Empirical Industrial Organization: Models, Methods, and Applications

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1. Introduction

1.1 Empirical Industrial Organization

Industrial Organization (IO) deals with the behavior of firms in markets. We are interested in understanding how firms interact strategically, and how their actions affect market outcomes. IO economists are particularly interested in three aspects related to market allocation: *market structure*, *firms' market power*, and *firms' strategies*. These are key concepts in IO. *Market structure* is a description of the number of firms in the market, their market shares, and the products they sell. A monopoly is an extreme case of market structure where a single firm concentrates the total output in the market. At the other extreme we have an atomistic market structure where industry output is equally shared by a very large number of very small firms. Between these two extremes, we have a whole spectrum of possible oligopoly market structures. *Market power* (or *monopoly power*) is the ability of a firm, or group of firms, to gain extraordinary profits above those needed to remunerate all the inputs at market prices. A *firm's strategy* is a description of the firms' actions (for instance, pricing, production and market entry decisions) contingent on the state of demand and cost conditions. We say that a firm behaves strategically if it takes into account that its actions affect other firms' profits and behavior.

A significant part of the research in IO deals with understanding the determinants of market power, market structure, and firms' strategies in actual markets and industries. IO economists propose models where these variables are determined endogenously and depend on multiple exogenous factors such as consumer demand, input supply, technology, regulation, as well as firms' beliefs about the behavior of competitors and the nature of competition. The typical model in IO treats demand, technology, and institutional features as given, and postulates some assumptions about how firms compete in a market. Based on these assumptions, we study firms' strategies. In particular, we are interested in finding a firm's profit maximizing strategy, given its beliefs about the behavior of other firms, and in determining equilibrium strategies: the set of all firms' strategies which are consistent with profit maximization and rational beliefs about each others' behavior. We use Game Theory tools to find these equilibrium strategies, and to

study how changes in exogenous factors affect firms' strategies, market structure, firms' profits, and consumer welfare.

The models of Perfect Competition and of Perfect Monopoly are two examples of IO models. However, they are extreme cases and they do not provide a realistic description of many markets and industries in today's economy. Many interesting markets are characterized by a relatively small number of firms who behave strategically and take into account how their decisions affect market prices and other firms' profits. We refer to these markets as *oligopoly markets*, and they are the focus of IO.

Most of the issues that we study in IO have an important empirical component. To answer questions related to competition between firms in an industry, we typically need information on consumer demand, firms' costs, and firms' strategies or actions in that industry. **Empirical Industrial Organization (EIO) deals with the combination of data, models, and econometric methods to answer empirical questions related to the behavior of firms in markets**. The tools of EIO are used in practice by firms, government agencies, consulting companies, and academic researchers. Firms use these tools to improve their strategies, decision making, and profits. For instance, EIO methods are useful tools to determine a firm's optimal prices, to evaluate the value added of a merger, to predict the implications of introducing a new product in the market, or to measure the benefits of price discrimination. Government agencies use the tools of industrial organization to evaluate the effects of a new policy in an industry (for instance, an increase in the sales tax, or an environmental policy), or to identify anti-competitive practices such as collusion, price fixing, or predatory conducts. Academic researchers use the tools of EIO to improve our understanding of industry competition. The following are some examples of these types of questions.

Example 1: Estimating the demand for a new product. A company considers launching a new product, for instance, a new smartphone. To estimate the profits that the new product will generate to the company, and to decide the initial price that maximizes these profits, the company needs to predict the demand for this new product, and the response (that is, price changes) of the other firms competing in the market of smartphones. Data on sales, prices, and product attributes from firms and products that are already active in the market can be used together models and methods in EIO to estimate the demand and the profit maximizing price of the new product, and to predict the response of competing firms.

Example 2: Evaluating the effects of a policy change. A government has introduced a new environmental policy that imposes new restrictions on the emissions of pollutants from factories in an industry. The new policy encourages firms in this industry to adopt a new technology that is environmentally cleaner. This alternative technology reduces variable costs but increases fixed costs. These changes in the cost structure affect competition. In particular, we expect a decline in the number of firms and a higher output-per-firm in the industry. The government wants to know how this new policy has affected competition and welfare in the industry. Using data on prices, quantities, and number of firms in the industry, together with a model of oligopoly competition, we can evaluate the effects of this policy change.

Example 3: Explain the persistence of market power. For many years, the industry of micro-processors for personal computers has been characterized by the duopoly of

Intel and AMD, with a clear leadership by Intel that enjoys more than two-thirds of the world market and a substantial degree of market power. There are multiple factors that may contribute to explain this market power and its persistence over time. For instance, large entry costs, economies of scale, learning-by-doing, consumer brand loyalty, or anti-competitive behavior are potential explanations. What is the relative contribution of each of these factors to explain the observed market structure and market power? Data on prices, quantities, product characteristics, and firms' investment in capacity can help us to understand and to measure the contribution of these factors.

1.2 Data in Empirical IO

Early research in empirical IO between the 1950s and 1970s was based on aggregate industry level data from multiple industries (Bain, 1951 and 1954, Demsetz, 1973). Studies in this literature looked at the empirical relationship between a measure of market power and a measure of market structure that captures the degree of concentration of output in a few firms (*market concentration*). In these studies, the typical measure of market power was the *Lerner Index (LI)* which is defined as price minus marginal cost divided by price, $LI \equiv (P - MC)/P$. And a common measure of market concentration is the *Herfindahl-Hirschman Index (HHI)*, defined as the sum of the squares of the market shares of the firms in the market: $HHI = \sum_{i=1}^{N} (q_i/Q)^2$, where q_i is firm i's output, and Q represents total industry output. Given a sample of N industries (indexed by n) with information on the Lerner and the Herfindahl-Hirschman indexes for each indistry, these studies related the two indexes using a linear regression model as follows,

$$LI_n = \beta_0 + \beta_1 \, HHI_n + \varepsilon_n \tag{1.1}$$

This linear regression model was estimated using industry-level cross-sectional data from very diverse industries, and they typically found a positive and statistically significant relationship between concentration and market power, that is, the OLS estimate of β_1 was statistically greater than zero. One of the main purposes of these empirical studies was to identify a relationship between market concentration and market structure that could be applied to most industries. Furthermore, the estimated regression function was a causal relationship. That is, the parameter β_1 is interpreted as the increase in the Lerner Index of a one-unit increase in market concentration as measured by the HHI. This interpretation does not take into account that both market power (LI) and concentration (HHI) are endogenous variables which are jointly determined in equilibrium and affected by the same exogenous variables, and some of these variables (ε) are unobservable to the researcher.

In the 1980s, the seminal work of Bresnahan (1981, 1982, 1987), Porter, (1983), Schmalensee (1989), Sutton (1991), among others, configured the basis for the so called *New Empirical IO*. These authors pointed out the serious limitations in the previous empirical literature based on aggregate industry-level data. One of the criticisms to the previous literature was that industries, even those apparently similar, can be very different in their exogenous or primitive characteristics such as demand, technology, and regulation. This heterogeneity implies that the relationship between market concentration and price-cost margins can also vary greatly across industries. In reality, the parameters of these linear regression models are heterogeneous across industries (that is,

we have β_{1n} instead of β_1) but they are estimated as constants in this previous literature. A second important criticism to the old EIO literature was that industry concentration, or market structure, cannot be considered as an exogenous explanatory variable. Market power and market structure are both endogenous variables that are jointly determined in an industry. The regression equation of market power on market structure should be interpreted as an equilibrium condition where there are multiple exogenous factors, both observable and unobservable to the researcher, that simultaneously affect these two endogenous variables. Overlooking the correlation between the explanatory variable (market structure) and the error term (unobserved heterogeneity in industry fundamentals) in this regression model implies a spurious estimation of *causal effect* or *ceteris paribus effect* of market structure on market power.

Given these limitations of the old EIO, the proponents of the *New Empirical IO* emphasize the need to study competition by looking at each industry separately using richer data at a more disaggregate level and combining these data with games of oligopoly competition. Since then, the typical empirical application in IO has used data of a single industry, with information at the level of individual firms, products, and markets, on prices, quantities, number of firms, and exogenous characteristics affecting demand and costs.

In the old EIO, sample variability in the data came from looking at multiple industries. This source of sample variation is absent in the typical empirical study in the New EIO. Furthermore, given that most studies now look at oligopoly industries with a few firms, sample variation across firms is also very limited and it is not enough to obtain consistent and precise estimates of parameters of interest. This leads to the question: what are the main sources of sample variability in empirical studies in modern EIO? Most of the sample variation in these studies come from observing multiple products and local markets within the same industry. For instance, in some industries the existence of transportation costs implies that firms compete for consumers at the level of local geographic markets. The particular description of a geographic local market (for instance, a city, a county, a census tract, or a census block) depends on the specific industry under study. Prices and market shares are determined at the local market level. Therefore, having data from many local markets can help to identify the parameters of our models. Sample variation at the product level is also extremely helpful. Most industries in today's economies are characterized by product differentiation. Firms produce and sell many varieties of a product. Having data at the level of very specific individual products and markets is key to identifying and estimating most IO models that we study in this book.

