice probabilities

JEL classifications: C35; C51; C61

INTRODUCTION

The estimation of *Dynamic Discrete Choice* (*DDC*) *structural models* requires the computation of expectations (value functions) defined as integrals or summations over the space of state variables. In most empirical applications, the range of variation of the vector of state variables is continuous or discrete with a very large number of values. In these cases the exact solution of expectations or value functions is an intractable problem. To deal with this dimensionality problem, applied researchers use approximation techniques such as discretization, Monte Carlo simulation, polynomials, sieves, neural networks, etc.[1] These approximation techniques are needed not only in full-solution estimation techniques but also in any two-step or sequential estimation method that requires the computation of value functions.[2] Replacing *true* expected values with approximations introduces an approximation error, and this error induces a statistical bias in the estimation of the parameter of interests. Though there is a rich literature on the asymptotic properties of these simulation-based estimators,[3] little is known about how to measure this *approximation-induced estimation bias* for a given finite sample.[4]

In this context, the main contribution of this article is in the derivation of marginal conditions of optimality (Euler equations) for a general class of *DDC* models. We show that these Euler equations provide moment conditions that can be used to estimate structural parameters without solving or approximating value functions. The estimator based on these Euler equations is not subject to bias induced by the approximation of value functions. Our result extends to discrete choice models the *GMM-Euler equation* approach that Hansen and Singleton (1982) proposed for the estimation of dynamic models with continuous decision variables. The *GMM-Euler equation* approach has been applied extensively to the estimation of dynamic structural models with continuous decision variables, such as problems of household consumption, savings, and portfolio choices, or firm investment decisions, among others. The conventional wisdom was that this method could not be applied to discrete choice models because, obviously, there are not marginal conditions of optimality with respect to discrete choice variables. In this article, we show that the optimal decision rule in a dynamic (or static) discrete choice model can be derived from a decision problem where the choice variables are probabilities that have continuous support. Using this representation of a discrete choice model, we obtain Euler equations by combining marginal conditions of optimality and Envelope Theorem conditions in a similar way as in dynamic models with continuous decision variables. Just as in the Hansen–Singleton approach, these Euler equations can be used to construct moment conditions and to estimate the structural parameters of the model by GMM without having to evaluate/approximate value functions.

Our derivation of Euler equations for DDC models extends previous work by Hotz and Miller (1993), Aguirregabiria and Mira (2002), and Arcidiacono and Miller (2011). These papers derive representations of optimal decision rules using *Conditional Choice Probabilities* (CCPs) and show how these representations can be applied to estimate DDC models using simple two-step methods that provide substantial computational savings relative to full-solution methods. In these papers, we can distinguish three different types of CCP representations of optimal decisions rules: (1) the *present-value* representation which consists of using CCPs to obtain a closed-form expression for the expected and discounted stream of future payoffs associated with each choice alternative; (2) the *terminal-state* representation which applies only to optimal stopping problems with a terminal state; and (3) the *finite-dependence* representation which was introduced by Arcidiacono and Miller (2011) and applies to a particular class of DDC models with the finite dependence property.[5]

Our article presents a new CCP representation that we call *CCP-Euler-equation* representation. This representation has several advantages over the previous ones. The *present-value representation* is the CCP approach more commonly used in empirical applications because it can be applied to a general class of DDC models. However, that representation requires the computation of present values and therefore it is subject to the curse of dimensionality and to biases induced by approximation error (e.g., discretization, Monte Carlo simulation). The *terminal-state*, the *finite-dependence*, and the CCP-Euler-equation representations do not involve the computation of present values, or even the estimation of CCPs at every possible state, and this implies substantial computational savings as well as avoiding biases induced by approximation errors. Furthermore, relative to *terminal-state* and *finite-dependence* representations, our *Euler equation* applies to a general class of DDC models. We can derive Euler equations for any DDC model where the unobservables satisfy the conditions of additive separability (AS) in the payoff function, and conditional independence (CI) in the transition of the state variables.

The estimation based on the moment conditions provided by the *Euler equation*, or *terminal-state*, or *finite-dependence* representations imply an efficiency loss relative to estimation based on *present-value* representation. As shown by Aguirregabiria and Mira (2002, Proposition 4), the two-step pseudo maximum likelihood (PML) estimator based on the CCP *present-value* representation is asymptotically efficient (equivalent to the maximum likelihood (ML) estimator). However, this efficiency property is not shared by the other CCP representations. Therefore, there is a trade-off in the choice between CCP estimators based on Euler equations and on present-value representations. The present-value representation is the best choice in models that do not require approximation methods. However, in models with large state spaces that require approximation methods, the Euler equations CCP estimator can provide more accurate estimates.

We present an empirical application where we estimate a model of firm investment. We compare estimates using CCP Euler equations, CCP present-value, and ML methods.

EULER EQUATIONS IN DYNAMIC DECISION MODELS

Dynamic Decision Model

Time is discrete and indexed by t. Every period t, an agent chooses an action a_t within the set of feasible actions $\mathcal{A}$ that, for the moment, can be either a continuous or a discrete choice set. The agent makes this decision to maximize his expected intertemporal payoff $\mathbb{E}_t\left[\sum_{j=0}^{T-t} \beta^j \Pi_t(a_{t+j}, \mathbf{s}_{t+j})\right]$, where $\beta \in (0,1)$ is the discount factor, T is the time horizon that can be finite or infinite, $\Pi_t(.)$ is the one-period payoff function at period t, and $\mathbf{s}_t$ is the vector of state variables at period t. These state variables follow a controlled Markov process, and the transition probability density function at period t is $f_t(\mathbf{s}_{t+1}|a_t, \mathbf{s}_t)$. By Bellman's principle of optimality, the sequence of value functions $\{V_t(.) : t \geq 1\}$ can be obtained using the recursive expression:

$$V_t(\mathbf{s}_t) = \max_{a_t \in A} \left\{ \Pi_t(a_t, \mathbf{s}_t) + \beta \int V_{t+1}(\mathbf{s}_{t+1}) f_t(\mathbf{s}_{t+1} \mid a_t, \mathbf{s}_t) \, d\mathbf{s}_{t+1} \right\} \quad (1)$$

The sequence of optimal decision rules $\{\alpha_t^*(.) : t \geq 1$ are defined as the argmax in $a_t \in \mathcal{A}$ of the expression within brackets in Eq. (1).

Suppose that the primitives of the model $\{\Pi_t, f_t, \beta\}$ can be characterized in terms of vector of structural parameters θ. The researcher has panel data for N agents (e.g., individuals, firms) over $\tilde{T}$ periods of time, with information on agents' actions and a subset of the state variables. The estimation problem is to use these data to consistently estimate the vector of parameters θ. In this section, we first describe this approach in the context of continuous-choice models, as proposed in the seminal work by Hansen and Singleton (1982). Second, we show how a general class of discrete choice models can be represented as continuous choice models where the decision variable is a vector of choice probabilities. Finally, we show that it is possible to construct Euler equations using this alternative representation of discrete choice models, and that these Euler equations can be used to construct moment conditions and a GMM estimator of the structural parameters θ.

Euler Equations in Dynamic Continuous Decision Models

Suppose that the decision a_t is a vector of continuous variables in the K dimensional Euclidean space: $a_t \in \mathcal{A} \subseteq \mathbb{R}^K$. The vector of state variables $\mathbf{s}_t \equiv (y_t, z_t)$ contains both *exogenous* (z_t) and *endogenous* (y_t) variables. Exogenous state variables follow a stochastic process that does not depend on the agent's actions $\{a_t\}$, for example, the price of capital in a model of firm investment under the assumption that firms are price takers in the capital market. In contrast, the evolution over time of the endogenous state variables, y_t, depends on the agent's actions, for example, the stock of capital in a model of firm investment. More precisely, the transition probability function of the state variables is

$$f_t(\mathbf{s}_{t+1} \mid a_t, \mathbf{s}_t) = 1\{y_{t+1} = Y(a_t, \mathbf{s}_t, z_{t+1})\} f_t^z(z_{t+1} \mid z_t) \quad (2)$$

where $1\{.\}$ is the indicator function, $Y(.)$ is a vector-valued function that represents the transition rule of the endogenous state variables, and f_t^z is the transition density function for the exogenous state variables. For the derivation of Euler equations in a continuous decision model, it is convenient to represent the transition rule of the endogenous state variables using the expression $1\{y_{t+1} = Y(a_t, \mathbf{s}_t, z_{t+1})\}$. This expression establishes that y_{t+1} is a deterministic function of $(a_t, \mathbf{s}_t, z_{t+1})$. However, this structure allows for a stochastic transition in the endogenous state variables because z_{t+1} is an argument of function $Y(.)$.[6] The following assumption provides sufficient conditions for the derivation of Euler equations in dynamic continuous decision models.

Assumption EE-Continuous. (A) The payoff function Π_t and the transition function $Y(.)$ are continuously differentiable in all their arguments. (B) a_t and y_t are both vectors in the K-dimension Euclidean space and for any value of $(a_t, \mathbf{s}_t, z_{t+1})$ we have that

$$\frac{\partial Y(a_t, \mathbf{s}_t, z_{t+1})}{\partial y_t'} = H(a_t, \mathbf{s}_t) \frac{\partial Y(a_t, \mathbf{s}_t, z_{t+1})}{\partial a_t'} \tag{3}$$

where $H(a_t, \mathbf{s}_t)$ is a $K \times K$ matrix.

For the derivation of the Euler equations, we consider the following *constrained optimization problem*. We want to find the decisions rules at periods t and $t+1$ that maximize the one-period-forward expected profit $\Pi_t + \beta\, \mathbb{E}_t(\Pi_{t+1})$ under the constraint that the probability distribution of the endogenous state variables y_{t+2} conditional on $\mathbf{s}_t$ implied by the new decision rules $\alpha_t(.)$ and $\alpha_{t+1}(.)$ is identical to that distribution under the optimal decision rules of our original DP problem, $\alpha_t^*(.)$ and $\alpha_{t+1}^*(.)$. By construction, this optimization problem depends on payoffs at periods t and $t+1$ only, and not on payoffs at $t+2$ and beyond. And by definition of optimal decision rules, we have that $\alpha_t^*(.)$ and $\alpha_{t+1}^*(.)$ should be the optimal solutions to this constrained optimization problem. For a given value of the state variables $\mathbf{s}_t$, we can represent this constrained optimization problem as

$$\begin{array}{l} \max_{\{a_t,\, a_{t+1}\} \in \mathcal{A}^2} \left\{ \Pi_t(a_t, \mathbf{s}_t) + \beta \int \Pi_{t+1}(a_{t+1}, Y(a_t, \mathbf{s}_t, z_{t+1}), z_{t+1})\, f_t^z(z_{t+1}|z_t) dz_{t+1} \right\} \\ \text{subject to: } Y(a_{t+1}, Y(a_t, \mathbf{s}_t, z_{t+1}), z_{t+1}, z_{t+2}) = \kappa_{t+2}^*(\mathbf{s}_t, z_{t+1}, z_{t+2}) \end{array} \tag{4}$$

where $Y(a_{t+1}, Y(a_t, \mathbf{s}_t, z_{t+1}), z_{t+1}, z_{t+2})$ represents the realization of y_{t+2} under arbitrary choice (a_t, a_{t+1}), and $\kappa_{t+2}^*(\mathbf{s}_t, z_{t+1}, z_{t+2})$ is a function that represents the realization of y_{t+2} under the optimal decision rules $\alpha_t^*(\mathbf{s}_t)$ and $\alpha_{t+1}^*(\mathbf{s}_{t+1})$, and it does not depend on (a_t, a_{t+1}). This constrained optimization problem can be solved using the Lagrangian method. It is possible to show that the optimal solution should satisfy the following marginal condition of optimality:[7]

$$\mathbb{E}_t\left(\frac{\partial \Pi_t}{\partial a_t'} + \beta\left[\frac{\partial \Pi_{t+1}}{\partial y_{t+1}'} - H(a_{t+1}, \boldsymbol{s}_{t+1})\frac{\partial \Pi_{t+1}}{\partial a_{t+1}'}\right]\frac{\partial Y_{t+1}}{\partial a_t'}\right) = 0 \tag{5}$$

where $\mathbb{E}_t(.)$ represents the expectation over the distribution of $\{a_{t+1}, \boldsymbol{s}_{t+1}\}$ conditional on $(a_t, \boldsymbol{s}_t)$. This system of equations is the *Euler equations* of the model.

Example 1. (*Optimal consumption and portfolio choice*; Hansen & Singleton, 1982). The vector of decision variables is $(c_t,\ q_{1t},\ q_{2t}, \ldots,\ q_{Jt})$ where c_t represents the individual's consumption expenditure, and q_{jt} denotes the number of shares of asset/security j that the individual holds in his portfolio at period t. The utility function depends only on consumption, that is, $\Pi_t(a_t, \boldsymbol{s}_t) = U_t(c_t)$. The consumer's budget constraint establishes that $c_t + \sum_{j=1}^{J} r_{jt}q_{jt} \leq w_t + \sum_{j=1}^{J} r_{jt}$, where w_t is labor earnings, and r_{jt} is the price of asset j at time t. Given that the budget constraint is satisfied with equality, we can write the utility function as $\Pi_t(a_t, \boldsymbol{s}_t) = U_t\left(w_t - \sum_{j=1}^{J} r_{jt}[q_{jt} - q_{jt-1}]\right)$, and the decision problem can be represented in terms of the decision variables $a_t = (q_{1t}, q_{2t}, \ldots, q_{Jt})$. The vector of exogenous state variables is $z_t = (w_t, r_{1t}, r_{2t}, \ldots, r_{Jt})$, and the vector of endogenous state variables consists of the individual's asset holdings at $t-1$, $y_t = (q_{1t-1}, q_{2t-1}, ..., q_{Jt-1})$. Therefore, the transition rule of the endogenous state variables is trivial, that is, $y_{t+1} = a_t$, such that $\partial Y_{t+1}/\partial y_t' = 0$, $\partial Y_{t+1}/\partial a_t' = I$, and the matrix $H(a_t, \boldsymbol{s}_t)$ is a matrix of zeros. Also, given the form of the utility function, we have that $\partial \Pi_t/\partial q_{jt} = -U_t'(c_t)r_{jt}$ and $\partial \Pi_t/\partial q_{jt-1} = U_t'(c_t)r_{jt}$. Plugging these expression in the general formula (5), we obtain the following system of Euler equations: for any asset $j = 1,\ 2, \ldots, J$:

$$\mathbb{E}_t\left(U_t'(c_t)r_{jt} - \beta U_{t+1}'(c_{t+1})r_{jt+1}\right) = 0 \tag{6}$$

Random Utility Model as a Continuous Optimization Problem

Before considering DDC models, in this section we describe how the optimal decision rule in a static discrete choice model can be represented using marginal conditions of optimality in an optimization problem where decision variables are (choice) probabilities. Later, we apply this result in our derivation of Euler equations in DDC models.