The typical dataset in EIO consists of cross-sectional or panel data of many products and/or local markets from the same industry, with information on selling prices, produced quantities, product attributes, and local market characteristics. Ideally, we would like to have data on firms' costs. However, this information is very rare. Firms are very secretive about their costs and strategies. Therefore, we typically have to infer firms' costs from our information on prices and quantities. There are several approaches we can take. When we have information on firms' inputs, inference on firms' costs can take the form of estimating production functions. When information on firms' inputs is not available, or not rich enough, we exploit our models of competition and profit maximization to infer firms' costs. Similarly, we will have to estimate price-cost margins (market power) and firms' profits using this information.

1.3 Structural models in Empirical IO

To study competition in an industry, EIO researchers propose and estimate structural models of demand and supply where firms behave strategically. These models typically have the following components or submodels: a model of consumer behavior or demand; a specification of firms' costs; a static equilibrium model of firms' competition in prices or quantities; a dynamic equilibrium model of firms' competition in some form of investment such as capacity, advertising, quality, or product characteristics; and a model of firm entry (and exit) in a market. The parameters of the model are structural in the sense that they describe consumer preferences, production technology, and institutional constraints. This class of econometric models provides us with useful tools for understanding competition, business strategies, and the evolution of an industry. They also help us to identify collusive and anti-competitive behavior, or to evaluate the effects of public policies in oligopoly industries, to mention some of their possible applications.

To understand the typical structure of an EIO model, and to illustrate and discuss some important economic and econometric issues in this class of models, the following section presents a simple empirical model of oligopoly competition. Though simple, this model incorporates some important features related to modelling and econometric issues such as specification, endogeneity, identification, estimation, and policy experiments. We will study these issues in detail throughout this book. This example is inspired by Ryan (2012), and the model below can be seen as a simplified version of the model in that paper.

1.3.1 Empirical question

We start with an empirical question. Suppose that we want to study competition in the cement industry of a country or region. It is well-known that this industry is energy intensive and generates a large amount of air pollutants. For these reasons, the government or regulator in this example is evaluating whether to pass a new law that restricts the amount of emissions a cement plant can make. This law would imply the adoption of a type of technology that few plants currently use. The "new" technology implies lower marginal costs but larger fixed costs than the "old" technology. The government would like to evaluate the implications of the new environmental regulation on firms' profits, competition, consumer welfare, and air pollution. As we discuss below, this evaluation can be *ex-ante* (that is, before the new policy is actually implemented) or *ex-post* (that is, after the implementation of the policy change).

1.3.2 Model

The next step is to specify a model that incorporates the **key features of the industry** that are important to answer our empirical question. The researcher needs to have some knowledge about competition in this industry, and about the most important features of demand and technology that characterize the industry. The model that I propose here incorporates four basic but important features of the cement industry. First, it is a homogeneous product industry. There is very little differentiation in the cement product. Nevertheless, the existence of large transportation costs per dollar value of cement makes the spatial location of cement plants a potentially important dimension for competition.

In this simple example, we ignore spatial competition, that we will analyze in chapters 4 and 5. Second, there are substantial fixed costs of operating a cement plant. The cost of buying (or renting) cement kilns, and the maintenance of this equipment, does not depend on the amount of output the plant produces and it represents a substantial fraction of the total cost of a cement plant. Third, variable production costs increase more than proportionally when output approaches the maximum installed capacity of the plant. Fourth, transportation costs of cement (per dollar value of the product) are very high. This explains why the industry is very local. Cement plants are located in proximity to the point of demand (that is, construction cites in cities or small towns) and they do not compete with cement plants located in other towns. For the moment, the simple model that we present here, ignores an important feature of the industry that will become relevant for our empirical question: installed capacity is a dynamic decision that depends on the plant's capacity investments and on depreciation.

1.3.3 Data

The specification of the model depends importantly on the **data** that is available for the researcher. The level of aggregation of the data (for instance, consumer and firm level vs. market level data), its frequency, or the availability or not of panel data are important factors that the researcher should consider when she specifies the model. Model features that are important to explain firm-level data might be quite irrelevant, or they may be under-identified, when using market level data. In this example, we consider a panel (longitudinal) dataset with aggregate information at the level of local markets. Later in this chapter we discuss the advantages of using richer firm-level data. The dataset consists of M local markets (for instance, towns) observed over T consecutive quarters.[1] We index markets by m and quarters by t. For every market-quarter observation, the dataset contains information on the number of plants operating in the market (N_{mt}), aggregate amount of output produced by all the plants (Q_{mt}), market price (P_{mt}), and some exogenous market characteristics ($\mathbf{X}_{mt}$) such as population, average income, etc.

$$Data = \{ P_{mt} \ , \ Q_{mt} \ , \ N_{mt}, \ \mathbf{X}_{mt} : m = 1,2,...,M; \ t = 1,2,...,T \} \tag{1.2}$$

Note that the researcher does not observe output at the plant level. Though the absence of data at the firm level is not ideal it is not uncommon either, especially when using publicly available data from census of manufacturers or businesses. Without information on output at the firm-level, our model has to impose strong restrictions on the form of the heterogeneity in firms' demand and costs. Later in this chapter, we discuss potential biases generated by these restrictions and how we can avoid them when we have firm-level data.

1.3.4 Components of the model

Our model of oligopoly competition has four main components: (a) demand equation; (b) cost function; (c) model of Cournot competition; and (d) model of market entry. An important aspect in the construction of an econometric model is the specification of unobservables. Including unobservable variables in our models is a way to acknowledge

[1] The definition of what is a local market represents an important modelling decision for this type of data and empirical application. We will examine this issue in detail in chapter 5.

the rich amount of heterogeneity in the real world (between firms, markets, or products, and over time), as well as the limited information of the researcher relative to the information available to actual economic agents in our models. Unobservables also account for measurement errors in the data. In general, the richer the specification of unobservables in a model, the more robust the empirical findings. Of course, there is a limit to the degree of unobserved heterogeneity that we can incorporate in our models, and this limit is given by the identification of the model.

1.3.5 Endogeneity and identification

A key econometric issue in the estimation of parameters in our econometric models is the endogeneity of the explanatory variables. For instance, prices and quantities that appear in a demand equation are jointly determined in the equilibrium of the model and they both depend on the exogenous variables affecting demand and costs. Some of these exogenous variables are unobservable to the researcher and are part of the error terms in our econometric models. Therefore, these error terms are correlated with some of the explanatory variables in the econometric model. For instance, the error term in the demand equation is correlated with the explanatory variable price. Ignoring this correlation can imply serious biases in the estimation of the parameters of the model and in the conclusions of the research. Dealing with this endogeneity problem is a fundamental element in EIO and in econometrics in general.

1.3.6 Demand equation

In this simple model we assume cement is a homogeneous product. We also abstract from spatial differentiation of cement plants.[2] We postulate a demand equation that is linear in prices and in parameters.

$$Q_{mt} = S_{mt} \left(\mathbf{X}^D_{mt} \, \beta_X - \beta_P \, P_{mt} + \varepsilon^D_{mt} \right) \tag{1.3}$$

β_X and $\beta_P \geq 0$ are parameters. S_{mt} represents demand size or population size. $\mathbf{X}^D_{mt}$ is a subvector of $\mathbf{X}_{mt}$ that contains observable variables that affect the demand of cement in a market, such as average income, population growth, or age composition of the population. ε^D_{mt} is an unobservable shock in demand per capita. This shock implies vertical parallel shifts in the demand curve.[3] A possible interpretation of this demand equation is that $\mathbf{X}^D_{mt} \, \beta_X - \beta_P \, P_{mt} + \varepsilon^D_{mt}$ is the downward sloping demand curve of a *representative consumer* in market m at period t. According to this interpretation, $\mathbf{X}^D_{mt} \, \beta_X + \varepsilon^D_{mt}$ is the willingness to pay of this representative consumer for the first unit that she buys of the product, and β_P captures the decreasing marginal utility from additional units. An alternative interpretation is based on the assumption that there is a continuum of consumers in the market with measure S_{mt}. [4]

[2] See Miller and Osborne (2013) for an empirical study of spatial differentiation and competition of cement plants.

[3] A more general specification of the linear demand equation includes an unobservable shock that affects the slope of the demand curve.

[4] Each consumer can buy at most one unit of the product. A consumer with willingness to pay v has a demand equal to one unit if $(v - P_{mt}) \geq 0$ and his demand is equal to zero if $(v - P_{mt}) < 0$. Then, the aggregate market demand is $Q_{mt} = S_{mt} \, (1 - G_{mt}(P_{mt}))$ where $G_{mt}(v)$ is the distribution function of consumers' willingness to pay in market m at period t, such that $\Pr(v \geq P_{mt}) = 1 - G_{mt}(P_{mt})$. Suppose

For some of the derivations below, it is convenient to represent the demand using the *inverse demand curve*:

$$P_{mt} = A_{mt} - B_{mt}\, Q_{mt} \tag{1.4}$$

where the intercept A_{mt} is $\left(\mathbf{X}^D_{mt}\ \beta_X + \varepsilon^D_{mt}\right)/\beta_P$, and the slope B_{mt} is $1/(\beta_P\ S_{mt})$. Using the standard representation of the demand curve in the plane, with Q in the horizontal axis and P in the vertical axis, we have that this curve moves upward when A_{mt} increases (vertical parallel shift) or when B_{mt} declines (counter-clockwise rotation).[5]

1.3.7 Cost function

The **cost function** of a firm has two components, variable cost and fixed cost: $C(q) = VC(q) + FC$, where q is the amount of output produced by a single firm, $C(q)$ is the total cost of a firm active in the market, and $VC(q)$ and FC represent variable cost and fixed cost, respectively.

If we had firm-level data on output, inputs, and input prices, we could estimate a production function and then use the dual approach to construct the variable cost and fixed cost function. For instance, suppose that the production function has the Cobb-Douglas form $q = L^{\alpha_L}\ K^{\alpha_K}\ \exp\{\varepsilon\}$ where L and K are the amounts of labor and capital inputs, respectively, α_L and α_K are parameters, and ε represents total factor productivity which is unobservable to the researcher. We can take a logarithm transformation of this production function to have the linear in parameters regression model, $\ln q = \alpha_L \ln L + \alpha_K \ln K + \varepsilon$. In chapter 3, we present methods for the estimation of the parameters in this production function. Suppose that labor is a variable input and capital is a fixed input. The variable cost function $VC(q)$ is the minimum variable cost (in this case, labor cost) to produce an amount of output q. For this production function, we have that:[6]

$$VC(q) \quad = \quad p_L \left[\frac{q}{\exp\{\varepsilon\} K^{\alpha_K}}\right]^{1/\alpha_L} \tag{1.5}$$

and

$$FC \quad = \quad p_K\, K \tag{1.6}$$

where p_L and p_K represent the price of labor and capital, respectively.

Here we consider a common situation where the researcher does not have data on inputs at the firm level. Costs cannot be identified/estimated from a production function. We will estimate costs using *revealed preference* .

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that the distribution function G_{mt} is uniform with support $[(A_{mt}-1)/\beta_P\ ,\ A_{mt}/\beta_P]$ and $A_{mt} \equiv \mathbf{X}^D_{mt}\ \beta_X + \varepsilon^D_{mt}$. Then, the aggregate market demand has the form in equation (1.3).

[5] In principle, market size S^*_{mt} could enter the vector $\mathbf{X}^D_{mt}$ to take into account that the distribution of consumers willingness to pay may change with the size of the population in the market. In that case, an increase in market size implies both a vertical shift and a rotation in the demand curve.

[6] Since capital is fixed, the production function implies a one-to-one relationship between output and labor. That is, to produce q units of output (given fixed K), the firm needs $L = \left[\frac{q}{\exp\{\varepsilon\} K^{\alpha_K}}\right]^{1/\alpha_L}$ units of labor. Therefore, if p_L is the price of labor, we have that $VC(q) = p_L\, L = p_L \left[\frac{q}{\exp\{\varepsilon\} K^{\alpha_K}}\right]^{1/\alpha_L}$.

1.3.8 Revealed Preference

Under the assumption that agents make decisions to maximize a utility or payoff, observed agents' choices reveal information to us about their payoff functions. In this case, a firm's choice of output reveals information about its marginal costs, and its decision to be active in the market or not reveals information about its fixed costs.

We start by assuming that every firm, either an incumbent or a potential entrant, has the same cost function. For convenience, we specify a quadratic variable cost function:

$$VC_{mt}(q) = \left(\mathbf{X}_{mt}^{MC}\,\gamma_x^{MC} + \varepsilon_{mt}^{MC}\right)\,q + \frac{\gamma_q^{MC}}{2}q^2 \tag{1.7}$$

γ_x^{MC} and γ_q^{MC} are parameters. $\mathbf{X}_{mt}^{MC}$ is a subvector of $\mathbf{X}_{mt}$ that contains observable variables that affect the marginal cost of cement production, including the prices of variable inputs such as limestone, energy, or labor. ε_{mt}^{MC} is a market shock in marginal cost that is unobserved to the researcher but observable to firms.

Given this variable cost function, the marginal cost is $MC_{mt}(q) = \overline{MC}_{mt} + \gamma_q^{MC} q$, where $\overline{MC}_{mt} \equiv \mathbf{X}_{mt}^{MC}\,\gamma_x^{MC} + \varepsilon_{mt}^{MC}$ represents the exogenous part of the marginal cost – that is, the part of the marginal cost that does not depend on the amount of output. Since $q \geq 0$, we have that $\overline{MC}_{mt}$ is the minimum possible value for the marginal cost. The component $\gamma_q^{MC} q$ captures how the marginal cost increases with output.

The fixed cost is associated with inputs that are used in a fixed amount, regardless the level of output. These inputs can be land, the physical plant, or some equipment. This fixed cost is specified as $FC_{mt} = \mathbf{X}_{mt}^{FC}\,\gamma^{FC} + \varepsilon_{mt}^{FC}$, where γ^{FC} is a vector of parameters. $\mathbf{X}_{mt}^{FC}$ is a vector of observable variables that affect fixed costs such as the rental price of fixed capital equipment. ε_{mt}^{FC} is an unobservable market specific shock. By including the market-specific shocks ε_{mt}^{MC} and ε_{mt}^{FC} we allow for market heterogeneity in costs that is unobservable to the researcher.

1.3.9 Cournot competition

Suppose that there are N_{mt} plants active in local market m at quarter t. For the moment, we treat the number of active firms as given, though this variable is endogenous in the model and we explain later how it is determined in the equilibrium of the model. We assume that firms active in a local market compete with each other à la Cournot. The assumption of Cournot competition is far from being innocuous for the predictions of the model, and we reexamine this assumption at the end of this chapter.

The profit function of firm i is:

$$\Pi_{mt}(q_i, \widetilde{Q}_i) = P_{mt}(q_i + \widetilde{Q}_i)\,q_i - VC_{mt}(q_i) - FC_{mt} \tag{1.8}$$

where q_i is firm i's own output, and $\widetilde{Q}_i$ represents the firm i's beliefs about the total amount of output of the other firms in the market. Under the assumption of *Nash-Cournot* competition, each firm i takes as given the quantity produced by the rest of the firms, $\widetilde{Q}_i$, and chooses her own output q_i to maximize her profit.

The profit function $\Pi_{mt}(q_i, \widetilde{Q}_i)$ is globally concave in q_i for any positive value of $\widetilde{Q}_i$. Therefore, there is a unique value of q_i that maximizes the firm's profit. That is, a firm's

best response is a function. This best response output is characterized by the following condition of optimality which establishes that marginal revenue equals marginal cost:

$$P_{mt} + P'_{mt}(q_i + \widetilde{Q}_i)\, q_i = MC_{mt}(q_i) \tag{1.9}$$

where $P'_{mt}(Q)$ is the derivative of the inverse demand function.

Given the linear demand function $P_{mt} = A_{mt} - B_{mt} Q$, the derivative $P'_{mt}(Q) = -B_{mt}$, and that the equilibrium is symmetric ($q_i = q$ for every firm i) such that $Q_{mt} = q + \widetilde{Q} = N_{mt}\, q$, we can get the following expression for output-per-firm in the Cournot equilibrium with N active firms:

$$q_{mt}(N) = \frac{A_{mt} - \overline{MC}_{mt}}{B_{mt}\,(N+1) + \gamma_q^{MC}} \tag{1.10}$$

This equation shows that, keeping the number of active firms fixed, output per firm increases with the intercept in the demand curve (A_{mt}), declines with marginal cost and the slope of the demand curve (B_{mt}), and it does not depend on fixed cost. The later is a general result that does not depend on the specific functional form that we have chosen for demand and variable costs: by definition, fixed costs do not have any influence on marginal revenue or marginal costs when the number of firms in the market is fixed. However, as we show below, fixed costs do have an indirect effect on output per firm through its effect on the number of active firms: the larger the fixed cost, the lower the number of firms, and the larger the output per firm.

Price over average variable cost is $P_{mt} - AVC_{mt} = [A_{mt} - B_{mt}\, N_{mt}\, q_{mt}(N)] - [\overline{MC}_{mt} + \gamma_q^{MC}/2\, q_{mt}(N)] = [A_{mt} - \overline{MC}_{mt}] - [B_{mt}\, N_{mt} + \gamma_q^{MC}/2]\, q_{mt}(N)$. Plugging expression (1.10) into this equation, we get the following relationship between price-cost margin and output-per-firm in the Cournot equilibrium:

$$P_{mt} - AVC_{mt} = \frac{\left(B_{mt} + \gamma_q^{MC}/2\right)\ \left(A_{mt} - \overline{MC}_{mt}\right)}{B_{mt}\ (N_{mt}+1) + \gamma_2^{MC}} = \left(B_{mt} + \gamma_q^{MC}/2\right)\ q_{mt}(N) \tag{1.11}$$

As the number of plants goes to infinity, the equilibrium price-cost margin converges to zero, and price becomes equal to the minimum marginal cost, $\overline{MC}_{mt}$, that is achieved by having infinite plants each with an atomist size. Plugging this expression into the profit function we get that in a Cournot equilibrium with N firms, the profit of an active firm is:

$$\begin{aligned} \Pi^*_{mt}(N) &= (P_{mt} - AVC_{mt})\ q_{mt}(N) - FC_{mt} \\ &= \left(B_{mt} + \gamma_q^{MC}/2\right) \left(\frac{A_{mt} - \overline{MC}_{mt}}{B_{mt}\,(N+1) + \gamma_q^{MC}} \right)^2 - FC_{mt} \end{aligned} \tag{1.12}$$

This Cournot equilibrium profit function is continuous and strictly decreasing in the number of active firms, N. These properties of the equilibrium profit function are important for the determination of the equilibrium number of active firms that we present in the next section.