Consider the following *Additive Random Utility Model* (ARUM) (McFadden, 1981). The set of feasible choices $\mathcal{A}$ is discrete and finite and it includes $J+1$ choice alternatives: $\mathcal{A}=\{0,1,\ldots,J\}$. Let $a\in\mathcal{A}$ represent the agent's choice. The payoff function has the following structure:

$$\Pi(a,\varepsilon)=\pi(a)+\varepsilon(a) \tag{7}$$

where $\pi(.)$ is a real valued function, and $\varepsilon\equiv\{\varepsilon(0),\ \varepsilon(1),\ldots,\varepsilon(J)\}$ is a vector of exogenous variables affecting the agent's payoff. The vector ε has a cumulative distribution function (CDF) G that is absolutely continuous with respect to Lebesgue measure, strictly increasing and continuously differentiable in all its arguments, and with finite means. The agent observes ε and chooses the action a that maximizes his payoff $\pi(a)+\varepsilon(a)$. The optimal decision rule of this model is a function $\alpha^*(\varepsilon)$ from the state space $\mathbb{R}^{J+1}$ into the action space $\mathcal{A}$ such that: $\alpha^*(\varepsilon)=\text{argmax}_{a\in\mathcal{A}}\{\pi(a)+\varepsilon(a)\}$. By the AS of the ε's, this optimal decision rule can be written as follows: for any $a\in\mathcal{A}$

$$\left\{\alpha^*(\varepsilon)=a\right\}\quad\text{iff}\quad\left\{\varepsilon(j)-\varepsilon(a)\leq\pi(a)-\pi(j)\ \text{ for any } j\neq a\right\} \tag{8}$$

Given this form of the optimal decision rule, we can restrict our analysis to decision rules with the following threshold form: $\left\{\alpha(\varepsilon)=a\right\}$ if and only if $\{\varepsilon(j)-\varepsilon(a)\leq\mu(a)-\mu(j)$ for any $j\neq a\}$, where $\mu(a)$ is an arbitrary real valued function. We can represent decision rules within this class using a CCP function $P(a)$, that is the decision rule integrated over the vector of random variables ε, that is, $P(a)\equiv\int 1\left\{\alpha(\varepsilon)=a\right\}G(\varepsilon)d\varepsilon$. Therefore, we have that

$$\begin{aligned} P(a)= & \int 1\left\{\varepsilon(j)-\varepsilon(a)\leq\mu(a)-\mu(j)\ \text{ for any } j\neq a\right\} \\ & dG(\varepsilon)=\tilde{G}_a(\mu(a)-\mu(j):\text{ for any } j\neq a) \end{aligned} \tag{9}$$

where $1\{.\}$ is the indicator function, and $\tilde{G}_a$ is the CDF of the vector $\{\varepsilon(j)-\varepsilon(a)$: for any $j\neq a\}$.

Lemma 1 establishes that in an ARUM we can represent decision rules using a vector of CCPs $\mathbf{P}\equiv\{P(1),P(2),\ldots,P(J)\}$ in the J-dimension simplex.

Lemma 1. (McFadden, 1981). Consider an ARUM where the distribution of ε is G that is absolutely continuous with respect to Lebesgue measure, strictly increasing and continuously differentiable in all its arguments. *Let* $\alpha(.)$ be a discrete-valued function from $\mathbb{R}^{J+1}$ into $\mathcal{A} = \{0, 1, \ldots, J\}$; let $\mathbf{m} \equiv \{\mu(1), \mu(2), \ldots, \mu(J)\}$ be a vector in the J-dimension Euclidean space, and consider the normalization $\mu(0) = 0$; and let $\mathbf{P} \equiv \{P(1), P(2), \ldots, P(J)\}$ be a vector in the J-dimension simplex $\mathcal{S}$. We can say that $\alpha(.)$, μ, and $\mathbf{P}$ represent the same decision rule in the ARUM if and only if the following conditions hold:

$$\alpha(\varepsilon) = \sum_{a=0}^{J} a 1\{\varepsilon(j) - \varepsilon(a) \leq \mu(a) - \mu(j) \text{ for any } j \neq a\} \tag{10}$$

and for any $a \in \mathcal{A}$

$$P(a) = \tilde{G}_a(\mu(a) - \mu(j)\text{: for any } j \neq a) \tag{11}$$

where $\tilde{G}_a$ is the CDF of the vector $\{\varepsilon(j) - \varepsilon(a)\text{: for any } j \neq a\}$.

Lemma 2 establishes the invertibility of the relationship between the vector of CCPs $\mathbf{P}$ and the vector of threshold values μ.

Lemma 2. (Hotz & Miller, 1993) Let $\tilde{\mathbf{G}}(.)$ be the vector-valued mapping $\{\tilde{G}_1(.), \tilde{G}_2(.), \ldots, \tilde{G}_J(.)\}$ from $\mathbb{R}^J$ into $\mathcal{S}$. Under the conditions of Lemma 1, the mapping $\tilde{\mathbf{G}}(.)$ is invertible everywhere. We represent the inverse mapping as $\tilde{G}^{-1}(.)$.

Given an arbitrary decision rule, represented in terms $\alpha(.)$, or μ, or $\mathbf{P}$, let Π^e be the expected payoff before the realization of the vector ε if the agent behaves according to this arbitrary decision rule. By definition

$$\Pi^e \equiv \int \{\pi(\alpha(\varepsilon)) + \varepsilon(\alpha(\varepsilon))\} dG(\varepsilon) = \mathbb{E}[\pi(\alpha(\varepsilon)) + \varepsilon(\alpha(\varepsilon))] \tag{12}$$

where the expectation $\mathbb{E}(.)$ is over the distribution of ε. By Lemmas 1–2, we can represent this expected payoff as a function either of $\alpha(.)$, or μ, or $\mathbf{P}$. For our analysis, it is most convenient to represent it as a function of CCPs, that is, $\Pi^e(\mathbf{P})$. Given its definition, this expected payoff function can be written as

$$\Pi^e(\mathbf{P}) = \sum_{a=0}^{J} P(a)\{\pi(a) + e(a, \mathbf{P})\} = \pi(0) + e(0, \mathbf{P}) + \sum_{a=1}^{J} P(a)\{\pi(a) - \pi(0) + e(a, \mathbf{P}) - e(0, \mathbf{P})\} \tag{13}$$

where $e(a, \mathbf{P})$ is defined as the expected value of $\varepsilon(a)$ conditional on alternative a being chosen under decision rule $\alpha(\varepsilon)$. That is, $e(a, \mathbf{P}) \equiv \mathbb{E}(\varepsilon(a) \mid \alpha(\varepsilon) = a)$, and as a function of $\mathbf{P}$ we have that

$$e(a, \mathbf{P}) = \mathbb{E}\left(\varepsilon(a) \mid \varepsilon(j) - \varepsilon(a) \leq \tilde{G}^{-1}(a, \mathbf{P}) - \tilde{G}^{-1}(j, \mathbf{P}) \text{ for any } j \neq a\right) \tag{14}$$

The conditions of the ARUM imply that functions $e(a, \mathbf{P})$ and $\Pi^e(\mathbf{P})$ are continuously differentiable with respect to $\mathbf{P}$ everywhere on the simplex $\mathcal{S}$. Therefore, this expected payoff function $\Pi^e(\mathbf{P})$ has a maximum on $\mathcal{S}$. We can define $\mathbf{P}^*$ as the vector of CCPs that maximizes this expected payoff function:

$$\mathbf{P}^* = \underset{\mathbf{P} \in \mathcal{S}}{\operatorname{argmax}} \left\{\Pi^e(\mathbf{P})\right\} \tag{15}$$

Then, we have two representations of the ARUM, and two apparently different decision problems. On the one hand, we have the discrete choice model with the optimal decision rule $\alpha^*(.)$ in Eq. (8) that maximizes the payoff $\pi(a) + \varepsilon(a)$ after ε is realized and known to the agent. We denote this as the ex-post decision problem to emphasize that the decision is *after* the realization of ε is known to the agent. Associated to α^*, we have its corresponding CCP, that we can represent as $\mathbf{P}^{\alpha^*}$, that is equal to $\tilde{\mathbf{G}}(\tilde{\pi})$ where $\tilde{\pi}$ is the vector of differential payoffs $\{\tilde{\pi}(a) \equiv \pi(a) - \pi(0) :$ for any $a \neq 0$. For econometric analysis of ARUM, we are interested in the $\mathbf{P}^{\alpha^*}$ representation because these are CCPs from the point of view of the econometrician (who does not observe ε) describing the behavior of an agent who knows π and ε and maximizes his payoff. On the other hand, we have the optimization problem represented by Eq. (15) where the agent chooses the vector of CCPs $\mathbf{P}$ to maximize his ex-ante expected payoff Π^e before the realization of ε. In principle, this second optimization problem is not the one the ARUM assumes the individual is solving. In the ARUM we assume that the individual makes his choice after observing the realization of the vector of ε's. Proposition 1 establishes that these two optimization problems are equivalent, that the choice probabilities $\mathbf{P}^{\alpha^*}$ and $\mathbf{P}^*$ are the same, and that $\mathbf{P}^*$ can be described in terms of the marginal conditions of optimality associated to the continuous optimization problem in Eq. (15).

Proposition 1. Let $\mathbf{P}^{\alpha^*}$ be the vector of CCPs associated with the optimal decision rule α^* in the discrete decision problem (8), and let $\mathbf{P}^*$ be the vector of CCPs that solves the continuous optimization problem (15). Then, (i) the vectors $\mathbf{P}^{\alpha^*}$ and $\mathbf{P}^*$ are the same; and (ii) $\mathbf{P}^*$ satisfies the marginal conditions of optimality $\partial\Pi^e(\mathbf{P}^*)/\partial P(a) = 0$ for any $a > 0$, and the marginal expected payoff $\partial\Pi^e(\mathbf{P})/\partial P(a)$ has the following form:

$$\frac{\partial\Pi^e(\mathbf{P})}{\partial P(a)} = \pi(a) - \pi(0) + e(a,\mathbf{P}) - e(0,\mathbf{P}) + \sum_{j=0}^{J} P(j)\frac{\partial e(j,\mathbf{P})}{\partial P(a)} \tag{16}$$

Proof in the appendix.

Proposition 1 establishes a characterization of the optimal decision rule in terms of marginal conditions of optimality with respect to CCPs. In the third section, we show that these conditions can be used to construct moment conditions and a two-step estimator of the structural parameters.

Example 2. (*Multinomial logit*). Suppose that the unobservable variables $\varepsilon(a)$ are i.i.d. with an extreme value type 1 distribution. For this distribution, the function $e(a,\mathbf{P})$ has the following simple form: $e(a,\mathbf{P}) = \gamma - \ln P(a)$, where γ is Euler's constant (see the appendix to Chapter 2 in Anderson, de Palma, and Thisse (1992), for a derivation of this property). Plugging this expression into Eq. (16), we get the following marginal condition of optimality:

$$\frac{\partial\Pi^e(\mathbf{P}^*)}{\partial P(a)} = \pi(a) - \pi(0) - \ln P^*(a) + \ln P^*(0) = 0 \tag{17}$$

because in this model, for any a, the term $\sum_{j=0}^{J} P(j)\,[\partial e(j,\mathbf{P})/\partial P(a)]$ is zero.[8]

Example 3. (*Binary probit model*). Suppose that the decision model is binary, $\mathcal{A} = \{0,1\}$, and $\varepsilon(0)$ and $\varepsilon(1)$ are independently and identically distributed with a normal distribution with zero mean and variance σ^2. Let $\phi(.)$ and $\Phi(.)$ denote the density and the CDFs for the standard normal, respectively, and let $\Phi^{-1}(.)$ be the inverse function of Φ. Given this distribution, it is possible to show that $e(0,P(1)) = \frac{\sigma}{\sqrt{2}}\frac{\phi(\Phi^{-1}[1-P(1)])}{1-P(1)}$, and $e(1,P(1)) = \frac{\sigma}{\sqrt{2}}\frac{\phi(\Phi^{-1}[P(1)])}{P(1)}$. Using these expressions, we have that[9]

$$\begin{aligned}\frac{\partial e(0,P(1))}{\partial P(1)} &= \frac{\sigma}{\sqrt{2}}\left[\frac{-\Phi^{-1}(1-P(1))}{1-P(1)} + \frac{\phi\left(\Phi^{-1}[P(1)]\right)}{[1-P(1)]^2}\right]\\ \frac{\partial e(1,P(1))}{\partial P(1)} &= \frac{\sigma}{\sqrt{2}}\left[\frac{-\Phi^{-1}(P(1))}{P(1)} - \frac{\phi\left(\Phi^{-1}[P(1)]\right)}{P(1)^2}\right]\end{aligned} \tag{18}$$

Solving these expressions into the first order condition in Eq. (16) and taking into account that by symmetry of the Normal distribution

$\Phi^{-1}(1-P(1)) = -\Phi^{-1}(P(1))$, we get the following marginal condition of optimality:

$$\frac{\partial \Pi^e(\mathbf{P}^*)}{\partial P(1)} = \pi(1) - \pi(0) - \sqrt{2}\sigma\,\Phi^{-1}(P(1)) = 0 \tag{19}$$

Euler Equations in Dynamic Discrete Choice Models

Consider the dynamic decision model in section "Dynamic Decision Model" but suppose now that the set of feasible actions is discrete and finite: $\mathcal{A} = \{0, 1, \ldots, J\}$. There are two sets of state variables: $\boldsymbol{s}_t = (\boldsymbol{x}_t, \varepsilon_t)$, where $\boldsymbol{x}_t$ is the vector of state variables observable to the researcher, and ε_t represents the unobservables for the researcher. The set of observable state variables $\boldsymbol{x}_t$ itself is comprised by two types of state variables, exogenous variables z_t and endogenous variables y_t. They are distinguished by the fact that the transition probability of the endogenous variables depends on the action a_t, while the transition probability of the exogenous variables does not depend on a_t. The vector of unobservables satisfies the assumptions of AS and CI (Rust, 1994).