1.3.10 Market entry

Now, we specify a model for how the number of active firms in a local market is determined in equilibrium. Remember that the profit of a firm that is not active in the

industry is zero.[7] The equilibrium entry condition establishes that every active firm and every potential entrant is maximizing profits. Therefore, active firms should be making non-negative profits, and potential entrants are not leaving positive profits on the table. Active firms should be better off in the market than in the outside alternative. That is, the profit of every active firms should be non-negative: $\Pi^*_{mt}(N_{mt}) \geq 0$. Potential entrants should be better off in the outside alternative than in the market. That is, if a potential entrant decides to enter in the market, it gets negative profits. Additional entry implies negative profits: $\Pi^*_{mt}(N_{mt}+1) < 0$.

Figure 1.1 presents the Cournot equilibrium profit of a firm as a function of the number of firms in the market, N, for an example where the demand function is $P = \$100 - 0.1Q$, the variable cost function is $VC(q) = \$20q + q^2/2$, and the fixed cost is $\$1,400$. As shown in equation (1.12), the equilibrium profit function is continuous and strictly decreasing in N. These properties imply that there is a unique value of N that satisfies the equilibrium conditions $\Pi^*_{mt}(N) \geq 0$ and $\Pi^*_{mt}(N+1) < 0$.[8] In the example of Figure 1.1, the equilibrium number is 5 firms. In this particular model, solving for the equilibrium number of firms is straightforward. Let N^*_{mt} be the real number that (uniquely) solves the condition $\Pi^*_{mt}(N) = 0$. Given the form of the equilibrium profit function $\Pi^*_{mt}(N)$, we have that:

$$N^*_{mt} \equiv -\left(1 + \frac{\gamma^{MC}_q}{B_{mt}}\right) + \left(A_{mt} - \overline{MC}_{mt}\right)\sqrt{\frac{1 + \gamma^{MC}_q/2B_{mt}}{FC_{mt}\ B_{mt}}} \tag{1.13}$$

The equilibrium number of firms is the largest integer that is smaller or equal to N^*_{mt}. We represent this relationship as $N_{mt} = int(N^*_{mt})$ where int is the integer function, that is, the largest integer that is smaller or equal than the argument. This expression shows that the number of active firms increases with demand and declines with marginal and fixed costs.

We can combine the equilibrium output per firm in equation 1.10 and the profit function in equation 1.12), to obtain the following expression for the Cournot equilibrium profit: $\Pi^*_{mt}(N) = (B_{mt} + \gamma^{MC}_q/2)\ q_{mt}(N)^2 - FC_{mt}$. This provides the following expression for the entry equilibrium condition – $\Pi^*_{mt}(N^*_{mt}) = 0$, that is particularly useful for the estimation of the model:[9]

$$\left(\frac{Q_{mt}}{N_{mt}}\right)^2 = \frac{FC_{mt}}{B_{mt} + \gamma^{MC}_q/2} \tag{1.14}$$

This equation shows how taking into account the endogenous determination of the number of firms in a market has important implications on firm size (output per firm). Firm size increases with fixed costs and declines with the slope of the demand curve, and

[7] In this model, the normalization to zero of the value of the outside option is innocuous. This normalization means that the 'fixed cost' FC_{mt} is actually the sum of the fixed cost in this market and the firm's profit in the best outside alternative.

[8] Suppose that there are two different integer values N_A and N_B that satisfy the entry equilibrium conditions $\Pi^*_{mt}(N) \geq 0$ and $\Pi^*_{mt}(N+1) < 0$. Without loss of generality, suppose that $N_B > N_A$. Since $N_B \geq N_A + 1$, strict monotonicity of Π^* implies that $\Pi^*(N_B) \leq \Pi^*(N_A+1) < 0$. But $\Pi^*(N_B) < 0$ contradicts the equilibrium condition for N_B.

[9] To derive this equation, we consider that the ratio $N^*_{mt}/int(N^*_{mt})$ is approximately equal to one.

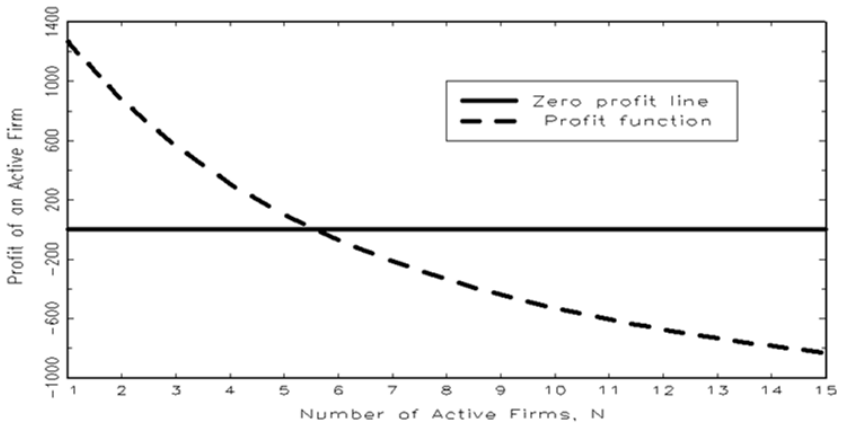


Figure 1.1: Cournot equilibrium profit as function of number of firms

with the degree of increasing marginal costs. Industries with large fixed costs, inelastic demand curves, and rapidly increasing marginal costs, have larger firms and a smaller number of them. In the extreme case, we can have a *natural monopoly*. The opposite case, in terms of market structure, is an industry with small fixed costs, very elastic demand, and constant marginal costs. An industry with these exogenous demand and cost characteristics will have an atomist market structure with a large number of very small firms. It is clear that exogenous demand and cost are key in determining the industry market structure and market power.

1.3.11 Structural equations

For simplicity, in some of the discussions in this chapter, we treat the number of firms N_{mt} as a continuous variable: $N_{mt} \equiv int(N^*_{mt}) = N^*_{mt}$. Then, we can replace the two inequalities $\Pi^*_{mt}(N_{mt}) \geq 0$ and $\Pi^*_{mt}(N_{mt}+1) < 0$ by the equality condition $\Pi^*_{mt}(N_{mt}) = 0$. This approximation is not necessarily innocuous, and we do not use it later in the book. For the moment, we keep it, because it provides simple expressions for the equilibrium values which are linear in parameters, and this simplifies our analysis of model identification and estimation. In this subsection, we omit the market and time subindexes.

The model can be described as a system of three equations: the demand equation; the Cournot equilibrium condition; and the entry equilibrium condition. The system has three endogenous variables: the number of firms in the market, N; the market price, P;

and output per-firm, $q \equiv Q/N$,

$$\begin{aligned} \textbf{Demand equation:} \quad P &= A - B\,N\,q \\ \textbf{Cournot Equilibrium Condition:} \quad q &= \frac{A - \overline{MC}}{B\,(N+1) + \gamma_q^{MC}} \\ \textbf{Entry Equilibrium Condition:} \quad q^2 &= \frac{FC}{B + \gamma_q^{MC}/2} \end{aligned} \tag{1.15}$$

This is a system of simultaneous equations. The system of equations in (1.15) is denoted as the *structural equations* of the model. Given a value of the exogenous variables, $\mathbf{X}$ and $\varepsilon \equiv (\varepsilon^D, \varepsilon^{MC}, \varepsilon^{FC})$, and of the structural parameters, $\theta \equiv \{\beta_x, \beta_p, \gamma_x^{MC}, \gamma_q^{MC}, \gamma^{FC}\}$, an *equilibrium* of the model is a vector of endogenous variables $\{N, P, q\}$ that solves this system of equations.

In this model, we can show that an equilibrium always exists and it is unique. To show this, notice that the entry equilibrium condition determines output per firm as a function of the exogenous variables:

$$q = \sqrt{\frac{FC}{B + \gamma_q^{MC}/2}} \tag{1.16}$$

This expression provides the equilibrium value for output per-firm. Plugging this expression for q into the Cournot equilibrium condition and solving for N, we can obtain the equilibrium value for the number of firms as:

$$N = -\left(1 + \frac{\gamma_q^{MC}}{B}\right) + \left(\frac{A - \overline{MC}}{B}\right)\sqrt{\frac{B + \gamma_q^{MC}/2}{FC}} \tag{1.17}$$

Finally, plugging the equilibrium expressions for N and q into the demand equation, we can obtain the equilibrium price as:

$$P = \overline{MC} + (\gamma_q^{MC} + B)\sqrt{\frac{FC}{B + \gamma_q^{MC}/2}} \tag{1.18}$$

Equations (1.16), (1.17), and (1.18) present the equilibrium values of the endogenous variables as functions of exogenous variables and parameters only. These three equations are called the *reduced form equations* of the model. In this model, because the equilibrium is always unique, the reduced form equations are functions. More generally, in models with multiple equilibria, reduced form equations are correspondences such that for a given value of the exogenous variables there are multiple values of the vector of endogenous variables, each value representing a different equlibria.