Additive Separability (AS): The one-period payoff function is additively separable in the unobservables: $\Pi_t(a_t, \boldsymbol{s}_t) = \pi_t(a_t, \boldsymbol{x}_t) + \varepsilon_t(a_t)$, *where* $\varepsilon_t \equiv \{\varepsilon_t(a): a \in \mathcal{A}\}$ *is a vector of unobservable random variables.*

Conditional Independence (CI): The transition probability (density) function of the state variables factors as: $f_t(\boldsymbol{s}_{t+1} \mid a_t, \boldsymbol{s}_t) = f_{xt}(\boldsymbol{x}_{t+1} \mid a_t, \boldsymbol{x}_t)\, dG(\varepsilon_{t+1})$, *where* $G(.)$ *is the CDF of* ε_t *which is absolutely continuous with respect to Lebesgue measure, strictly increasing and continuously differentiable in all its arguments, and with finite means.*

Under these assumptions the optimal decision rules $\alpha_t^*(\boldsymbol{x}_t, \varepsilon_t)$ have the following form:

$$\left\{\alpha_t^*(\boldsymbol{x}_t, \varepsilon_t) = a\right\} \quad \text{iff} \quad \left\{\varepsilon_t(j) - \varepsilon_t(a) \le v_t(a, \boldsymbol{x}_t) - v_t(j, \boldsymbol{x}_t) \text{ for any } j \ne a\right\} \tag{20}$$

where $v_t(a, \boldsymbol{x}_t)$ is the conditional-choice value function that is defined as $v_t(a, \boldsymbol{x}_t) \equiv \pi_t(a, \boldsymbol{x}_t) + \beta \int_{\boldsymbol{x}_{t+1}} \overline{V}_{t+1}(\boldsymbol{x}_{t+1})\, f_{xt}(\boldsymbol{x}_{t+1} \mid a, \boldsymbol{x}_t) d\boldsymbol{x}_{t+1}$, and $\overline{V}_t(\boldsymbol{x}_t)$ is the *integrated value function*, $\overline{V}_t(\boldsymbol{x}_t) \equiv \int_{\varepsilon_t} V_t(\boldsymbol{x}_t, \varepsilon_t)\, dG(\varepsilon_t)$. Furthermore, the integrated value function satisfies the following integrated Bellman equation:

$$\overline{V}_t(\boldsymbol{x}_t) = \int_{\varepsilon_t} \max_{a_t \in \mathcal{A}} \left\{ \pi_t(a_t, \boldsymbol{x}_t) + \varepsilon_t(a_t) + \beta \int \overline{V}_{t+1}(\boldsymbol{x}_{t+1}) f_{xt}(\boldsymbol{x}_{t+1} \mid a, \boldsymbol{x}_t)\, d\boldsymbol{x}_{t+1} \right\} dG_t(\varepsilon_t) \tag{21}$$

We can restrict our analysis to decision rules $\alpha_t(\boldsymbol{x}_t, \varepsilon_t)$ with the following "threshold" structure: $\{\alpha_t(\boldsymbol{x}_t, \varepsilon_t) = a\}$ if and only if $\{\varepsilon_t(j) - \varepsilon_t(a) \leq \mu_t(a, \boldsymbol{x}_t) - \mu_t(j, \boldsymbol{x}_t)$ for any $j \neq a\}$, where $\mu_t(a, \boldsymbol{x}_t)$ is an arbitrary real valued function. As in the ARUM, we can represent decision rules using a discrete valued function $\alpha_t(\boldsymbol{x}_t, \varepsilon_t)$, a real valued function $\mu_t(a, \boldsymbol{x}_t)$, or a probability valued function $P_t(a|\boldsymbol{x}_t)$.

$$\begin{aligned} P_t(a|\boldsymbol{x}_t) &\equiv \int 1\left\{\alpha_t(\boldsymbol{x}_t, \varepsilon_t) = a\right\} G_t(\varepsilon_t)\, d\varepsilon_t \\ &= \tilde{G}_a\left(\mu_t(a, \boldsymbol{x}_t) - \mu_t(j, \boldsymbol{x}_t) \text{: for any } j \neq 0,\ a\right) \end{aligned} \tag{22}$$

where $\tilde{G}_a$ has the same interpretation as in the ARUM, that is, the CDF of the vector $\{\varepsilon(j) - \varepsilon(a)$: for any $j \neq a\}$. Lemmas 1 and 2 from the ARUM extend to this DDC model (Proposition 1 in Hotz & Miller, 1993). In particular, at every period t, there is a one-to-one relationship between the vector of value differences $\tilde{\mu}_t(\boldsymbol{x}_t) \equiv \{\mu_t(a, \boldsymbol{x}_t) - \mu_t(0, \boldsymbol{x}_t)$: $a > 0$ and the vector of CCPs $\mathbf{P}_t(\boldsymbol{x}_t) \equiv \{P_t(a|\boldsymbol{x}_t) : a \neq 0$. We represent this mapping as $\mathbf{P}_t(\boldsymbol{x}_t) = \tilde{G}(\tilde{\mu}_t(\boldsymbol{x}_t))$, and the corresponding inverse mapping as $\tilde{\mu}_t(\boldsymbol{x}_t) = \tilde{G}^{-1}(\mathbf{P}_t(\boldsymbol{x}_t))$.

Given an arbitrary sequence of decision rules, represented in terms of either $\alpha \equiv \{\alpha_t(.) : t \geq 1\}$, or $\tilde{\mu} \equiv \{\tilde{\mu}_t(.) : t \geq 1\}$, or $\mathbf{P} \equiv \{\mathbf{P}_t(.) : t \geq 1\}$, let $W_t^e(\boldsymbol{x}_t)$ be the expected intertemporal payoff function at period t before the realization of the vector ε_t if the agent behaves according to this arbitrary sequence of decision rules. By definition

$$\begin{aligned} W_t^e(\boldsymbol{x}_t) &\equiv \mathbb{E}\left(\sum_{r=0}^{T-t} \beta^r [\pi_{t+r}(\alpha_{t+r}(\boldsymbol{x}_{t+r}, \varepsilon_{t+r}), \boldsymbol{x}_{t+r}) + \varepsilon_{t+r}(\alpha_{t+r}(\boldsymbol{x}_{t+r}, \varepsilon_{t+r}))] \,|\, \boldsymbol{x}_t\right) \\ &= \mathbb{E}\left[\pi_t(\alpha_t(\boldsymbol{x}_t, \varepsilon_t), \boldsymbol{x}_t) + \varepsilon_t(\alpha_t(\boldsymbol{x}_t, \varepsilon_t)) + \beta \int W_{t+1}^e(\boldsymbol{x}_{t+1}) f_{xt}(\boldsymbol{x}_{t+1}|\alpha_t(\boldsymbol{x}_t, \varepsilon_t), \boldsymbol{x}_t) d\boldsymbol{x}_{t+1}\right] \end{aligned} \tag{23}$$

We denote $W_t^e(\boldsymbol{x}_t)$ as the *valuation function* to distinguish it from the optimal value function and to emphasize that $W_t^e(\boldsymbol{x}_t)$ provides the valuation of any arbitrary decision rule. We are interested in the representation of this valuation function as a function of CCPs. Therefore, we use the notation $W_t^e(\boldsymbol{x}_t, \mathbf{P}_t, \mathbf{P}_{t'>t})$. Given its definition, this function can be written using the recursive formula:

$$\begin{aligned} W_t^e(\boldsymbol{x}_t, \mathbf{P}_t, \mathbf{P}_{t'>t}) = \Pi_t^e(\boldsymbol{x}_t, \mathbf{P}_t) + \beta \int & W_{t+1}^e(\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1}, \mathbf{P}_{t'>t+1}) \\ & \times f_t^e(y_{t+1}|\boldsymbol{x}_t, \mathbf{P}_t) f_z(z_{t+1}\,|\,z_t) d\boldsymbol{x}_{t+1} \end{aligned} \tag{24}$$

where $\Pi_t^e(\boldsymbol{x}_t, \mathbf{P}_t)$ is the expected one-period profit $\sum_{a=0}^{J} P_t(a|\boldsymbol{x}_t)[\pi_t(a, \boldsymbol{x}_t) + e_t(a, \mathbf{P}_t(\boldsymbol{x}_t))]$; $e_t(a, \mathbf{P}_t(\boldsymbol{x}_t))$ has the same definition as in the static model, that is, it is the expected value of $\varepsilon_t(a)$ conditional on alternative a being chosen under decision rule $\alpha_t(\boldsymbol{x}_t, \varepsilon_t)$[10]; and $f_t^e(y_{t+1}|\boldsymbol{x}_t, \mathbf{P}_t)$ is the transition probability of the endogenous state variables y induced by the CCP function $\mathbf{P}_t(\boldsymbol{x}_t)$, that is, $\sum_{a=0}^{J} P_t(a|\boldsymbol{x}_t) f_{yt}(y_{t+1}|a, \boldsymbol{x}_t)$.

The valuation function $W_t^e(\boldsymbol{x}_t, \mathbf{P}_t, \mathbf{P}_{t'>t})$ is continuously differentiable with respect to the choice probabilities over the simplex. Then, we can define $\mathbf{P}^*$ as the sequence of CCP functions $\{\mathbf{P}_t^*(x) : t \geq 1,\ x \in \mathcal{X}\}$ such that for any (t, x) the vector of CCPs $\mathbf{P}_t^*(x)$ maximizes the values $W_t^e(x, \mathbf{P}_t, \mathbf{P}_{t'>t})$ given that future CCPs $\mathbf{P}_{t'>t}$ are fixed at their values in $\mathbf{P}^*$.

$$\mathbf{P}_t^*(x) = \arg\max_{\mathbf{P}_t(x) \in \mathcal{S}} \left\{ W_t^e\left(x, \mathbf{P}_t, \mathbf{P}_{t'>t}^*\right) \right\} \tag{25}$$

As in the ARUM, we have apparently two different optimal CCP functions. We have the CCP functions associated with the sequence of optimal decision rules $\alpha_t^*(.)$, that we represent as $\{\mathbf{P}_t^{\alpha^*}: t \geq 1\}$. And we have sequence of CCP functions $\{\mathbf{P}_t^*: t \geq 1\}$ defined in Eq. (25). Proposition 2 establishes that the two sequences of CCPs are the same one, and that these probabilities satisfy the marginal conditions of optimality associated to the continuous optimization problem in Eq. (25).

Proposition 2. Let $\{\mathbf{P}_t^{\alpha^*}: t \geq 1\}$ be the sequence of CCP functions associated with the sequence of optimal decision rules $\{\alpha_t^*: t \geq 1\}$ as defined in the DDC problem (20), and let $\{\mathbf{P}_t^*: t \geq 1\}$ be sequence of CCP functions that solves the continuous optimization problem (25). Then, for every (t, x): (i) the vectors $\mathbf{P}_t^{\alpha^*}(x)$ and $\mathbf{P}_t^*(x)$ are the same; and (ii) $\mathbf{P}_t^*(x)$ satisfies the marginal conditions of optimality

$$\frac{\partial W_t^e\left(x, \mathbf{P}_t^*, \mathbf{P}_{t'>t}^*\right)}{\partial P_t(a|x)} = 0 \tag{26}$$

for any $a > 0$, and the marginal value $\partial W_t / \partial P_t$ has the following form:

$$\frac{\partial W_t^e}{\partial P_t(a|x)} = v_t(a, \boldsymbol{x}_t, \mathbf{P}_{t'>t}) - v_t(0, \boldsymbol{x}_t, \mathbf{P}_{t'>t}) + e_t(a, \mathbf{P}_t(x)) - e_t(0, \mathbf{P}_t(x)) + \sum_{j=0}^{J} P_t(j|x) \frac{\partial e_t(j, \mathbf{P}_t(x))}{\partial P_t(a|x)} \tag{27}$$

where $v_t(a, \boldsymbol{x}_t, \mathbf{P}_{t'>t})$ is the conditional-choice value function $\pi_t(a, \boldsymbol{x}_t) + \beta \int W_{t+1}(\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1}, \mathbf{P}_{t'>t+1}) f_t(\boldsymbol{x}_{t+1}|a, \boldsymbol{x}_t)\, d\boldsymbol{x}_{t+1}$.

Proof in the appendix.

Proposition 2 shows that we can treat the DDC model as a dynamic continuous optimization problem where optimal choices, in the form of choice probabilities, satisfy marginal conditions of optimality. Nevertheless, the marginal conditions of optimality in Eq. (27) involve value functions. We are looking for conditions of optimality in the spirit of Euler equations that involve only payoff functions at two consecutive periods, t and $t+1$. To obtain these conditions, we construct a constrained optimization problem similar to the one for the derivation of Euler equations in section "Euler Equations in Dynamic Continuous Decision Models".

By Bellman's principle, the optimal choice probabilities at periods t and $t+1$ come from the solution to the optimization problem $\max_{\mathbf{P}_t, \mathbf{P}_{t+1}} \times W_t^e(x, \mathbf{P}_t, \mathbf{P}_{t+1}, \mathbf{P}^*_{t'>t+1})$, where we have fixed at its optimum the individual's behavior at any period after $t+1$, $\mathbf{P}^*_{t'>t+1}$. In general, the CCPs $\mathbf{P}_t$ and $\mathbf{P}_{t+1}$ affect the distribution of the state variables at periods after $t+1$ such that the optimality conditions of the problem $\max_{\mathbf{P}_t, \mathbf{P}_{t+1}} \times W_t^e(x, \mathbf{P}_t, \mathbf{P}_{t+1}, \mathbf{P}^*_{t'>t+1})$ involve payoff functions and state variables at every period in the future. Instead, suppose that we consider a similar optimization problem but where we now impose the constraint that the probability distribution of the endogenous state variables at $t+2$ should be the one implied by the optimal CCPs at periods t and $t+1$. Since $(\mathbf{P}^*_t, \mathbf{P}^*_{t+1})$ satisfy this constraint, it is clear that these CCPs represent also the unique solution to this constrained optimization problem. That is

$$\{\mathbf{P}^*_t(x), \mathbf{P}^*_{t+1}\} = \arg \max_{\{\mathbf{P}_t(x), \mathbf{P}_{t+1}\}} \Delta_t = \left\{ W_t^e(x, \mathbf{P}_t, \mathbf{P}_{t+1}, \mathbf{P}^*_{t'>t+1}) - W_t^e(x, \mathbf{P}^*_t, \mathbf{P}^*_{t+1}, \mathbf{P}^*_{t'>t+1}) \right\}$$
$$\text{subject to:} f^e_{t \to t+2}(.|x, \mathbf{P}_t, \mathbf{P}_{t+1}) = f^e_{t \to t+2}(.|x, \mathbf{P}^*_t, \mathbf{P}^*_{t+1}) \qquad (28)$$

where we use function $f^e_{t \to t+2}(.|x, \mathbf{P}_t, \mathbf{P}_{t+1})$ to represent the distribution of y_{t+2} conditional on $\boldsymbol{x}_t = x$ and induced by the CCPs $\mathbf{P}_t(x)$ and $\mathbf{P}_{t+1}$, that can be written as

$$f_{t \to t+2}(y_{t+2}|\boldsymbol{x}_t, \mathbf{P}_t, \mathbf{P}_{t+1}) = \int f^e_{t+1}(y_{t+2}|\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1})\ f^e_t(y_{t+1}|\boldsymbol{x}_t, \mathbf{P}_t)\ f_z(z_{t+1}|z_t) d\boldsymbol{x}_{t+1} \qquad (29)$$

and, as defined above, $f_t^e(.|x, \mathbf{P}_t)$ is the one-period-forward transition probability of the endogenous state variables y induced by the CCP function $\mathbf{P}_t(x)$, that is, $\sum_{a=0}^{J} P_t(a|\boldsymbol{x}_t)\ f_t(y_{t+1}|a, \boldsymbol{x}_t)$.

By the definition of the valuation function W_t^e, we have that

$$W_t^e(\boldsymbol{x}_t, \mathbf{P}) = \Pi_t^e(\boldsymbol{x}_t, \mathbf{P}_t) + \beta \int \Pi_{t+1}^e(\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1}) f_t^e(y_{t+1}|\boldsymbol{x}_t, \mathbf{P}_t) f_z(z_{t+1}|z_t) d\boldsymbol{x}_{t+1}$$
$$+ \beta^2 \int W_{t+2}^e(\boldsymbol{x}_{t+2}, \mathbf{P}_{t'>t+1}) f_{t\to t+2}(y_{t+2}|\boldsymbol{x}_t, \mathbf{P}_t, \mathbf{P}_{t+1})\ f_z(z_{t+2}|z_t) d\boldsymbol{x}_{t+2} \tag{30}$$

The last term in this expression is exactly the same for $W_t^e(x, \mathbf{P}_t, \mathbf{P}_{t+1}, \mathbf{P}_{t'>t+1}^*)$ and for $W_t^e(x, \mathbf{P}_t^*, \mathbf{P}_{t+1}^*, \mathbf{P}_{t'>t+1}^*)$ because we have the same function W_{t+2}^e and because we restrict the distribution of y_{t+2} to be the same. Therefore, subject to this constraint we have that Δ_t is equal to $\Pi_t^e(x, \mathbf{P}_t) + \beta \int \Pi_{t+1}^e(\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1}) f_t^e(\boldsymbol{x}_{t+1}|x, \mathbf{P}_t) d\boldsymbol{x}_{t+1}$, and the optimal CCPs at periods t and $t+1$ solve the following optimization problem:

$$\{\mathbf{P}_t^*(x), \mathbf{P}_{t+1}^*\} = \arg \max_{\{\mathbf{P}_t(x), \mathbf{P}_{t+1}\}} \Delta_t$$
$$= \left\{ \Pi_t^e(x, \mathbf{P}_t) + \beta \int \Pi_{t+1}^e(\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1}) f_t^e(\boldsymbol{x}_{t+1}|x, \mathbf{P}_t) d\boldsymbol{x}_{t+1} \right\}$$
$$\text{subject to}: f_{t\to t+2}^e(.|x, \mathbf{P}_t, \mathbf{P}_{t+1}) = f_{t\to t+2}^e(.|x, \mathbf{P}_t^*, \mathbf{P}_{t+1}^*) \tag{31}$$

Suppose that the space of the vector of endogenous state variables $\mathcal{Y}$ is discrete and finite. Therefore, the set of restrictions on $f_{t\to t+2}^e \times (y_{t+2}|x, \mathbf{P}_t, \mathbf{P}_{t+1})$ in the constrained optimization problem (31) includes at most $|\mathcal{Y}| - 1$ restrictions, where $|\mathcal{Y}|$ is the number of points in the support set $\mathcal{Y}$. Therefore, the number of Lagrange multipliers, and the matrix that we have to invert to get these multipliers is of at most as large as $|\mathcal{Y}| - 1$. In fact, in many models, the number of Lagrange multipliers that we must solve for can be much smaller than the dimension of the vector of endogenous state variables. This is because in many models the transition probability of the endogenous state variable is such that, given the state variable at period t, the state variable at period $t+2$ can take only a limited and small number of possible values. We present several examples below.

Let $\mathcal{Y}_{+s}(\boldsymbol{x}_t)$ be the set of values that the endogenous state variables can reach with positive probability s periods in the future given that the state today is $\boldsymbol{x}_t$. To be precise, $\mathcal{Y}_{+s}(\boldsymbol{x}_t)$ includes all these possible values except one of them because we can represent the probability distribution of y_{t+s}

using the probabilities of each possible value except one. Let $\lambda_t(\boldsymbol{x}_t) = \{\lambda_t(y_{t+2}|\boldsymbol{x}_t) : y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)\}$ be the $|\mathcal{Y}_{+2}(\boldsymbol{x}_t)| \times 1$ vector of Lagrange multipliers associated to this set of restrictions. The Lagrangian function for this optimization problem is

$$\mathcal{L}_t(\mathbf{P}_t(\boldsymbol{x}_t),\mathbf{P}_{t+1}) = \Pi_t^e(x,\mathbf{P}_t) + \beta \sum_{\boldsymbol{x}_{t+1}} \Pi_{t+1}^e(\boldsymbol{x}_{t+1},\mathbf{P}_{t+1}) f_t^e(y_{t+1} \,|\, \boldsymbol{x}_t,\mathbf{P}_t) f_z(z_{t+1}|z_t)$$
$$- \sum_{y_{t+2}} \lambda_t(y_{t+2}|\boldsymbol{x}_t) \left[\sum_{\boldsymbol{x}_{t+1}} f_{t+1}^e(y_{t+2}|\boldsymbol{x}_{t+1},\mathbf{P}_{t+1}) f_t^e(y_{t+1}|\boldsymbol{x}_t,\mathbf{P}_t) f_z(z_{t+1}|z_t) \right] \quad (32)$$

Given this Lagrangian function, we can derive the first order conditions of optimality with respect to $\mathbf{P}_t(\boldsymbol{x}_t)$ and $\mathbf{P}_{t+1}$ and combine these conditions to obtain Euler equations.

Proposition 3. The marginal conditions for the maximization of the Lagrangian function in Eq. (32) imply the following Euler equations. For every value of $\boldsymbol{x}_t$:

$$\frac{\partial \Pi_t^e}{\partial P_t(a|\boldsymbol{x}_t)} + \beta \sum_{\boldsymbol{x}_{t+1}} \left[\Pi_{t+1}^e(\boldsymbol{x}_{t+1}) - \mathbf{m}(x_{t+1})' \frac{\partial \mathbf{\Pi}_{t+1}^e(z_{t+1})}{\partial \mathbf{P}_{t+1}(z_{t+1})} \right] \tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t) f_z(z_{t+1}|z_t) = 0 \quad (33)$$

where $\tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t) \equiv f_t(y_{t+1}|a,\boldsymbol{x}_t) - f_t(y_{t+1}|0,\boldsymbol{x}_t)$; $\partial \mathbf{\Pi}_{t+1}^e(z_{t+1}) / \partial \mathbf{P}_{t+1}(z_{t+1})$ is a column vector with dimension $J|\mathcal{Y}_{+1}(\boldsymbol{x}_t)| \times 1$ that contains the partial derivatives $\{\partial \Pi_{t+1}^e(y_{t+1},z_{t+1}) / \partial P_{t+1}(a|y_{t+1},z_{t+1})\}$ for every action $a>0$ and every value $y_{t+1} \in \mathcal{Y}_{+1}(\boldsymbol{x}_t)$ that can be reach from $\boldsymbol{x}_t$, and fixed value for z_{t+1}; and $\mathbf{m}(\boldsymbol{x}_{t+1})$ is a $J|\mathcal{Y}_{+1}(\boldsymbol{x}_t)| \times 1$ vector such that $\mathbf{m}(\boldsymbol{x}_{t+1}) \equiv \mathbf{f}_{t+1}^e(x_{t+1})' [\tilde{\mathbf{F}}_{t+1}(z_{t+1})' \tilde{\mathbf{F}}_{t+1}(z_{t+1})]^{-1} \tilde{\mathbf{F}}_{t+1}(z_{t+1})'$ where $\mathbf{f}_{t+1}^e(\boldsymbol{x}_{t+1})$ is the vector of transition probabilities $\{f_{t+1}^e(y_{t+2}|\boldsymbol{x}_{t+1}) : y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)\}$, and $\tilde{\mathbf{F}}_{t+1}(z_{t+1})$ is matrix with dimension $J|\mathcal{Y}_{+1}(\boldsymbol{x}_t)| \times |\mathcal{Y}_{+2}(\boldsymbol{x}_t)|$ that contains the probabilities $\tilde{f}_{t+1}(y_{t+2}|a,\boldsymbol{x}_{t+1})$ for every $y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)$, every $y_{t+1} \in \mathcal{Y}_{+1}(\boldsymbol{x}_t)$, and every action $a>0$, with fixed z_{t+1}.

Proof in the appendix.

Proposition 3 shows that in general we can derive marginal conditional of optimality that involve only payoffs and states at two consecutive periods. The derivation of this Euler equation, described in the appendix, is based on the combination of the Lagrangian conditions $\partial \mathcal{L}_t / \partial P_t(a|\boldsymbol{x}_t) = 0$

and $\partial \mathcal{L}_t / \partial P_{t+1}(a|\boldsymbol{x}_{t+1}) = 0$. Using the group of conditions $\partial \mathcal{L}_t / \partial P_{t+1}(a|\boldsymbol{x}_{t+1}) = 0$ we can solve for the vector of Lagrange multipliers as $[\tilde{\mathbf{F}}_{t+1}(z_{t+1})' \tilde{\mathbf{F}}_{t+1}(z_{t+1})]^{-1} \tilde{\mathbf{F}}_{t+1}(z_{t+1})' \partial \mathbf{\Pi}^e_{t+1}(z_{t+1}) / \partial \mathbf{P}_{t+1}(z_{t+1})$ and then we can plug this solution into the first Lagrangian conditions, $\partial \mathcal{L}_t / \partial P_t(a|\boldsymbol{x}_t) = 0$. This provides the expression for the Euler equation in (33). The main computational cost in the derivation of this expression comes from inverting the matrices $[\tilde{\mathbf{F}}_{t+1}(z_{t+1})' \tilde{\mathbf{F}}_{t+1}(z_{t+1})]$. The dimension of these matrices is $|\mathcal{Y}_{+2}(\boldsymbol{x}_t)| \times |\mathcal{Y}_{+2}(\boldsymbol{x}_t)|$, where $\mathcal{Y}_{+2}(\boldsymbol{x}_t)$ is the set of possible values that the endogenous state variable y_{t+2} can take given $\boldsymbol{x}_t$. In most applications, the number of elements in the set $\mathcal{Y}_{+2}(\boldsymbol{x}_t)$ is substantially smaller that the whole number of values in the space of the endogenous state variable, and several orders of magnitude smaller than the dimension of the complete state space that includes the exogenous state variables. This property implies very substantial computational savings in the estimation of the model. We now provide some examples of models where the form of the Euler equations is particularly simple. In these examples, we have simple closed form expressions for the Lagrange multipliers. These examples correspond to models that are commonly estimated in applications of DDC models.

Example 4. (*Dynamic binary choice model of entry and exit*). Consider a binary decision model, $\mathcal{A} = \{0, 1\}$, where a_t is the indicator of being active in a market or in some particular activity. The endogenous state variable y_t is the lagged value of the decision variable, $y_t = a_{t-1}$, and it represents whether the agent was active at previous period. The vector of state variables is then $\boldsymbol{x}_t = (y_t, z_t)$ where z_t are exogenous state variables. Suppose that $\varepsilon_t(0)$ and $\varepsilon_t(1)$ are extreme value type 1 distributed with dispersion parameter σ_ε. In this model, the one-period expected payoff function is $\Pi^e_t(\boldsymbol{x}_t, P_t) = P_t(0|\boldsymbol{x}_t)\ [\pi(0, \boldsymbol{x}_t) - \sigma_\varepsilon \ln P_t(0|\boldsymbol{x}_t)] + P_t(1|\boldsymbol{x}_t)\,[\pi(1, \boldsymbol{x}_t) - \sigma_\varepsilon \ln P_t(1|\boldsymbol{x}_t)]$. The transition of the endogenous state variable induced by the CCP is the CCP itself, that is, $f^e_t(y_{t+1}|\boldsymbol{x}_t, P_t) = P_t(y_{t+1}|\boldsymbol{x}_t)$. Therefore, we can write the Δ_t function in the constrained optimization problem as

$$\begin{aligned}\Delta_t = \Pi^e_t(\boldsymbol{x}_t, P_t) + \beta \sum_{z_{t+1}} f_z(z_{t+1}|z_t)\ [P_t(0|\boldsymbol{x}_t) \Pi^e_{t+1}(0, z_{t+1}, P_{t+1}) \\ + P_t(1|\boldsymbol{x}_t)\ \Pi^e_{t+1}(1, z_{t+1}, P_{t+1})]\end{aligned} \tag{34}$$

Given $\boldsymbol{x}_t$, the state variable y_{t+2} can take two values, 0 or 1. Therefore, there is only one free probability in $f^e_{t \to t+2}$ and one restriction in the Lagrangian problem. This probability is

$$f^e_{t\to t+2}(1|\boldsymbol{x}_t, P_t, \mathbf{P}_{t+1}) = \sum_{z_{t+1}} f_z(z_{t+1}|z_t)\,[P_t(0|\boldsymbol{x}_t)P_{t+1}(1|0, z_{t+1}) + P_t(1|\boldsymbol{x}_t)P_{t+1}(1|1, z_{t+1})] \quad (35)$$

Let $\lambda(\boldsymbol{x}_t)$ be the Lagrange multiplier for this restriction. For a given $\boldsymbol{x}_t$, the free probabilities that enter in the Lagrangian problem are $P_t(1|\boldsymbol{x}_t)$, $P_{t+1}(1|0, z_{t+1})$, and $P_{t+1}(1|1, z_{t+1})$ for any possible value of z_{t+1} in the support set $\mathcal{Z}$. The first order condition for the maximization of the Lagrangian with respect to $P_t(1|\boldsymbol{x}_t)$ is

$$\frac{\partial \Pi^e_t}{\partial P_t(1|\boldsymbol{x}_t)} + \beta \sum_{z_{t+1}} [\Pi^e_{t+1}(1) - \Pi^e_{t+1}(0) - \lambda(\boldsymbol{x}_t)\{P_{t+1}(1|1, z_{t+1}) - P_{t+1}(1|0, z_{t+1})\}]\, f_z(z_{t+1}|z_t) = 0 \quad (36)$$

The marginal condition with respect to one of the probabilities $P_{t+1}(1|\boldsymbol{x}_{t+1})$ (for a given value of $\boldsymbol{x}_{t+1}$) is $\beta \frac{\partial \Pi^e_{t+1}(0, z_{t+1}, P_{t+1})}{\partial P_{t+1}(1|0, z_{t+1})} = \beta \frac{\partial \Pi^e_{t+1}(1, z_{t+1}, P_{t+1})}{\partial P_{t+1}(1|1, z_{t+1})} = \lambda(\boldsymbol{x}_t)$. Substituting the marginal condition with respect to $P_{t+1}(1|\boldsymbol{x}_{t+1})$ into the marginal condition with respect to $P_t(1|\boldsymbol{x}_t)$ we get the Euler equation:

$$\frac{\partial \Pi^e_t}{\partial P_t(1|\boldsymbol{x}_t)} + \beta \mathbb{E}_t\left(\Pi^e_{t+1}(1, z_{t+1}) - \Pi^e_{t+1}(0, z_{t+1})\right) + \beta \mathbb{E}_t\left(P_{t+1}(1|0, z_{t+1}) \frac{\partial \Pi^e_{t+1}(0, z_{t+1}, P_{t+1})}{\partial P_{t+1}(1|0, z_{t+1})} - P_{t+1}(1|1, z_{t+1}) \frac{\partial \Pi^e_{t+1}(1, z_{t+1}, P_{t+1})}{\partial P_{t+1}(1|1, z_{t+1})}\right) = 0 \quad (37)$$

where we use $\mathbb{E}_t(.)$ to represent in a compact form the expectation over the distribution of $f_z(z_{t+1}|z_t)$. Finally, for the logit version of this model and as shown in Example 2, the marginal expected profit $\partial \Pi^e_t / \partial P_t(1|\boldsymbol{x}_t)$ is equal to $\pi(1, \boldsymbol{x}_t) - \pi_t(0, \boldsymbol{x}_t) - \sigma_\varepsilon(\ln P_t(1|\boldsymbol{x}_t) - \ln P_t(0|\boldsymbol{x}_t))$. Taking this into account and operating in the Euler equation, we can obtain this simpler formula for this Euler equation:

$$\left[\pi(1, y_t, z_t) - \pi(0, y_t, z_t) - \sigma_\varepsilon \ln\left(\frac{P_t(1|y_t, z_t)}{P_t(0|y_t, z_t)}\right)\right] + \beta \mathbb{E}_t\left[\pi(1, 1, z_{t+1}) - \pi_t(1, 0, z_{t+1}) - \sigma_\varepsilon \ln\left(\frac{P_{t+1}(1|1, z_{t+1})}{P_{t+1}(1|0, z_{t+1})}\right)\right] = 0 \quad (38)$$