1.4 Identification and estimation

Suppose that the researcher has access to a panel dataset that follows M local markets over T quarters. For every market-quarter the dataset includes information on market

price, aggregate output, number of firms, and some exogenous market characteristics such as population, average household income, and input prices: $\{P_{mt}\ ,\ Q_{mt}\ ,\ N_{mt}, \mathbf{X}_{mt}\}$. The researcher wants to use these data and the model described above to learn about different aspects of competition in this industry and to evaluate the effects of the policy change described above. Before we study the identification and estimation of the structural parameters of the model, it is interesting to examine some empirical predictions of the model that can be derived from the reduced form equations.

1.4.1 Reduced form equations

From an empirical point of view, the reduced form equations establish relationships between exogenous market characteristics, such as market size, and the observable endogenous variables of the model: price, number of firms, and firm size. Can we learn about competition in this industry, and about some of the structural parameters, by estimating the reduced form equations? As we show below, there is very important evidence that can be obtained from the estimation of these equations. However, providing answers to some other questions requires the estimation of the structural model. For instance, the estimation of the structural model is helpful to answer our policy question.

Relationship between market size and firm size

The reduced form equation for output-per-firm in (1.16), implies the following relationship between firm size (or output per firm) q and market size S, given that $B = 1/\beta_P S$:

$$\ln(q) = \frac{1}{2}\left[\ln(\beta_p FC) + \ln(S) - \ln\left(1 + \frac{\beta_p\ \gamma_q^{MC}\ S}{2}\right)\right] \tag{1.19}$$

We can distinguish three different cases for this relationship. When fixed cost is zero $(FC = 0)$ there is no relationship between firm size and market size. The model becomes one of perfect competition and the equilibrium is characterized by a very large number of firms $(N = \infty)$ each with an atomistic size $(q = 0)$. When the fixed cost is strictly positive $(FC > 0)$ there is a positive relationship between market size and firm size. Markets with larger demand have larger firms. We can distinguish between two different cases when the fixed cost is strictly positive. When the marginal cost is constant $(\gamma_q^{MC} = 0)$, the relationship between firm size and market size is $\ln(q) = \frac{1}{2}[\ln(\beta_p FC) + \ln(S)]$ such that firm size always increases proportionally with market size. When the marginal cost is increasing $(\gamma_q^{MC} > 0)$, the limit of firm size when market size goes to infinity is equal to $\sqrt{2FC/\gamma_q^{MC}}$, and this constant represents the maximum size of a firm in the industry. The value $\sqrt{2FC/\gamma_q^{MC}}$ is the level of output-per-firm that minimizes the Average Total Cost, and it is denoted the *Minimum Efficient Scale* (MES). Figure 1.2 illustrates these two cases for the relationship between firm size and market size. The values of the parameters that generate these curves are $FC = 1{,}400$, $\beta_p = 1$, $\gamma_q^{MC} = 0$ and $\gamma_q^{MC} = 1$.

Equation (1.19) and figure 1.2 show that the shape of the relationship between market size and firm size reveals information on the relative magnitude of the fixed cost and the convexity of the variable cost. Given a cross-section of local markets in an homogeneous product industry, the representation of the scatter-plot of sample points of (S_{mt}, q_{mt}) in the plane, and the estimation of a nonlinear (or nonparametric) regression of q_{mt} on S_{mt}

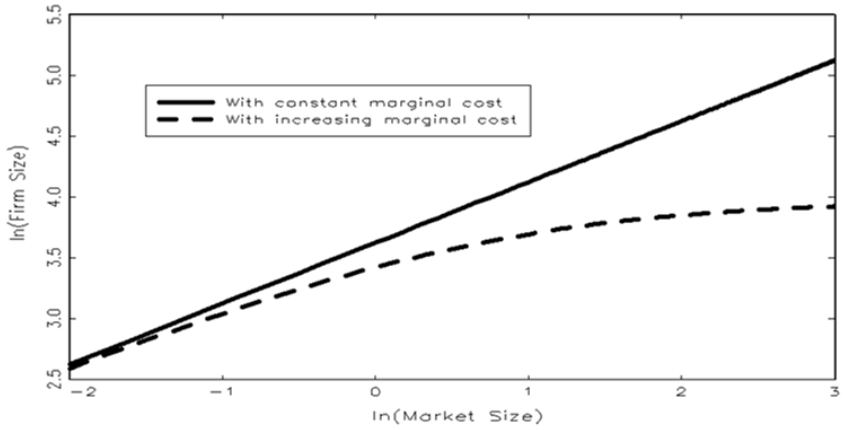


Figure 1.2: Relationship between firm size and market size

provides empirical evidence on this aspect of cost structure. Campbell and Hopenhayn (2005) look at this empirical relationship in thirteen retail industries using a sample of 225 US cities. Figure 1.3 presents the scatter-plot and the estimated regression line for the logarithm of firm size on the logarithm of market size in the *Women's Clothing* retail industry. In this example, the relationship in logarithms is linear, which is consistent with $FC > 0$ and $\gamma_q^{MC} = 0$ for this industry. In logarithms, for small γ_q^{MC}, we have that $\ln(q_{mt}) = \alpha_0 + \alpha_1 \ln(S_{mt}) + \alpha_q S_{mt} + u_{mt}$, where $\alpha_1 \equiv 1/2$, and $\alpha_2 \equiv -\beta_p \gamma_q^{MC}/2$. Therefore, testing the null hypothesis $\alpha_2 = 0$ is equivalent to testing for non-convexity in the variable cost, that is, $\gamma_q^{MC} = 0$. Note that market size is measured with error and this creates an endogeneity problem in the estimation of this relationship. Campbell and Hopenhayn take into account this issue and try to correct for endogeneity bias using Instrumental Variables.

This testable prediction on the relationship between market size and firm size is not shared by other models of firm competition such as models of monopolistic competition or models of perfect competition, where market structure, market power, and firm size do not depend on market size. In all the industries studied by Campbell and Hopenhayn, this type of evidence is at odds with models of monopolistic and perfect competition.

Relationship between market size and price

Are prices higher in small or in larger markets? This is an interesting empirical question per se. The model shows that the relationship between price and market size can reveal some interesting information about competition in an industry. We can distinguish three cases depending on the values of FC and γ_q^{MC}. If the industry is such that the fixed cost is zero or negligible, then the model predicts that there should not be any relationship between market size and price. In fact, price should be always equal to the minimum

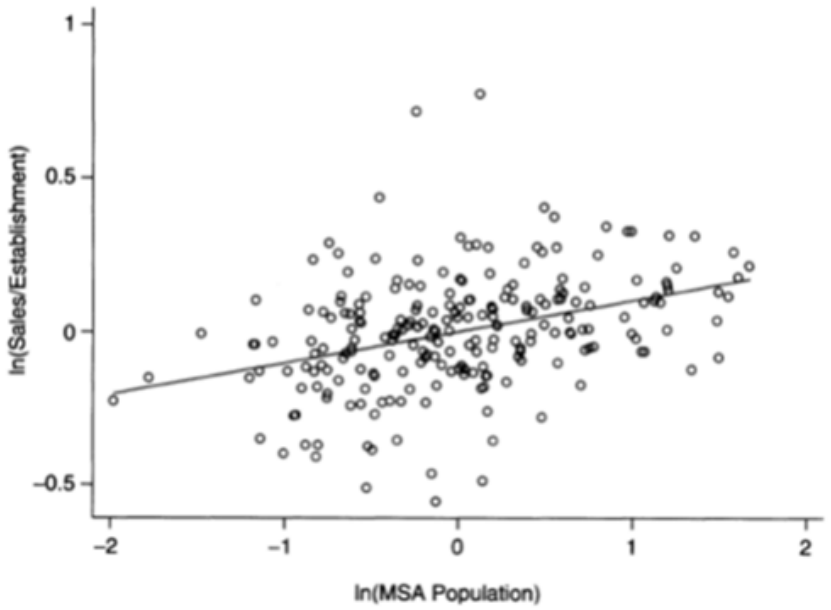


Figure 1.3: Market size matters (Campbell and Hopenhayn, 2005)

marginal cost, $\overline{MC}_{mt}$. When the fixed cost is strictly positive and the variable cost is linear in output, the reduced form equation for price becomes $P = \overline{MC} + \sqrt{FC/\beta_p S^*}$. In this case, an increase in market size always has a negative effect on price, though the marginal effect is decreasing. When market size goes to infinity, price converges to the minimum marginal cost $\overline{MC}$. This is also the relationship that we have between market size and price when the variable cost function is strictly convex, with the only difference that now as market size goes to infinity the price converges to $\overline{MC} + \sqrt{2\gamma_q^{MC} FC}$, which is the marginal cost when output-per-firm is at the *Minimum Efficient Scale* (MES).