Example 5. (*Machine replacement model*). Consider a model where the binary choice variable a_t is the indicator for a firm's decision to replace an old machine or equipment by a new one. The endogenous state variable y_t is the age of the "old" machine that takes discrete values $\{1, 2, \ldots\}$ and it follows the transition rule $y_{t+1} = 1 + (1 - a_t)y_t$, that is, if the firm replaces the machine at period t (i.e., $a_t = 1$), then at period $t+1$ it has a brand new machine with $y_{t+1} = 1$, otherwise the firm continues with the old machine that at $t+1$ will be one period older. Given y_t, we have that y_{t+1} can take only two values, $y_{t+1} \in \{1, y_t + 1\}$. Thus, the Δ_t function is

$$\Delta_t = \Pi_t^e(\boldsymbol{x}_t) + \beta \sum_{z_{t+1}} f_z(z_{t+1}|z_t)[P_t(0|\boldsymbol{x}_t)\Pi_{t+1}^e(y_t + 1, z_{t+1}) + P_t(1|\boldsymbol{x}_t)\Pi_{t+1}^e(1, z_{t+1})] \tag{39}$$

Given y_t, we have that y_{t+2} can take only three values, $y_{t+1} \in \{1, 2, y_t + 1\}$. There are only two free probabilities in the distribution of $f_{t\to t+2}^e(y_{t+2}|\boldsymbol{x}_t)$. Without loss of generality, we use the probabilities $f_{t\to t+2}^e(1|\boldsymbol{x}_t)$ and $f_{t\to t+2}^e(2|\boldsymbol{x}_t)$ to construct the Lagrange function. These probabilities have the following form:

$$\begin{aligned} f_{t\to t+2}^e(1|\boldsymbol{x}_t) &= \sum_{z_{t+1}} f_z(z_{t+1}|z_t)[P_t(0|\boldsymbol{x}_t)P_{t+1}(1|y_t + 1, z_{t+1}) \\ &\qquad + P_t(1|\boldsymbol{x}_t)P_{t+1}(1|1, z_{t+1})] \\ f_{t\to t+2}^e(2|\boldsymbol{x}_t) &= P_t(1|\boldsymbol{x}_t) \sum_{z_{t+1}} f_z(z_{t+1}|z_t)P_{t+1}(0|1, z_{t+1}) \end{aligned} \tag{40}$$

The Lagrangian function depends on the CCPs $P_t(1|\boldsymbol{x}_t)$, $P_{t+1}(1|1, z_{t+1})$, and $P_{t+1}(1|y_t + 1, z_{t+1})$. The Lagrangian optimality condition with respect to $P_t(1|\boldsymbol{x}_t)$ is

$$\begin{aligned} &\frac{\partial \Pi_t^e}{\partial P_t(1|\boldsymbol{x}_t)} + \beta \sum_{z_{t+1}} f_z(z_{t+1}|z_t)\left[\Pi_{t+1}^e(1, z_{t+1}) - \Pi_{t+1}^e(y_t + 1, z_{t+1})\right] \\ &\qquad - \lambda(1) \sum_{z_{t+1}} f_z(z_{t+1}|z_t)[P_{t+1}(1|1, z_{t+1}) - P_{t+1}(1|y_t + 1, z_{t+1})] \\ &\qquad\qquad - \lambda(2) \sum_{z_{t+1}} f_z(z_{t+1}|z_t)P_{t+1}(0|1, z_{t+1}) = 0 \end{aligned} \tag{41}$$

And the Lagrangian conditions with respect to $P_{t+1}(1|1, z_{t+1})$ and $P_{t+1}(1|y_t + 1, z_{t+1})$ are $\beta \frac{\partial \Pi_{t+1}^e(1, z_{t+1})}{\partial P_{t+1}(1|1, z_{t+1})} - \lambda(1) + \lambda(2) = 0$, and $\beta \frac{\partial \Pi_{t+1}^e(y_t + 1, z_{t+1})}{\partial P_{t+1}(1|y_t + 1, z_{t+1})} -$

$\lambda(1)=0$, respectively. We can use the second set of conditions to solve trivially for the Lagrange multipliers, and then plug in the expression for this multipliers in the first set of Lagrangian conditions. We obtain the Euler equation:

$$\frac{\partial \Pi_t^e}{\partial P_t(1|\mathbf{x}_t)} + \beta\,\mathbb{E}_t\left[\Pi_{t+1}^e(1, z_{t+1}) - \Pi_{t+1}^e(y_t+1, z_{t+1})\right]$$
$$+\beta\,\mathbb{E}_t\left[\frac{\partial \Pi_{t+1}^e(1, z_{t+1})}{\partial P_{t+1}(1|1, z\,t+1)} P_{t+1}(0|1, z_{t+1}) - \frac{\partial \Pi_{t+1}^e(y_t+1, z_{t+1})}{\partial P_{t+1}(1|y_t+1, z_{t+1})} P_{t+1}(0|y_t+1, z_{t+1})\right] = 0 \tag{42}$$

Finally, taking into account that for the logit specification of the unobservables the marginal expected profit $\partial \Pi_t^e / \partial P_t(1|\mathbf{x}_t)$ is equal to $\pi(1,\mathbf{x}_t) - \pi(0,\mathbf{x}_t) - \sigma_\varepsilon[\ln\ P_t(1|\mathbf{x}_t) - \ln P_t(0|\mathbf{x}_t)]$, and operating in the previous expression, it is possible to obtain the following Euler equation:

$$\left[\pi(1, y_t, z_t) - \pi(0, y_t, z_t) - \sigma_\varepsilon \ln\left(\frac{P_t(1|y_t, z_t)}{P_t(0|y_t, z_t)}\right)\right]$$
$$+\beta\,\mathbb{E}_t\left[\pi(1, 1, z_{t+1}) - \pi(1, y_t+1, z_{t+1}) - \sigma_\varepsilon \ln\left(\frac{P_{t+1}(1|1, z_{t+1})}{P_{t+1}(1|y_t+1, z_{t+1})}\right)\right] = 0 \tag{43}$$

Relationship between Euler Equations and Other CCP Representations

Our derivation of Euler equations for DDC models above is related to previous work by Hotz and Miller (1993), Aguirregabiria and Mira (2002), and Arcidiacono and Miller (2011). These papers derive representations of optimal decision rules using CCPs and show how these representations can be applied to estimate DDC models using simple two-step methods that provide substantial computational savings relative to full-solution methods. In these previous papers, we can distinguish three different types of CCP representations of optimal decisions rules: (1) the *present-value* representation; (2) the *terminal-state* representation; and (3) the *finite-dependence* representation.

The *present-value* representation consists of using CCPs to obtain an expression for the expected and discounted stream of future payoffs associated with each choice alternative. In general, given CCPs, the valuation function $W_t^e(x, \mathbf{P})$ can be obtained recursively using its definition, $W_t^e(\boldsymbol{x}_t, \mathbf{P}) = \Pi_t^e(\boldsymbol{x}_t, \mathbf{P}) + \beta \int W_{t+1}^e(\boldsymbol{x}_{t+1}, \mathbf{P}) f_t^e(\boldsymbol{x}_{t+1}|\boldsymbol{x}_t, \mathbf{P})\, d\boldsymbol{x}_{t+1}$. And given this valuation function we can construct the agent's optimal decision rule (or best response) at period t given that he believes that in the future he will behave according to the CCPs in the vector $\mathbf{P}$. This *present-value* representation is the CCP approach more commonly used in empirical applications because it can be applied to a general class of DDC models. However, this representation requires the computation of present values and therefore it is subject to the curse of dimensionality. In applications with large state spaces, this approach can be implemented only if it is combined with an approximation method such as the discretization of the state space, or Monte Carlo simulation (e.g., Bajari et al., 2007; Hotz et al., 1994). In general, these approximation methods introduce a bias in parameter estimates.

The *terminal-state* representation was introduced by Hotz and Miller (1993) and it applies only to optimal stopping problems with a terminal state. The *finite-dependence* representation was introduced by Arcidiacono and Miller (2011) and applies to a particular class of DDC models with the finite dependence property. A DDC model has the finite dependence property if given two values of the decision variable at period t and their respective paths of the state variables after this period, there is always a finite period $t' > t$ (with probability one) where the state variables in the two paths take the same value. The *terminal-state* and the *finite-dependence* CCP representations do not involve the computation of present values, or even the estimation of CCPs at every possible state. This implies substantial computational savings as well as avoiding biases induced by approximation errors.

The system of *Euler equations* that we have derived in Proposition 3 can be seen also as a CCP representation of the optimal decision rule in a DDC model. Our representation shares all the computational advantages of the *terminal-state* and *finite-dependence* representations. However, in contrast to the *terminal-state* and *finite-dependence*, our Euler equation representation applies to a general class of DDC models. We can derive Euler equations for any DDC model where the unobservables satisfy the conditions of AS in the payoff function, and CI in the transition of the state variables.

GMM ESTIMATION OF EULER EQUATIONS

Suppose that the researcher has panel data of N agents over $\tilde{T}$ periods of time, where he observes agents' actions $\{a_{it}: i = 1, 2, \ldots, N; t = 1, 2, \ldots, \tilde{T}\}$, and a subvector x of the state variables, $\{x_{it}: i=1, 2, \ldots, N;\ t=1, 2, \ldots, T\}$. The number of agents N is large, and the number of time periods is typically short. The researcher is interested in using this sample to estimate the structural parameters of the model, θ. We describe here the GMM estimation of these structural parameters using moment restrictions from the Euler equations derived in section "Euler Equations in Dynamic Decision Models."

GMM Estimation of Euler Equations in Continuous Decision Models

The GMM estimation of the structural parameters is based on the combination of the Euler equation(s) in (5), the assumption of rational expectations, and some assumptions on the unobservable state variables (Hansen & Singleton, 1982). For the unobservables, this literature has considered the following type of assumption.

Assumption GMM-EE continuous decision. (A) The partial derivatives of the payoff function are $\partial\Pi(a_t, \boldsymbol{s}_t)/\partial a_t = \pi_a(a_t, \boldsymbol{x}_t)$ and $\partial\Pi(a_t, \boldsymbol{s}_t)/\partial y_t = \pi_y(a_t, \boldsymbol{x}_t) + \varepsilon_t$, where $\pi_a(a_t, \boldsymbol{x}_t)$ and $\pi_y(a_t, \boldsymbol{x}_t)$ are known functions to the researcher up to a vector of parameters θ, and ε_t is a vector of unobservables with zero means, not serially correlated, and mean independent of $(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}, a_{t-1})$ such that $\mathbb{E}(\varepsilon_{t+1} | \boldsymbol{x}_{t+1}, \boldsymbol{x}_t, a_t) = 0$. (B) The partial derivatives of the transition rule, $\partial Y_{t+1}/\partial a_t'$ and $\partial Y_{t+1}/\partial y_t'$, and the matrix $H(a_t, \boldsymbol{s}_t)$ do not depend on unobserved variables, that is, $\partial Y_{t+1}/\partial a_t' = Y_a(a_t, \boldsymbol{x}_t)$, $\partial Y_{t+1}/\partial y_t' = Y_y(a_t, \boldsymbol{x}_t)$, and $H(a_t, \boldsymbol{s}_t) = H(a_t, \boldsymbol{x}_t)$.

Under these conditions, the Euler equation implies the following orthogonality condition in terms only of observable variables $\{a, x\}$ and structural parameters θ: $\mathbb{E}(\omega(a_t, \boldsymbol{x}_t, a_{t+1}, \boldsymbol{x}_{t+1}; \theta) | \boldsymbol{x}_t) = 0$, where

$$\begin{aligned}\omega(a_t, \boldsymbol{x}_t, a_{t+1}, \boldsymbol{x}_{t+1}; \theta) \equiv \pi_a(a_t, \boldsymbol{x}_t; \theta) + \beta[\pi_y(a_{t+1}, \boldsymbol{x}_{t+1}; \theta) \\ - H(a_{t+1}, \boldsymbol{x}_{t+1}; \theta)\pi_a(a_{t+1}, \boldsymbol{x}_{t+1}; \theta)] Y_a(a_t, \boldsymbol{x}_t; \theta)\end{aligned} \tag{44}$$

The GMM estimator $\hat{\theta}_N$ is defined as the value of θ that minimizes the criterion function $m_N(\theta)'\Omega_N m_N(\theta)$, where $m_N(\theta) \equiv \{m_{N,1}(\theta), m_{N,2}(\theta), \ldots, m_{N,T-1}(\theta)\}$ is the vector of sample moments

$$m_{N,t}(\theta) = \frac{1}{N}\sum_{i=1}^{N} Z(x_{it})\omega(a_{it}, x_{it}, a_{it+1}, x_{it+1}; \theta) \tag{45}$$

and $Z(x_{it})$ is a vector of instruments (i.e., known functions of the observable state variables at period t).

The GMM-Euler equation approach for dynamic models with continuous decision variables has been extended to models with corner solutions and censored decision variables (Aguirregabiria, 1997; Cooper, Haltiwanger, & Willis, 2010; Pakes, 1994), and to dynamic games (Berry & Pakes, 2000).[11]

GMM Estimation of Static Random Utility Models

Consider the ARUM in section "Random Utility Model as a Continuous Optimization Problem." Now, the deterministic component of the utility function for agent i is $\pi(a_i, x_i; \theta)$, where x_i is a vector of exogenous characteristics of agent i and of the environment which are observable to the researcher, and θ is a vector of structural parameters. Given a random sample of N individuals with information on $\{a_i, x_i\}$, the marginal conditions of optimality in Eq. (16) can be used to construct a *semiparametric two-step GMM estimator* of the structural parameters. The first step consists in the nonparametric estimation of the CCPs $P(a|x) \equiv \Pr(a_{it} = a | x_{it} = x)$. Let $\hat{\mathbf{P}}_N \equiv \{\hat{P}(a|x_i)\}$ be a vector of nonparametric estimates of CCPs for any choice alternative a and any value of x_i in the sample. For instance, $\hat{P}_t(a|x)$ can be a kernel (Nadaraya−Watson) estimator of the regression between $1\{a_i = a\}$ and x_i. In the second step, the vector of parameters θ is estimated using the following GMM estimator:

$$\hat{\theta}_N = \arg\min_{\theta \in \Theta} m'_N\left(\theta, \hat{\mathbf{P}}_N\right) \Omega_N m_N\left(\theta, \hat{\mathbf{P}}_N\right) \tag{46}$$

where $m_N(\theta, \mathbf{P}) \equiv \{m_{N,1}(\theta, \mathbf{P}), m_{N,2}(\theta, \mathbf{P}), \ldots, m_{N,J}(\theta, \mathbf{P})\}$ is the vector of sample moments, with

$$m_{N,a}(\theta, \mathbf{P}) = \frac{1}{N} \sum_{i=1}^{N} Z_i \Bigg[\pi(a, x_i; \theta) - \pi(0, x_i; \theta) + e(a, x_i, P) - e(0, x_i, P) + \sum_{j=0}^{J} P(j|x_i) \frac{\partial e(j, x_i, P)}{\partial P(a|x_i)} \Bigg] \tag{47}$$

This two-step semiparametric estimator is root-N consistent and asymptotically normal under mild regularity conditions (see Theorems 8.1 and 8.2 in Newey & McFadden, 1994). The variance matrix of this estimator can be estimated using the semiparametric method in Newey (1994), or as recently shown by Ackerberg, Chen, and Hahn (2012) using a computationally simpler parametric-like method as in Newey (1984).