As in the case of firm size, we can use cross-sectional data on prices and market size to test for the relationship between these variables. Finding a significant negative effect of market size on price implies the rejection of monopolistic and perfect competition models in favor of oligopoly competition.

Policy Question and Reduced Form Equations

Recall our initial objective: evaluating the effects of a policy which generates an increase in the fixed cost and a reduction in the marginal cost on firms in the cement industry. What do the reduced form equations say about the effects of this policy? Could the estimation of the reduced form equation provide enough information to answer our policy questions?

Equation (1.19) shows that an increase in the fixed cost FC and a reduction in the marginal cost parameter γ_q^{MC} imply a larger firm size. Therefore, the model predicts that the new policy will transform the industry into one with larger firms. However, without further information about the values of the parameters of the model, the reduced form equations do not provide a prediction about the effects on the number of firms, aggregate output, price, and consumer welfare. Not only the magnitude but even the sign of these

effects depend on the values of the structural parameters. A larger fixed cost reduces the number of firms and aggregate output, increases price, and has a negative effect on consumer welfare. A reduction in the marginal cost affects all the endogenous variables in the opposite direction. The net effects are ambiguous, and depend on the values of the demand and cost parameters and on the magnitude of the change in fixed cost and marginal cost.

Interestingly, the sign of the effect of the policy on number of firms, output, prices, and consumer welfare depends on market size. The effect of a reduction in marginal cost is quantitatively more important in large markets than in small ones. Therefore, in large markets this positive effect dominates the negative effect of the increase in the fixed costs. We may have that in large markets the policy increases the number of firms, reduces prices, and increases consumer welfare, and the effects on small markets are just the opposite. The welfare effects of this policy are not neutral with respect to market size.

It is relevant to distinguish between two cases or scenarios in terms of the information for the researcher about the policy change. In the first case, which we denote as a *factual policy change*, the sample includes observations both before and after the policy change. The second case represents a *counterfactual policy change*, and the data contains only observations without the new policy. The distinction is relevant because the identification assumptions are different in each. In the case of a factual policy change, and under some conditions, we may need only the identification of the parameters in the reduced form equations. Identification of reduced form parameters requires weaker assumptions than identification of structural parameters.

Many empirical questions in IO deal with predicting the effects of changes that have not yet occurred. For instance, when an industry regulator makes a recommendation on whether to approve a merger between two companies or not, she has to predict the effects of a merger that has not yet taken place. Similarly, a company that decides whether or not to introduce a new product in a market, or that designs the features of that new product, needs to predict the demand for that hypothetical product before it has been introduced in the market. In our example here, we first consider the case where the regulator has not yet implemented the new environmental regulation and wants to predict the effects of this regulation. To evaluate the effects of our policy change in a counterfactual setting, we make use of our structural model and a two step approach. First, we use our data to estimate the structural parameters of the model. And second, we use the estimated model to predict the responses to changes in some parameters or/and exogenous variables implied by the counterfactual policy change, under the assumption that the rest of the parameters remain constant. We now turn to the problem of identification of the structural parameters.[10]

1.4.2 Estimation of structural model

The researcher wants to use the available data to estimate the vector of structural parameters $\theta = \{\beta_x, \beta_p, \gamma_x^{MC}, \gamma_q^{MC}, \gamma^{FC}\}$. Given an estimate of the true θ, we can use

[10] Sometimes, for some counterfactual policy questions we need to know only some of the structural parameters. This idea goes back at least to the origins of the Cowles Foundation in the 1950s, and more specifically to the work of Marschak (1953), and it has been exploited recently in different studies. See also Chetty (2009) and Aguirregabiria (2010).

our model to evaluate/predict the effects of a hypothetical change in the cost parameters γ_x^{MC}, γ_q^{MC}, and γ^{FC} implied by the policy. For simplicity, we start by considering a version of the model without measurement error in the observable measure of market size, that is, $\exp\{\varepsilon_{mt}^S\} = 1$ for every market and period (m,t).

The econometric model can be represented using the following system of simultaneous equations:

$$\begin{aligned} \frac{Q_{mt}}{S_{mt}} &= \beta_x \, \mathbf{X}_{mt}^D - \beta_p \, P_{mt} + \varepsilon_{mt}^D \\ \left(P_{mt} - \frac{1}{\beta_p} \frac{q_{mt}}{S_{mt}} \right) &= \gamma_x^{MC} \, \mathbf{X}_{mt}^{MC} + \gamma_q^{MC} \, q_{mt} + \varepsilon_{mt}^{MC} \\ q_{mt}^2 \left(\frac{1}{\beta_p S_{mt}} + \frac{\gamma_q^{MC}}{2} \right) &= \gamma^{FC} \, \mathbf{X}_{mt}^{FC} + \varepsilon_{mt}^{FC} \end{aligned} \tag{1.20}$$

We complete the econometric model with an assumption about the distribution of the unobservables. It is standard to assume that the unobservables ε_{mt} are mean independent of the observable exogenous variables.

Assumption: The vector of unobservable variables in the structural model, ε_{mt}, is mean independent of S_{mt}: $\mathbb{E}(\varepsilon_{mt}|S_{mt}) = 0$.

We say the parameters of the model are identified if there is a feasible estimator of θ that is *consistent* in a statistical or econometric sense.[11]

To prove the vector of parameters is identified, a standard approach is using the moment restrictions implied by the model to show that we can **uniquely** determine the value of θ as a function of moments of the observable variables. For instance, in a classical linear regression model $Y = \beta_0 + \beta_1 \, X + \varepsilon$ under the assumption of no correlation between the error term and the regressor, we have that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(X \, \varepsilon)$ and these conditions imply that $\beta_1 = cov(X,Y)/var(X)$ and $\beta_0 = \mathbb{E}(Y) - [cov(X,Y)/var(X)] \, \mathbb{E}(X)$. These expressions show that the parameters β_0 and β_1 are identified using data of Y and X. In our model, Assumption 1, provides moment restrictions, but we show below that these restrictions are not sufficient to identify the parameters of the model.

Endogeneity

The key identification problem in our model is that the regressors in the three equations are endogenous variables that are correlated with the unobservables or error terms. In the presence of endogeneous regressors, OLS estimation produces biased and inconsistent parameter estimates. In the second equation, the left-hand-side is the price minus the

[11] Given our sample with large M and small T, and an estimator $\widehat{\theta}_M$ we say that $\widehat{\theta}_M$ is a consistent estimator of the true value θ if $\widehat{\theta}_M$ converges in probability to θ as the sample size M goes to infinity: $p\lim_{M\to\infty} \widehat{\theta}_M = \theta$, or using the definition of the limit in probability operator: for any scalar $\delta > 0$,

$$\lim_{M\to\infty} \Pr\left(\left| \widehat{\theta}_M - \theta \right| > \delta \right) = 0$$

A sufficient condition for the consistency of the estimator $\widehat{\theta}_M$ is that the bias and variance of the estimator ($\mathbb{E}(\widehat{\theta}_M - \theta)$ and $Var(\widehat{\theta}_M)$) converge to zero as M goes to infinity.

price-cost-margin and this should be equal to the marginal cost on the right-hand-side. In the third equation, the left-hand-side is total profit minus variable profit, and this should be equal to the fixed cost on the right-hand-side.

Given this representation of the system of equations, it is clear that we can follow a sequential approach to identify and estimate the model. First, we consider the identification of demand parameters. Given identification of the demand slope parameter β_1, the variable on the right-hand-side of the Cournot equilibrium equation is known, and we consider the identification of parameters in the variable cost. Finally, given β_1 and γ_2^{MC}, the variable on the right-hand-side of the entry-equilibrium equation is known and therefore the identification of the fixed cost parameter follows trivially from the moment condition $\mathbb{E}(\varepsilon_{mt}^{FC}) = 0$. Following this sequential approach, it should be clear that there are two endogeneity or identification problems: (1) in the demand equation, price is an endogenous regressor, that is, $\mathbb{E}(P_{mt}\ \varepsilon_{mt}^{D}) \neq 0$; and (2) in the Cournot equilibrium equation, output per firm is an endogenous regressor, that is, $\mathbb{E}(q_{mt}\ \varepsilon_{mt}^{MC}) \neq 0$.

How can we deal with this endogeneity problem? There is not such a thing as "the" method or approach to deal with endogeneity problems. There are different approaches, each with their relative advantages and limitations. These approaches are based on different assumptions that may be more or less plausible depending on the application. The advantages and plausibility of an approach should be judged in the context of an specific application.

We now use our simple model to illustrate some of the identification assumptions and strategies that have been used in many applications in empirical IO and that we will see throughout this book: (a) randomized experiments; (b) exclusion restrictions; (c) "natural experiments" as exclusion restrictions; and (d) restrictions on the covariance structure of the unobservables.

Randomized experiments

The implementation of an adequate randomized experiment is an ideal situation for the identification of an econometric model. The careful design of a useful randomized experiment is not a trivial problem. We illustrate some of the issues in the context of our model. We also want to emphasize here that the structural model is a useful tool in the design of the randomized experiment.

Suppose that we want to estimate first the demand equation. We need to design an experiment that generates sample variation in price that is not perfectly correlated with market size and is independent of the unobserved demand shock ε_{mt}^{D}. The experiment consists of a firm subsidy per unit of output produced and sold in the market. In market-quarter (m,t) this subsidy is of τ_{mt} dollars per unit of output, and τ_{mt} is randomly distributed over (m,t) and independently distributed of any market characteristic. For instance, it is determined as a random draw from some distribution. We also need to assume that the implementation of the experiment does not introduce any change in the behavior of consumers. Under this condition, we have that the following conditions hold: the subsidy is not correlated with the demand shock and with market size $\mathbb{E}(\tau_{mt}\ S_{mt}) = 0$, but it is correlated with price. That is,

$$\mathbb{E}(\tau_{mt}\varepsilon_{mt}^{D}) = 0,\ \mathbb{E}(\tau_{mt}S_{mt}) = 0,\ \text{but}\ \mathbb{E}(\tau_{mt}P_{mt}) \neq 0 \tag{1.21}$$

These conditions imply that we can use the amount of subsidy, τ_{mt}, as an instrument for P_{mt} in the demand equation, to identify all the parameters in the demand equation. More

precisely, the moment conditions

$$\mathbb{E}(\varepsilon_{mt}^{D}) = 0, \;\; \mathbb{E}(S_{mt}\varepsilon_{mt}^{D}) = 0, \;\; \text{and } \mathbb{E}(\tau_{mt}\varepsilon_{mt}^{D}) = 0 \tag{1.22}$$

identify the parameters β_0, β_S, and β_1 in the demand equation. Given the estimated demand parameters, we can use also the moment conditions

$$\mathbb{E}(\varepsilon_{mt}^{MC}) = 0, \;\; \mathbb{E}(S_{mt}\varepsilon_{mt}^{MC}) = 0, \;\; \text{and } \;\; \mathbb{E}(\tau_{mt}\varepsilon_{mt}^{MC}) = 0 \tag{1.23}$$

to identify variable cost parameters in the Cournot equation, and the moment conditions

$$\mathbb{E}(\varepsilon_{mt}^{FC}) = 0, \;\; \mathbb{E}(S_{mt}\varepsilon_{mt}^{FC}) = 0, \;\; \text{and } \mathbb{E}(\tau_{mt}\varepsilon_{mt}^{FC}) = 0 \tag{1.24}$$

to identify the fixed cost parameter in the entry equation.

A well known concern in any experiment, either in the lab or in the field, is that agents' behavior may change if they know that they are the subjects of an experiment. In the experiment that we have here, that is a potential concern for the behavior of firms. Firms involved in the experiment may change the way they compete during the time the experiment is implemented. For instance, they may decide to agree not to change their levels of output such that the subsidy will not pass through to the price and they will keep the subsidy as a pure transfer. However, as long as the subsidy has some effect on price (that is, there is at least a partial pass-through of the subsidy to price), this concern does not affect the identification of the demand parameters. A key aspect in this experimental design is to ensure consumers are not aware of this experiment such that they do not change their demand behaviour. In contrast, if some consumers were aware of the temporary nature of this experiment, they may decide to buy excess cement for inventory. If that is the case, the experiment will affect demand, and the estimates of the demand parameters based on this randomized experiment will be biased.

Exclusion restrictions – Instrumental Variables

The method of instrumental variables is the most common approach to deal with endogeneity in econometrics, and in empirical micro fields in particular. An instrumental variable is an observable variable that satisfies three restrictions in the equation we want to estimate: (i) it does not appear explicitly in the equation; (ii) it is correlated with the endogenous regressor(s); and (iii) it is not correlated with the error term (unobservables) of the equation. In the context of our model, for the estimation of demand parameters we need a variable that is not included in the demand equation, is not correlated with the demand shock, and is correlated with price.

According to our model, input prices are variables that may satisfy these conditions. For instance, limestone and coal are two important variable inputs in the production of cement. The prices of limestone and coal are potential instruments because they affect marginal cost, they should be correlated with price, but they do not enter in the demand equation. What is not so obvious is whether these variables are uncorrelated with the unobserved demand shock. If the demand for coal and limestone from the cement industry represents a small fraction of the total demand of these inputs in the local market, it seems plausible to argue that shocks in the demand of cement may not be correlated with the price of these inputs. However, if the cement industry represents 90% of the demand of limestone in a local market, this independence assumption seems completely implausible.

Natural experiments as exclusion restrictions

Consider an unexpected natural shock that affected the production cost of some markets in a particular period of time. Let I_{mt} be the indicator of the event "affected by the natural shock". This variable is zero for every market before period t^* when the natural event occurred; it is always zero for markets that do not experience the event, that is, the control group; and it goes from zero to one for markets in the experimental group. Since there are good reasons to believe that the natural event affected costs, it is clear that price depends on the dummy variable I_{mt}. For I_{mt} to be a valid instrument for price, the key identification assumption required is that demand was unaffected by the natural event. Under this assumption, the moment condition $\mathbb{E}(I_{mt}\ \varepsilon^D_{mt}) = 0$, together with the conditions $\mathbb{E}(\varepsilon^D_{mt}) = 0$ and $\mathbb{E}(S_{mt}\ \varepsilon^D_{mt}) = 0$, identify the demand parameters.

The condition that the natural event did not affect the demand is a strong assumption. Though the natural event is completely exogenous and unexpected, there is no reason why it may have occurred in markets that have relatively high (or low) levels of demand, or have taken place during a period of high (or low) demand. In contrast to the case of the randomized experiment described above, where by the own design of the experiment the subsidy was not correlated with the demand shock, there is nothing in the natural experiment implying that $\mathbb{E}(I_{mt}\ \varepsilon^D_{mt}) = 0$. To try to deal with this issue, most applications exploiting identification from 'natural experiments' assume a particular structure for the unobserved error:

$$\varepsilon^D_{mt} = \omega^D_m + \delta^D_t + u^D_{mt}, \tag{1.25}$$

We can control for ω^D_m using market dummies, and for δ_t using time dummies. The 'natural experiment' dummy I_{mt} can be correlated with ω^D_m and/or with δ^D_t. The identification assumption is that I_{mt} is not correlated with the shock u^D_{mt}.

Restrictions on unobservables

Suppose that the unobservables in the demand and in the marginal cost have the covariance structure:

$$\begin{aligned} \varepsilon^D_{mt} &= \omega^D_m + \delta^D_t + u^D_{mt}, \\ \varepsilon^{MC}_{mt} &= \omega^{MC}_m + \delta^{MC}_t + u^{MC}_{mt} \end{aligned} \tag{1.26}$$

These components of the variance specification of the unobservables, together with restrictions on the serial or/and the spatial correlation of the demand shocks u^D_{mt}, have been exploited to obtain exclusion restrictions and instrumental variables estimators. We distinguish two cases depending on whether the restrictions are on the serial correlation of the shock (that is, Arellano-Bond Instruments; Arellano and Bond, 1991), or on the spatial correlation (that is, Hausman-Nevo Instruments; Hausman, 1996, and Nevo, 2001).

Arellano-Bond instruments. Suppose that the shock u^D_{mt} is not serially correlated over time. That is, all the time persistence in unobserved demand comes from the time-invariant effect ω^D_m, and from the common industry shocks δ^D_t, but the idiosyncratic demand shock u^D_{mt} is not persistent over time. Under these conditions, in the demand equation in first-differences, $\Delta Q_{mt}/S_{mt} = \beta_S\ \Delta S_{mt} - \beta_1\ \Delta P_{mt} + \Delta\delta^D_t + \Delta u^D_{mt}$, the lagged endogenous variables $\{P_{mt-2},\ Q_{mt-2},\ N_{mt-2}\}$ are not correlated with the error Δu^D_{mt}, and they can be used as instruments to estimate demand parameters. The key identification

assumption is that the shocks u_{mt}^{MC} in the marginal cost are more persistent than the demand shocks u_{mt}^{D}.

Hausman-Nevo instruments. Suppose that we can classify the M local markets in R regions. Local markets in the same region may share a similar supply of inputs in the production of cement and similar production costs. However, suppose that the demand shock u_{mt}^{D} is not spatially correlated, such that local markets in the same region have independent demand shocks. All the spatial correlation in demand comes from observables variables, from correlation between the time-invariant components ω_{m}^{D}, or from the common shock δ_{t}^{D}. Let $\overline{P}_{(-m)t}$ be the average price of cement in markets that belong to the same region as market m but where the average excludes market m. Under these conditions, and after controlling for ω_{m}^{D} using market-dummies and for δ_{t}^{D} using time-dummies, the average price $\overline{P}_{(-m)t}$ is not correlated with the demand shock u_{mt}^{D} and it can be used as an instrument to estimate demand parameters. The key identification assumption is that the shocks u_{mt}^{MC} in the marginal cost have spatial correlation that is not present in demand shocks u_{mt}^{D}.