GMM Estimation of Euler Equations in DDC Models

The Euler equations that we have derived for DDC model implies the following orthogonality conditions: $\mathbb{E}(\xi(a_t, \boldsymbol{x}_t, \boldsymbol{x}_{t+1}; P_t, P_{t+1}, \theta)|a_t, \boldsymbol{x}_t) = 0$, where

$$\xi(a_t,\ \boldsymbol{x}_t,\ \boldsymbol{x}_{t+1}; P_t,\ P_{t+1}, \theta) \equiv \frac{\partial \Pi_t^e}{\partial P_t(a_t|\boldsymbol{x}_t)} + \beta \left[\Pi_{t+1}^e(\boldsymbol{x}_{t+1}) - \mathbf{m}(x_{t+1})' \frac{\partial \boldsymbol{\Pi}_{t+1}^e(z_{t+1})}{\partial \mathbf{P}_{t+1}(z_{t+1})} \right] \frac{\tilde{f}_t(y_{t+1}|a_t, \boldsymbol{x}_t)}{f_t(y_{t+1}|a_t, \boldsymbol{x}_t)} \tag{48}$$

Note that this orthogonality condition comes from the Euler equation (33) in Proposition 3, but we have made two changes. First, we have included the expectation $\mathbb{E}(.|a_t, \boldsymbol{x}_t)$ that replaces the sum $\sum_{x_{t+1}}$ and the distribution of $\boldsymbol{x}_{t+1}$ conditional on $(a_t, \boldsymbol{x}_t)$, that is, $f_t(y_{t+1}|a_t, \boldsymbol{x}_t)\ f_z(z_{t+1}|z_t)$. And second, the Euler equation applies to any hypothetical choice, a, at period t, but in the orthogonality condition $\mathbb{E}(\xi(a_t, \boldsymbol{x}_t, \boldsymbol{x}_{t+1}; P_t, P_{t+1}, \theta)|a_t, \boldsymbol{x}_t) = 0$ we consider only the actual/observed choice a_t.

Given these conditions, we can construct a consistent an asymptotically normal estimator of θ using a semiparametric two-step GMM similar to the one described above for the static model. For simplicity, suppose that the sample includes only two periods, t and $t+1$. Let $\hat{\mathbf{P}}_{t,N}$ and $\hat{\mathbf{P}}_{t+1,N}$ be vectors with the nonparametric estimates of $\{P_t(a|\boldsymbol{x}_t)\}$ and $\{P_{t+1}(a|\boldsymbol{x}_{t+1})\}$, respectively at any value of $\boldsymbol{x}_t$ and $\boldsymbol{x}_{t+1}$ observed in the sample. Note that we do not need to estimate CCPs at states which are not observed in the sample. In the second step, the GMM estimator of θ is

$$\hat{\theta}_N = \arg\min_{\theta \in \Theta} m_N'\left(\theta, \hat{\mathbf{P}}_{t,N}, \hat{\mathbf{P}}_{t+1,N}\right) \Omega_N m_N\left(\theta, \hat{\mathbf{P}}_{t,N}, \hat{\mathbf{P}}_{t+1,N}\right) \tag{49}$$

where $m_N(\theta, \mathbf{P}_t, \mathbf{P}_{t+1})$ is the vector of sample moments:

$$m_N(\theta, \mathbf{P}_t, \mathbf{P}_{t+1}) = \frac{1}{N} \sum_{i=1}^{N} Z(a_{it}, x_{it}) \xi(a_{it}, x_{it}, x_{it+1}; P_{it}, P_{it+1}, \theta) \tag{50}$$

$Z(a_{it}, x_{it})$ is a vector of instruments, that is, known functions of the observable decision and state variables at period t. As in the case of the static ARUM, this semiparametric two-step GMM estimator is consistent and asymptotically normal under mild regularity conditions.

Relationship with Other CCP Estimators

The estimation based on the moment conditions provided by the *Euler equation*, or *terminal-state*, or *finite-dependence* representations imply an efficiency loss relative to estimation based on *present-value* representation. As shown by Aguirregabiria and Mira (2002, Proposition 4), the two-step PML estimator based on the CCP *present-value* representation is asymptotically efficient (equivalent to the ML estimator). This efficiency property is not shared by the other CCP representations. Therefore, there is a trade-off in the choice between CCP estimators based on Euler equations and on present-value representations. The present-value representation is the best choice in models that do not require approximation methods. However, in models with large state spaces that require approximation methods, the Euler equations CCP estimator can provide more accurate estimates.

AN APPLICATION

This section presents an application of the Euler equations-GMM method to a binary choice model of firm investment. More specifically, we consider the problem of a dairy farmer who has to decide when to replace a dairy cow by a new heifer. The cow replacement model that we consider here is an example of asset or "machine" replacement model.[12] We estimate this model using data on dairy cow replacement decisions and milk production using a two-step PML estimator and the ML estimator, and compare these estimates to those of the Euler equations-GMM method.

Model

Consider a farmer that produces and sells milk using dairy cows. The farm can be conceptualized as a plant with a fixed number of stalls n, one for each dairy cow. We index time by t and stalls by i. In our model, one period of time is a lactation period of 13 months. Farmer profits at period t is the sum of profits across the stalls, $\sum_{i=1}^{n} \Pi_{it}$ where Π_{it} is the profit from stall i at period t, minus the fixed cost of operating a farm with n stalls/cows, $FC_t(n)$. In this application, we take the size of a farm, n, as exogenously given. Furthermore, profits are separable across stalls and we can view the problem as maximization of profit from an individual stall.

The farmer decides when (after which lactation period) to replace the existing cow by a new heifer. Let $a_{it} \in \{0, 1\}$ be the indicator for this replacement decision: $a_{it} = 1$ means that the existing cow is replaced at the end of the current lactation period. The profit from stall i at period t is

$$\Pi_{it} = \begin{cases} p_t^M M(y_{it}, \omega_{it}) - C(y_{it}) + \varepsilon_{it}(0) & \text{if} \quad a_{it} = 0 \\ p_t^M M(y_{it}, \omega_{it}) - C(y_{it}) - R(y_{it}, p_t^H) + \varepsilon_{it}(1) & \text{if} \quad a_{it} = 1 \end{cases} \tag{51}$$

$M(y_{it}, \omega_{it})$ is the production of milk of the cow in stall i at period t, where $y_{it} \in \{1, 2, ..., y^{\max}\}$ is the current cow's age or lactation number, and ω_{it} is a cow-stall idiosyncratic productivity. p_t^M is the market price of milk. $C(y_{it})$ is the maintenance cost that may depend on the age of the cow. $R(y_{it}, p_t^H)$ is the net cost of replacing the existing cow by a new heifer. This net cost is equal to the market price of a new heifer, p_t^H, plus some adjustment/transaction costs, minus the market value of the retired cow. This market value depends on the quality of the meat, and this quality depends on the age of the retired cow but not on her milk productivity. In what follows we assume that the prices p_t^M and p_t^H are constants and as such, do not constitute part of the vector of state variables. So the vector of observable state variables is $x_{it} = (y_{it}, \omega_{it})$ where y_{it} is the endogenous state variable, and $z_{it} = \omega_{it}$ is the vector of endogenous state variable.

The estimations that we present below are based on the following specification on the functions $C(.)$ and $R(.)$: $C(y_{it}) = \theta_C y_{it}$, and $R(y_{it}) = \theta_R$. That is, the maintenance cost of a cow is linear in the cow's age, and the replacement cost is fixed over time.[13] While the productivity shock ω_{it} is unobservable to the econometrician, as we show below, under some assumptions it can be recovered by estimation of the milk production function, $m_{it} = M(y_{it}, \omega_{it})$, where m_{it} is the amount of milk, in liters, produced by the cow in stall i at period t. The transition probability function for the productivity shock ω_{it} is

$$\Pr(\omega_{i,t+1} | \omega_{it}, a_{it}) = \begin{cases} p_\omega(\omega_{i,t+1} | \omega_{it}) & \text{if} \quad a_{it} = 0 \\ p_\omega^0(\omega_{i,t+1}) & \text{if} \quad a_{it} = 1 \end{cases} \tag{52}$$

An important feature of this transition probability is that the productivity of a new heifer is independent of the productivity of the retired cow. Once we have recovered ω_{it}, the transition function for the productivity shock can be identified from the data. The transition rule for the cow age is trivial: $y_{i,t+1} = 1 + (1 - a_{it})y_{it}$. The unobservables $\varepsilon_{it}(0)$ and $\varepsilon_{it}(1)$ are assumed i.i.d. over i and over t with type 1 extreme value distribution with dispersion parameter σ_ε.

Data

The dataset comes from Miranda and Schnitkey (1995). It contains information on the replacement decision, age and milk production of cows from five Ohio dairy farms over the period 1986–1992. There are 2,340 observations from a total of 1,103 cows: 103 cows from farmer 1; 187 cows from farmer 2; 365 from farmer 3; 282 from farmer 4; and 166 cows from the last farmer. The data were provided by these five farmers through the Dairy Herd Improvement Association.

Here we use the sample of cows which entered in the production process before 1987. The reason for this selection is that for these initial cohorts we have complete lifetime histories for every cow, while for the later cohorts we have censored durations. Our working sample consists of 357 cows and 783 observations.

Table 1. Descriptive Statistics (Working Sample: 357 Cows with Complete Spells).

		Cow Lactation Period (Age)				
		1	2	3	4	5
Distribution of cows (%) by age of replacement		113 (31.7%)	126 (35.3%)	68 (19.0%)	37 (10.4%)	13 (3.6%)
Hazard rate for the replacement decision		0.317	0.516	0.571	0.740	1.000
Mean Milk Production	1	14.90	18.13	18.76	18.42	16.85
(thousand pounds) by age	2	–	17.42	19.80	20.46	19.40
(row) and age at replacement	3	–	–	20.06	23.74	22.28
(column)	4	–	–	–	20.07	21.60
	5	–	–	–	–	16.99

In Table 1 we provide some basic descriptive statistics from our working sample. The hazard rate for the replacement decision increases monotonically with the age of the cow. Average milk production (per cow and period) presents an inverted-U shape pattern both with respect to the current age of the cow and with respect to the age of the cow at the moment of replacement. This evidence is consistent with a causal effect of age of milk output but also with a selection effect, that is, more productive cows tend to be replaced at older ages.

Estimation

In this section we estimate the structural parameters of the profit function using our Euler equations method, as well as two more standard methods for estimation of DDC models, the two-step PML method and ML method for illustrative purposes.

Estimation of Milk Production Function

Regardless of the method we use to estimate the structural parameters in the cost functions, we first estimate the milk production function, $m_{it} = M(y_{it}, \omega_{it})$, outside the dynamic programming problem. We consider a specification for milk production that is nonparametric in age, and log-additive in the productivity shock ω_{it}:

$$\ln(m_{it}) = \sum_{j=1}^{y^{\max}} \alpha_j 1\{y_{it} = j\} + \omega_{it} \tag{53}$$

A potentially important issue in the estimation of this production function is that we expect age y_{it} to be positively correlated with the productivity shock ω_{it}. Less productive cows are replaced at early ages, and high productivity cows at later ages. Therefore, OLS estimates of α will not have a causal interpretation, as the age of the cow y_{it} is positively correlated with unobserved productivity ω_{it}. Specifically, we would expect that $E[\omega_{it}|y_{it}]$ is increasing in y_{it} as more productive cows survive longer than less productive ones. This would tend to bias downward the α's at early ages and upward bias the α's at old ages.[14]

To overcome this endogeneity problem, we consider the following approach. First, note that if the productivity shock were not serially correlated, there would be no endogeneity problem because age is a predetermined variable which is not correlated with an unanticipated shock at period t. Therefore, if we can transform the production function such that the unobservable is not serially correlated, then the unobservable in the production function will not be correlated with age. Note that the productivity shock ω_{it} is cow specific and is not transferred to another cow in the same stall. Therefore, if the age of the cow is 1, we have that ω_{it} is not correlated with $1\{y_{it}=1\}$. That is,

$$\alpha_1 = \mathbb{E}[\ln(m_{it})|y_{it}=1] \tag{54}$$

and we can estimate consistently α_1 using the frequency estimator $[\sum_{i,t}1\{y_{it}=1\}\ \ln(m_{it})]/[\sum_{i,t}1\{y_{it}=1\}]$. For ages greater than 1, we assume that ω_{it} follows an AR(1) process, $\omega_{it}=\rho\ \omega_{it-1}+\xi_{it}$, where ξ_{it} is an i.i.d. shock. Then, we can transform the production function to obtain the following sequence of equations. For $y_{it}\geq 2$

$$\ln(m_{it})=\rho\ \ln(m_{it-1})+\sum_{j=2}^{y^H}\gamma_j 1\{y_{it}=j\}+\xi_{it} \tag{55}$$

where $\gamma_j \equiv \alpha_j-\rho\ \alpha_{j-1}$. OLS estimation of this equation provides consistent estimates of ρ and $\gamma's$. Finally, using these estimates and the estimator of α_1, we obtain consistent estimates of ρ and $\alpha's$. We can also iterate in this procedure to obtain Cochrane–Orcutt FGLS estimator.

Table 2. Estimation of Milk Production Function (Working Sample: 357 Cows with Complete Spells).

Explanatory Variables	Estimates (Standard Errors)		
	Not controlling for selection	Controlling for selection	
		γ parameters	α parameters
$\ln(m_{it-1})$	–	0.636 (0.048)	
1{Age = 1}	2.823 (0.011)	–	2.823 (0.010)
1{Age = 2}	2.905 (0.014)	1.068 (0.139)	2.863 (0.014)
1{Age = 3}	3.047 (0.019)	1.150 (0.144)	2.971 (0.020)
1{Age = 4}	3.001 (0.030)	1.004 (0.152)	2.894 (0.030)
1{Age = 5}	2.809 (0.059)	0.862 (0.155)	2.702 (0.057)
R^2	0.13	0.364	
Number of observations	783	426	

Table 2 presents estimates of the production function. In column 1 we provide OLS estimates of Eq. (53) in levels. Column 2 presents OLS estimates of semi-difference transformed Eq. (55). And column 3, provides the estimates of the α parameters implied by the estimates in column 2, where their standard errors have been obtained using the delta method. The comparison of the estimates in columns 1 and 3 is fully consistent with the bias we expected. In column 1 we ignore the tendency for more productive cows to survive longer and we estimate a larger effect of age on milk production than when we do account for this in column 3. The difference is particularly large when the cow is age 4 or 5.

Structural Estimation of Payoff Parameters
We now proceed to the estimation of the structural parameters in the maintenance cost, replacement cost/value, and variance of ε, that is, $\boldsymbol{\theta} = \{\sigma_\varepsilon, \theta_C, \theta_R\}$. We begin by deriving the Euler equations of this model. This Euler equations correspond to the ones in the machine replacement model in Example 5 above. That is,

$$\left[\pi(1, y_t, \omega_t) - \pi(0, y_t, \omega_t) - \sigma_\varepsilon \quad \ln\left(\frac{P(1|y_t, \omega_t)}{P(0|y_t, \omega_t)}\right)\right] + \beta\,\mathbb{E}_t\left[\pi(1, 1, \omega_{t+1}) - \pi(1, y_t+1, \omega_{t+1}) - \sigma_\varepsilon \;\ln\left(\frac{P(1|1, \omega_{t+1})}{P(1|y_t+1, \omega_{t+1})}\right)\right] = 0 \tag{56}$$

where we have imposed the restriction that the model is stationary such as the functions $\pi(.)$ and $P(.)$ are time-invariant. Using our parameterization of the payoff function, we have that $\pi(1, y_t, \omega_t) - \pi(0, y_t, \omega_t) = -\theta_R$, and $\pi(1, 1, \omega_{t+1}) - \pi(1, y_t+1, \omega_{t+1}) = [p^M M(1, \omega_{t+1}) - p^M M(y_t+1, \omega_{t+1})] + \theta_C y_t$, such that we can get the following simple formula for this Euler equation:

$$\mathbb{E}_t\left(\tilde{M}_{t+1} - \theta_R + \theta_C \beta y_t + \sigma_\varepsilon \tilde{e}_{t+1}\right) = 0 \tag{57}$$

where $\tilde{M}_{t+1} \equiv \beta p^M [M(1, \omega_{t+1}) - M(y_t+1, \omega_{t+1})]$, and $\tilde{e}_{t+1} \equiv [\ln P(0|\boldsymbol{x}_t) + \beta \ln P(1|y_t+1, \omega_{t+1})] - [\ln P(1|\boldsymbol{x}_t) + \beta \ln P(1|1, \omega_{t+1})]$. We estimate $\boldsymbol{\theta} = \{\sigma_\varepsilon, \theta_C, \theta_R\}$ using a GMM estimator based on the moment conditions $\mathbb{E}_t(Z_t \{\tilde{M}_{t+1} - \theta_R + \theta_C y_t + \sigma_\varepsilon \tilde{e}_{t+1}\})$ where the vector of instruments Z_t is $\{1, y_t, \omega_t\, M(1, \omega_t) - M(y_t+1, \omega_t), \ln P(0|\boldsymbol{x}_t) - \ln P(1|\boldsymbol{x}_t), \ln P(1|y_t+1, \omega_t) - \ln P(1|1, \omega_t)\}'$.

Table 3. Estimation of Maintenance Cost and Replacement Cost Parameters (Working Sample: 357 Cows with Complete Spells).

Structural Parameters	Estimates			
	Two-Step PML	MLE	GMM-Euler equation	
			1-step	2-step (Opt. Wei. matrix)
Dispersion of unobs. σ_ε	0.296 (0.035)	0.288 (0.031)	0.133 (0.042)	0.138 (0.038)
Maintenance cost θ_C	0.136 (0.029)	0.131 (0.029)	0.103 (0.035)	0.105 (0.031)
Replacement cost θ_R	0.363 (0.085)	0.342 (0.079)	0.209 (0.087)	0.241 (0.085)
Number of observations	770	770	770	770
Pseudo R^2	0.707	0.707		

Table 3 presents estimates of these structural parameters using GMM-Euler equations, and using two other standard methods of estimation, the two-step PML (see Aguirregabiria & Mira, 2002), and the ML estimator. We use the nested pseudo likelihood (NPL) method of Aguirregabiria and Mira (2002) to obtain the ML estimates.[15] For these PML and ML estimations, we discretize the state variable ω_{it} in 201 values using a uniform grid in the interval $[-5\hat{\sigma}_\omega,\ 5\hat{\sigma}_\omega]$. The two-step PML and the MLE are very similar both in terms of point estimates and standard errors. Note that these estimators are asymptotically equivalent (Proposition 4, Aguirregabiria & Mira, 2002). However, in small samples and with large state spaces the finite sample properties of these estimators can be very different, and more specifically the two-step PML can have a substantially larger small sample bias (Kasahara & Shimotsu, 2008). In this application, it seems that the dimension of the state space is small relative to the sample size such that the initial nonparametric estimates of CCPs are precise enough, and the finite sample bias of the two-step PML is also small.

Table 3 presents two different GMM estimates based on the Euler equations: a 1-step GMM estimator where the weighting matrix is $(\sum_{i,t} Z_{it}\, Z'_{it})^{-1}$, and 2-step GMM estimator using the optimal weighting matrix. Both GMM estimates are substantially different to the MLE estimates, but the optimal GMM estimator is closer. A possible simple explanation for the difference between the GMM-EE and the MLE estimates is that the GMM estimates is asymptotically less efficient, that is, it is not using the optimal set of instruments. Other possible factor that may generate differences between these estimates is that the GMM estimator is not invariant to normalizations. In particular, we can get quite different estimates of $\boldsymbol{\theta} = \{\sigma_\varepsilon, \theta_C, \theta_R\}$ if we use a GMM estimator under the normalization that the coefficient of $\tilde{M}_{t+1}$ is equal to one (i.e., using moment conditions $\mathbb{E}(Z_t\{\tilde{M}_{t+1} - \theta_R + \theta_C \beta y_t + \sigma_\varepsilon \tilde{e}_{t+1}\}) = 0$) and if we use a GMM estimator under the normalization that the coefficient of $\tilde{e}_{t+1}$ is equal to one (i.e., using moment conditions $\mathbb{E}(Z_t\{(1/\sigma_\varepsilon)\tilde{M}_{t+1} - (\theta_R/\sigma_\varepsilon) + (\theta_C/\sigma_\varepsilon)\ \beta y_t + \tilde{e}_{t+1}\}) = 0)$. While the first normalization seems more "natural" because our parameters of interest appear linearly in the moment conditions, the second normalization is "closer" the moment conditions implied by the likelihood equations and MLE. We plan to explore this issue and obtain GMM-EE estimates under alternative normalizations.

The estimates of the structural parameters in Table 3 are measured in thousands of dollars. For comparison, it is helpful to take into account that the sample mean of the annual revenue generated by a cow's milk production is \$150, 000. According to the ML estimates, the cost of replacing a cow by a new heifer is \$34, 200 (i.e., 22.8% of a cow's annual revenue), and maintenance cost increases every lactation period by \$13, 100 (i.e., 8.7% of annual revenue). There is very significant unobserved heterogeneity in the cow replacement decision, as the standard deviation of these unobservables is equal \$28, 800.

Fig. 1 displays the predicted probability of replacement by age of the cow (replacement probability at age 5 is 1). The probabilities are

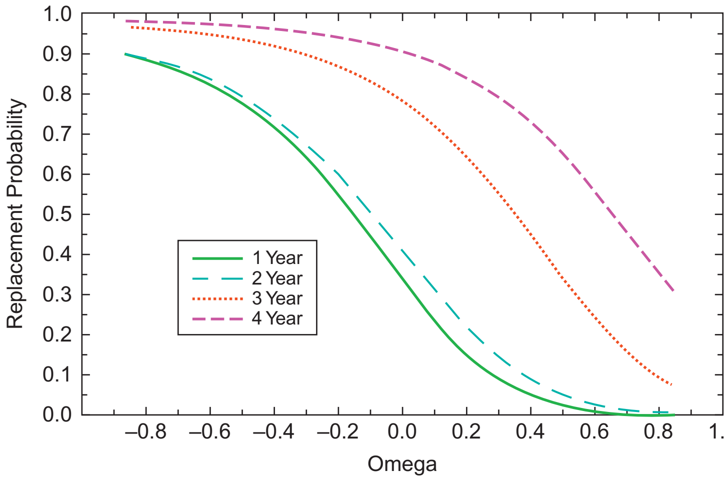

Fig. 1. Predicted probability of replacement.

constructed using the ML estimates. The results suggest that at any age, replacement is less likely the more productive the cow, and that for any given productivity older cows are more likely to be replaced. There is an especially large increase in the probability of replacement going from age 2 to age 3.

Because its simplicity, this empirical application provides a helpful framework for a first look at the estimation of DDC models using GMM-Euler equations. However, it is important to note that the small state space also implies that this example cannot show the advantages of this estimation method in terms of reducing the bias induced by the approximation of value functions in large state spaces. To investigate this issue, in our future work we plan to extend this application to include additional continuous state variables (i.e., price of milk, and the cost of a new heifer). We also plan to implement Monte Carlo experiments.

CONCLUSIONS

This article deals with the estimation of DDC structural models. We show that we can represent the DDC model as a continuous choice model where the decision variables are choice probabilities. Using this representation of the discrete choice model, we derive marginal conditions of optimality (Euler equations) for a general class DDC structural models, and based on these conditions we show that the structural parameters in the model can be estimated without solving or approximating value functions. This result generalizes the *GMM-Euler equation* approach proposed in the seminal work of Hansen and Singleton (1982) for the estimation of dynamic continuous decision models to the case of discrete choice models. The main advantage of this approach, relative to other estimation methods in the literature, is that the estimator is not subject to biases induced by the errors in the approximation of value functions.

NOTES

1. See Rust (1996) and the recent book by Powell (2007) for a survey of numerical approximation methods in the solution of dynamic programming problems. See also Geweke (1996) and Geweke and Keane (2001) for excellent surveys on integration methods in economics and econometrics with particular attention to dynamic structural models.

2. The Nested Fixed Point algorithm (NFXP) (Rust, 1987; Wolpin, 1984) is a commonly used full-solution method for the estimation of single-agent dynamic structural models. The Nested Pseudo Likelihood (NPL) method (Aguirregabiria & Mira, 2002, 2007) and the method of Mathematical Programming with Equilibrium Constraints (MPEC) (Su & Judd, 2012) are other full-solution methods. Two-step and sequential estimation methods include Conditional Choice Probabilities (CCP) (Hotz & Miller, 1993), K-step Pseudo Maximum Likelihood (Aguirregabiria & Mira, 2002, 2007), Asymptotic Least Squares (Pesendorfer & Schmidt-Dengler, 2008), and their simulated-based estimation versions (Bajari, Benkard, & Levin, 2007; Hotz et al., 1994).

3. Lerman and Manski (1981), McFadden (1989), and Pakes and Pollard (1989) are seminal works in this literature. See Gourieroux and Monfort (1993, 1997) Hajivassiliou and Ruud (1994), and Stern (1997) for excellent surveys.

4. In empirical applications, the most common approach to measure the importance of this bias is local sensitivity analysis. The parameter that represents the degree of accuracy of the approximation (e.g., the number of Monte Carlo simulations, the order of the polynomial, the number of grid points) is changed marginally around a selected value and the different estimations are compared. This approach may have low power to detect approximation-error-induced bias, especially when the approximation is poor and these biases can be very large.

5. A DDC model has the finite dependence property if given two values of the decision variable at period t and their respective paths of the state variables after this period, there is always a finite period $t' > t$ (with probability one) where the state variables in the two paths take the same value.

6. This representation is more general than it may look like because the vector of exogenous state variables in z_{t+1} can include any i.i.d. stochastic element that affects the transition rule of the endogenous state variables y. To see this, suppose that the transition probability of y_{t+1} is stochastic conditional on $(z_{t+1}, a_t, \mathbf{s}_t)$ such that $y_{t+1} = Y(\xi_{t+1}, z_{t+1}, a_t, \mathbf{s}_t)$ where ξ_{t+1} is a random variable that is unknown at period t and is i.i.d. over time. We can expand the vector of exogenous state variables to include ξ such that the new vector is $z_t^* \equiv (z_t, \xi_t)$. Then, $f^*(y_{t+1}, z_{t+1}^* | a_t, y_t, z_t^*) = f^{y*}(y_{t+1}|z_{t+1}^*, a_t, y_t, z_t^*) f^{z^*}(z_{t+1}^*|z_t^*)$ and by construction $f^{y*}(y_{t+1}|z_{t+1}^*, a_t, y_t, z_t^*) = 1\{y_{t+1} = Y(\xi_{t+1}, z_{t+1}, a_t, \mathbf{s}_t)\}$.

7. See Section 9.5 in Stokey, Lucas, and Prescott (1989) and Section 4 in Rust (1992).

8. Note that $\sum_{j=0}^{J} P(j)[\partial e(j, \mathbf{P})/\partial P(a)]$ is equal to $P(a)[-1/P(a)] + P(0)[1/P(0)] = 0$.

9. For the derivation of these expressions, it is useful to take into account that $\phi'(z) = -z\phi(z)$ and $d\Phi^{-1}(P)/dP = 1/\phi(\Phi^{-1}(P))$.

10. Therefore, we also have that $e_t(a, \mathbf{P}_t(\boldsymbol{x}_t))$ is equal to $\mathbb{E}(\varepsilon_t(a)|\varepsilon_t(j) - \varepsilon_t(a) \leq \tilde{G}^{-1}(a, \mathbf{P}_t(\boldsymbol{x}_t)) - \tilde{G}^{-1}(j, \mathbf{P}_t(\boldsymbol{x}_t))$ for any $j \neq a$).

11. The paper by Cooper et al. (2010) is "Euler Equation Estimation for Discrete Choice Models: A Capital Accumulation Application." However, that paper deals with the estimation of models with continuous but censored decision variables, and not with pure discrete choice models.

12. Dynamic structural models of machine replacement have been estimated before by Rust (1987), Sturm (1991), Das (1992), Kennet (1994), Rust and Rothwell (1995), Adda and Cooper (2000), Cho (2011), and Kasahara (2009), among others.

13. The latter may seem a strong assumption, but given that almost every cow in our sample is sold in the first few years of its life, the assumption may not be so strong over the range of ages observed in the data.

14. The nature of this type of bias is very similar to the one in the estimation of the effect of firm-age in a production function of manufacturing firms, or in the estimation of the effect of firm-specific experience in a wage equation.

15. In the context of single agent DDC models with a globally concave pseudo likelihood, the NPL operator is a contraction such that it always converges to its unique fixed point (Kasahara and Shimotsu) and this fixed point is the MLE (Aguirregabiria & Mira, 2002). In this application the NPL algorithm converged to the MLE after seven iterations using a convergence criterion of $\|\hat{\theta}_k - \hat{\theta}_{k-1}\| < 10^{-6}$.

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APPENDIX

Proof of Proposition 1.

Part (i). Let $\Pi(\alpha,\varepsilon)$ be the ex-post payoff function associated with a decision rule α, such that $\Pi(\alpha,\varepsilon)=\sum_{a=0}^{J}1\{\alpha(\varepsilon)=a\}[\pi(a)+\varepsilon(a)]$. By Lemmas 1–2, there is a one-to-one relationship between $\mathbf{P}$ and α. Given this relationship, we can represent the ex-post payoff function associated with a decision rule α using the following function of $\mathbf{P}$:

$$\Pi(\mathbf{P},\varepsilon)\equiv\sum_{a=0}^{J}1\left\{\varepsilon(j)-\varepsilon(a)\leq\tilde{G}^{-1}(a,\mathbf{P})-\tilde{G}^{-1}(j,\mathbf{P})\ \text{ for any }\ j\neq a\right\}[\pi(a)+\varepsilon(a)] \tag{A.1}$$

Given that α^* maximizes $\Pi(\alpha,\varepsilon)$ for every possible value of ε, then by construction, $\mathbf{P}^{\alpha^*}$ maximizes $\Pi(\mathbf{P},\varepsilon)$ for every possible value of ε. The proof of this is by contradiction. Suppose that there is a vector of CCPs $\mathbf{P}_0\neq\mathbf{P}^{\alpha^*}$ and a value ε_0 such that $\Pi(\mathbf{P}_0,\varepsilon_0)>\Pi(\mathbf{P}^{\alpha^*},\varepsilon_0)$. This implies that the optimal decision for ε_0 is the action a with the largest value of $\tilde{G}^{-1}(a,\mathbf{P}_0)+\varepsilon_0(a)$. But because $[\tilde{G}^{-1}(a,\mathbf{P}_0)-\tilde{G}^{-1}(j,\mathbf{P}_0)]\neq[\tilde{G}^{-1}(a,\mathbf{P}^{\alpha^*})-\tilde{G}^{-1}(j,\mathbf{P}^{\alpha^*})]=\pi(a)-\pi(j)$, the action that maximizes $\tilde{G}^{-1}(a,\mathbf{P}_0)+\varepsilon_0(a)$ is different to the action that maximizes $\pi(a)+\varepsilon(a)$. This contradicts that $\Pi(\mathbf{P}_0,\varepsilon_0)>\Pi(\mathbf{P}^{\alpha^*},\varepsilon_0)$.

Because $\mathbf{P}^{\alpha^*}$ maximizes $\Pi(\mathbf{P},\varepsilon)$ for every possible value of ε, it should be true that $\mathbf{P}^{\alpha^*}$ maximizes in $\mathbf{P}$ the "integrated" payoff function $\int\Pi(\mathbf{P},\varepsilon)dG(\varepsilon)$. It is straightforward to show that this integrated payoff function is the expected payoff function $\Pi^e(\mathbf{P})$. Therefore, $\mathbf{P}^{\alpha^*}$ maximizes the expected payoff function. By uniqueness of $\mathbf{P}^*$, this implies that $\mathbf{P}^{\alpha^*}=\mathbf{P}^*$.

Part (ii). The expected payoff function $\Pi^e(\mathbf{P})$ is continuously differentiable with respect to $\mathbf{P}$. Furthermore, $\Pi^e(\mathbf{P})$ goes to minus infinite as any of the choice probabilities in $\mathbf{P}$ goes to 0 or to 1, that is, when $\mathbf{P}$ goes to the frontier of the simplex $\mathcal{S}$. Therefore, the maximizer $\mathbf{P}^*$ should be in the interior of the simplex and it should satisfy the marginal conditions of optimality $\partial\Pi^e(\mathbf{P}^*)/\partial\mathbf{P}=0$. Finally, given the definition of the expected payoff function in Eq. (13), we have that

$$\frac{\partial\Pi^e(\mathbf{P})}{\partial P(a)}=\pi(a)-\pi(0)+e(a,\mathbf{P})-e(0,\mathbf{P})+\sum_{j=0}^{J}P(j)\frac{\partial e(j,\mathbf{P})}{\partial P(a)} \tag{A.2}$$

Proof of Proposition 2.

The proof of this proposition is a recursive application of Proposition 1. Let $W_t(\boldsymbol{x}_t, \varepsilon_t, \alpha_t, \mathbf{P}_{t'>t})$ be the ex-post valuation function associated with a current decision rule α_t and future CCPs $\mathbf{P}_{t'>t}$, such that

$$W_t(\boldsymbol{x}_t, \varepsilon_t, \alpha_t, \mathbf{P}_{t'>t}) = \sum_{a=0}^{J} 1\{\alpha_t(\boldsymbol{x}_t, \varepsilon_t) = a\}[v_t(a, \boldsymbol{x}_t, \mathbf{P}_{t'>t}) + \varepsilon_t(a)] \tag{A.3}$$

and $v_t(a, \boldsymbol{x}_t, \mathbf{P}_{t'>t})$ is the conditional choice value $\pi_t(a, \boldsymbol{x}_t) + \beta \int W_{t+1}(\boldsymbol{x}_{t+1}, \mathbf{P}_{t+1}(\boldsymbol{x}_{t+1}), \mathbf{P}_{t'>t+1})\, f_t(\boldsymbol{x}_{t+1}|a, \boldsymbol{x}_t) d\boldsymbol{x}_{t+1}$. By Lemmas 1−2, there is a one-to-one relationship between $\mathbf{P}_t(\boldsymbol{x}_t)$ and α_t. Given this relationship, we can represent the ex-post valuation function associated with a decision rule α_t using the following function of $\mathbf{P}_t(\boldsymbol{x}_t)$:

$$\begin{aligned} W_t(\boldsymbol{x}_t, \varepsilon_t, \mathbf{P}_t(\boldsymbol{x}_t), \mathbf{P}_{t'>t}) \equiv \sum_{a=0}^{J} & 1\{\varepsilon_t(j) - \varepsilon_t(a) \leq \tilde{G}^{-1}(a, \mathbf{P}_t(\boldsymbol{x}_t)) \\ & - \tilde{G}^{-1}(j, \mathbf{P}_t(\boldsymbol{x}_t)) \text{ for any } j \neq a\} \;\; [v_t(a, \boldsymbol{x}_t, \mathbf{P}_{t'>t}) + \varepsilon_t(a)] \end{aligned} \tag{A.4}$$

By definition of the optimal decision rule, given $\mathbf{P}^*_{t\prime>t}$ the decision rule α^*_t maximizes $W_t(\boldsymbol{x}_t, \varepsilon_t, \alpha_t, \mathbf{P}^*_{t\prime>t})$ for every possible value of ε_t. Then, as in Proposition 1, we have that by construction, $\mathbf{P}^{\alpha^*}_t(\boldsymbol{x}_t)$ maximizes $W_t(\boldsymbol{x}_t, \varepsilon_t, \mathbf{P}_t(\boldsymbol{x}_t), \mathbf{P}_{t'>t})$ for every possible value of ε_t. This implies that $\mathbf{P}^{\alpha^*}_t(\boldsymbol{x}_t)$ also maximizes the "integrated" valuation function $\int W_t(\boldsymbol{x}_t, \varepsilon_t, \mathbf{P}_t(\boldsymbol{x}_t), \mathbf{P}_{t'>t}) dG(\varepsilon_t)$. But this integrated function is equal to the expected valuation function $W^e_t(\boldsymbol{x}_t, \mathbf{P}_t(\boldsymbol{x}_t), \mathbf{P}_{t'>t})$. Therefore, $\mathbf{P}^{\alpha^*}_t(x)$ maximizes $W^e_t(x, \mathbf{P}_t(x), \mathbf{P}_{t'>t})$. By uniqueness of $\mathbf{P}^*_t(x)$, this implies that $\mathbf{P}^{\alpha^*}_t(x) = \mathbf{P}^*_t(x)$.

The expected valuation function $W^e_t(x, \mathbf{P}_t(x), \mathbf{P}_{t'>t})$ is continuously differentiable with respect to $\mathbf{P}_t(x)$. The maximizer $W^e_t(x, \mathbf{P}_t(x), \mathbf{P}_{t'>t})$ with respect to $\mathbf{P}_t(x)$ should be in the interior of the simplex and it should satisfy the marginal conditions of optimality $\partial W^e_t\left(x, \mathbf{P}^*_t(x), \mathbf{P}^*_{t\prime>t}\right)/\partial \mathbf{P}^*_t(x) = 0$. Given the definition of the expected value function, we have that

$$\frac{\partial W^e_t}{\partial P_t(a|x)} = v_t(a, \boldsymbol{x}_t, \mathbf{P}_{t'>t}) - v_t(0, \boldsymbol{x}_t, \mathbf{P}_{t'>t}) + e_t(a, \mathbf{P}_t) - e_t(0, \mathbf{P}_t) + \sum_{j=0}^{J} P_t(j|x) \frac{\partial e_t(j, \mathbf{P}_t)}{\partial P_t(a|x)} \tag{A.5}$$

Proof of Proposition 3.

For the derivation of the expressions below for the Lagrangian conditions, note that, by definition of $f_t^e(y_{t+1}|\boldsymbol{x}_t)$, we have that $\partial f_t^e(y_{t+1}|\boldsymbol{x}_t)/\partial P_t(a|\boldsymbol{x}_t) = \tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t) \equiv f_t(y_{t+1}|a,\boldsymbol{x}_t) - f_t(y_{t+1}|0,\boldsymbol{x}_t)$. For any $a>0$, the Lagrange condition $\partial \mathcal{L}_t/\partial P_t(a|\boldsymbol{x}_t) = 0$ implies that

$$\frac{\partial \Pi_t^e}{\partial P_t(a|\boldsymbol{x}_t)} + \beta \sum_{\boldsymbol{x}_{t+1}} \Pi_{t+1}^e(\boldsymbol{x}_{t+1})\ \tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t)\ f_z(z_{t+1}|z_t)$$
$$- \sum_{y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)} \lambda_t(y_{t+2},\boldsymbol{x}_t) \left[\sum_{x_{t+1}} f_{t+1}^e(y_{t+2}|x_{t+1})\ \tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t)\ f_z(z_{t+1}|z_t) \right] = 0 \tag{A.6}$$

We can also represent this expression as

$$\frac{\partial \Pi_t^e}{\partial P_t(a|\boldsymbol{x}_t)} + \beta \sum_{\boldsymbol{x}_{t+1}} \left[\Pi_{t+1}^e(x_{t+1}) - \mathbf{f}_{t+1}^e(x_{t+1})' \frac{\boldsymbol{\lambda}_t(\boldsymbol{x}_t)}{\beta} \right] \tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t)\ f_z(z_{t+1}|z_t) = 0 \tag{A.7}$$

where $\boldsymbol{\lambda}_t(\boldsymbol{x}_t)$ is the vector with dimension $|\mathcal{Y}_{+2}(\boldsymbol{x}_t)| \times 1$ with the Lagrange multipliers $\{\lambda_t(y_{t+2},\boldsymbol{x}_t) : y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)\}$, and $\mathbf{f}_{t+1}^e(.|\boldsymbol{x}_{t+1})$ is the vector of transition probabilities $\{f_{t+1}^e(y_{t+2}|\boldsymbol{x}_{t+1}) : y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)\}$. Similarly, for any $a>0$ and any $\boldsymbol{x}_{t+1} \in \mathcal{X}$, the Lagrange condition $\partial \mathcal{L}_t/\partial P_{t+1}(a|\boldsymbol{x}_{t+1}) = 0$ implies that

$$\beta \frac{\partial \Pi_{t+1}^e(\boldsymbol{x}_{t+1})}{\partial P_{t+1}(a|\boldsymbol{x}_{t+1})} - \sum_{y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)} \lambda_t(y_{t+2},\boldsymbol{x}_t)\ \tilde{f}_{t+1}(y_{t+2}|a,\boldsymbol{x}_{t+1}) = 0 \tag{A.8}$$

We can represent this system of equations in vector form as

$$\tilde{\mathbf{F}}_{t+1}(z_{t+1}) \frac{\boldsymbol{\lambda}_t(\boldsymbol{x}_t)}{\beta} = \frac{\partial \boldsymbol{\Pi}_{t+1}^e(z_{t+1})}{\partial \mathbf{P}_{t+1}(z_{t+1})} \tag{A.9}$$

$\boldsymbol{\lambda}_t(\boldsymbol{x}_t)$ is the vector of Lagrange multipliers defined above. $\partial \boldsymbol{\Pi}_{t+1}^e(z_{t+1})/\partial \mathbf{P}_{t+1}(z_{t+1})$ is a column vector with dimension $J|\mathcal{Y}_{+1}(\boldsymbol{x}_t)| \times 1$ that contains the partial derivatives $\{\partial \Pi_{t+1}^e(y_{t+1}, z_{t+1})/\partial P_{t+1}(a|y_{t+1}, z_{t+1})\}$ for every action $a>0$ and every value $y_{t+1} \in \mathcal{Y}_{+1}(\boldsymbol{x}_t)$ that can be reach from $\boldsymbol{x}_t$, and fixed value for z_{t+1}. And $\tilde{\mathbf{F}}_{t+1}(z_{t+1})$ is matrix with dimension $J|\mathcal{Y}_{+1}(\boldsymbol{x}_t)| \times |\mathcal{Y}_{+2}(\boldsymbol{x}_t)|$ that contains the probabilities $\tilde{f}_{t+1}(y_{t+2}|a,\boldsymbol{x}_{t+1})$ for every $y_{t+2} \in \mathcal{Y}_{+2}(\boldsymbol{x}_t)$, every $y_{t+1} \in \mathcal{Y}_{+1}(\boldsymbol{x}_t)$, and every action $a>0$, with fixed z_{t+1}. In general, the matrix $\tilde{\mathbf{F}}_{t+1}(z_{t+1})$ is full-column rank for any value of z_{t+1}. Therefore, for any value of z_{t+1}, the square matrix $\tilde{\mathbf{F}}_{t+1}(z_{t+1})' \; \tilde{\mathbf{F}}_{t+1}(z_{t+1})$ is non-singular and we can solve for the Lagrange multipliers as

$$\frac{\lambda_t(\boldsymbol{x}_t)}{\beta} = \left[\tilde{\mathbf{F}}_{t+1}(z_{t+1})'\tilde{\mathbf{F}}_{t+1}(z_{t+1})\right]^{-1}\left[\tilde{\mathbf{F}}_{t+1}(z_{t+1})' \frac{\partial \mathbf{\Pi}^e_{t+1}(z_{t+1})}{\partial \mathbf{P}_{t+1}(z_{t+1})}\right] \tag{A.10}$$

Solving this expression for the Lagrange multipliers into Eq. (A.7), we get the following Euler equation

$$\frac{\partial \Pi^e_t}{\partial P_t(a|\boldsymbol{x}_t)} + \beta \sum_{x_{t+1}} \left[\Pi^e_{t+1}(\boldsymbol{x}_{t+1}) - \mathbf{m}(\boldsymbol{x}_{t+1})' \frac{\partial \mathbf{\Pi}^e_{t+1}(z_{t+1})}{\partial \mathbf{P}_{t+1}(z_{t+1})}\right] \tilde{f}_t(y_{t+1}|a,\boldsymbol{x}_t)\, f_z(z_{t+1}|z_t) = 0 \tag{A.11}$$

where $\mathbf{m}(\boldsymbol{x}_{t+1})$ is a $J|\mathcal{Y}_{+1}(\boldsymbol{x}_t)| \times 1$ vector such that $\mathbf{m}(\boldsymbol{x}_{t+1})' = \mathbf{f}^e_{t+1}(\boldsymbol{x}_{t+1})'$ $[\tilde{\mathbf{F}}_{t+1}(z_{t+1})'\tilde{\mathbf{F}}_{t+1}(z_{t+1})]^{-1}, \tilde{\mathbf{F}}_{t+1}(z_{t+1})'$, and $\partial \mathbf{\Pi}^e_{t+1}(z_{t+1})/\partial \mathbf{P}_{t+1}(z_{t+1})$ is the vector of partial derivatives defined above.