Zero covariance between unobservables. In simultaneous equations models, an assumption of zero covariance between the unobservables of two structural equations provides a moment condition that can be used to identify structural parameters. In the context of our model, consider the restrictions $\mathbb{E}(\varepsilon_{mt}^{FC}\ \varepsilon_{mt}^{D}) = 0$ and $\mathbb{E}(\varepsilon_{mt}^{FC}\ \varepsilon_{mt}^{MC}) = 0$. These restrictions imply the moment conditions:

$$\mathbb{E}\left(\left[q_{mt}^{2}\left(\frac{1}{\beta_{p}S_{mt}}+\frac{\gamma_{2}^{MC}}{2}\right)-\gamma^{FC}\,\mathbf{X}_{mt}^{FC}\right]\left[\frac{Q_{mt}}{S_{mt}}-\beta_{x}\,\mathbf{X}_{mt}^{D}-\beta_{p}\,P_{mt}\right]\right)=0 \tag{1.27}$$

and

$$\mathbb{E}\left(\left[q_{mt}^{2}\left(\frac{1}{\beta_{p}S_{mt}}+\frac{\gamma_{2}^{MC}}{2}\right)-\gamma^{FC}\,\mathbf{X}_{mt}^{FC}\right]\left[P_{mt}-\frac{1}{\beta_{p}}\frac{q_{mt}}{S_{mt}}-\gamma_{1}^{MC}\,\mathbf{X}_{mt}^{MC}-\gamma_{2}^{MC}\,q_{mt}\right]\right)=0 \tag{1.28}$$

These moment restrictions, together with the restrictions $\mathbb{E}(\varepsilon_{mt}^{D}) = 0$, $\mathbb{E}(\varepsilon_{mt}^{MC}) = 0$, $\mathbb{E}(\varepsilon_{mt}^{FC}) = 0$, $\mathbb{E}(S_{mt}\ \varepsilon_{mt}^{D}) = 0$, $\mathbb{E}(S_{mt}\ \varepsilon_{mt}^{MC}) = 0$, and $\mathbb{E}(S_{mt}\ \varepsilon_{mt}^{FC}) = 0$, identify the structural parameters of the model.

We can consider a weaker version of this assumption: if $\varepsilon_{mt}^{FC} = \omega_{m}^{FC} + \delta_{t}^{FC} + u_{mt}^{FC}$ and $\varepsilon_{mt}^{D} = \omega_{m}^{D} + \delta_{t}^{D} + u_{mt}^{D}$, we can allow for correlation between the $\omega's$ and $\delta's$ and assume that only the market specific shocks u_{mt}^{FC} and u_{mt}^{D} are not correlated.

Multiple equilibria and Identification

Multiplicity of equilibria is a common feature in many models in IO. In our example, for any value of the parameters and exogenous variables, the equilibrium in the model is unique. There are three assumptions in our simple model that play an important role in generating this strong equilibrium uniqueness: (a) linearity assumptions, that is, linear demand; (b) homogeneous firms, that is, homogeneous product and costs; and (c) no dynamics. Once we relax any of these assumptions, multiple equilibria becomes the rule more than the exception: for some values of the exogenous variables and parameters, the model has multiple equilibria.

Is multiplicity of equilibria an important issue for estimation? It may or may not be, depending on the structure of the model and on the estimation method that we

choose. We will examine this issue in detail throughout this book, but let us provide here some general ideas about this issue.

Suppose that the fixed cost of operating a plant in the market is a decreasing function of the number of firms in the local market. For instance, the supply of equipment (fixed input) increases with the number of firms in the market, and the price of this fixed input declines. Then, $FC_{mt} = \gamma^{FC} - \delta\, N_{mt} + \varepsilon_{mt}^{FC}$, where δ is a positive parameter. Then, the equilibrium condition for market entry becomes:

$$\left(\frac{Q_{mt}}{N_{mt}}\right)^2 = \frac{\gamma^{FC} - \delta\, N_{mt} + \varepsilon_{mt}^{FC}}{B_{mt} + \gamma_q^{MC}/2} \tag{1.29}$$

This equilibrium equation can imply multiple equilibria for the number of firms in the market. The existence of positive synergies in the entry cost introduces some "coordination" aspects in the game of entry (Cooper, 1999). If δ is large enough, this coordination feature can generate multiple equilibria. Of course, multiplicity in the number of firms also implies multiplicity in the other endogenous variables, price, and output per firm. Therefore, the reduced form equations are now correspondences, instead of functions, that relate exogenous variables and parameters with endogenous variables.

Does this multiplicity of equilibria generate problems for the identification and estimation of the structural parameters of the model? Not necessarily. Note that, in contrast to the case of the reduced form equations, the three structural equations (demand, Cournot equilibrium, and entry condition) still hold with the only difference that we now have the term $-\delta\, N_{mt}$ in the structural equation for the entry equilibrium condition. That is,

$$q_{mt}^2 \left(\frac{1}{\beta_p S_{mt}} + \frac{\gamma_q^{MC}}{2}\right) \;=\; \gamma^{FC}\, \mathbf{X}_{mt}^{FC} - \delta\, N_{mt} + \varepsilon_{mt}^{FC} \tag{1.30}$$

The identification of the parameters in demand and variable costs is not affected. Suppose that those parameters are identified such that the left-hand-side in the previous equation is a known variable to the researcher. In the right hand side, we now have the number of firms as a regressor. This variable is endogenous and correlated with the error term ε_{mt}^{FC}. However, dealing with the endogeneity of the number of firms for the estimation of the parameters γ^{FC} and δ is an issue that does not have anything to do with multiple equilibria. We have that endogeneity problem whether or not the model has multiple equilibria, and the way of solving that problem does not depend on the existence of multiple equilibria. For instance, if we have valid instruments and estimate this equation using Instrumental Variables (IV), the estimation will be the same regardless of the multiple equilibria in the model.

In fact, multiple equilibria may even help for identification in some cases. For instance, if there is multiple equilibria in the data and equilibrium selection is random and independent of ε_{mt}^{FC}, then multiple equilibria helps for identification because it generates additional sample variation in the number of firms that is independent of the error term.

In some models, multiplicity of equilibria can be a nuisance for estimation. Suppose that we want to estimate the model using the maximum likelihood (ML) method. To use the ML method we need to derive the probability distribution of the endogenous variables conditional on the exogenous variables and the parameters of the model. However, in

a model with multiple equilibria there is no such thing as “the” distribution of the endogenous variables. There are multiple distributions, one for each equilibrium type. Therefore, we do not have a likelihood function but a likelihood correspondence. Is the MLE well defined in this case? How to compute it? Is it computationally feasible? Are there alternative methods that are computationally simpler? We will address all these questions later in the book.

1.4.3 Extensions

The rest of the book deals with empirical models of market structure that relax some of these assumptions. (a) *Product differentiation* and more general forms of demand (see chapter 2 on demand estimation). (b) *Heterogeneity in firms' costs*: exploiting information on firms' inputs to identify richer cost structures(see chapter 3, on production function estimation). (c) *Relaxing the assumption of Cournot competition*, and identification of the "nature of competition" from the data, for instance, collusion (see chapter 4 on models of price and quantity competition). (d) *Heterogeneity of entry costs* in oligopoly games of entry (see chapter 5 on static games of entry). (e) *Spatial differentiation* and plant spatial location. (see chapter 5 on games of spatial competition). (f) Competition in quality and other product characteristics (see chapter 5 on games of quality competition). (g) *Investment in capacity* and physical capital (see chapters 6 and 7 on dynamic structural models of firm investment decisions). (h) *Consumers intertemporal substitution and dynamic demand* of storable and durable products (see chapter 8 on dynamic demand). (i) Dynamic strategic interactions in firms' investment and innovation decisions (see chapter 9 dynamic games]. (j) *Mergers* (see chapter 5 on conduct parameters and chapter 9 on dynamic games). (k) *Firm networks*, chains, and competition between networks (see chapter 9 on dynamic games). (l) Firms' competition in auctions (see chapter 10 on auctions).

1.5 Summary

In this chapter, we have described Empirical Industrial Organization as a discipline that deals with the combination of data, models, and econometric methods to answer empirical questions related to the behavior of firms in markets. The answers to empirical questions IO are typically based on the estimation of structural models of competition. These models have four key components: demand, costs, price or quantity competition, and market entry. The identification and estimation of the structural parameters in these models are based on the principle of revealed preference. Endogeneity is an important issue in the estimation of the model parameters. We have described different approaches to deal with endogeneity problems, from randomized control trials and natural experiments, to instrumental variables, and restrictions on the structure of the unobserved variables. Multiplicity of equilibria is also a common feature in some empirical games.

1.6 Exercises

1.6.1 Exercise 1

Write a computer program in your favorite mathematical software (for instance, R, Gauss, Matlab, Stata, Julia, Python, etc) that implements the following tasks.

(a) Fix as constants in your program the values of the exogenous cost variables MC_{mt}, and FC_{mt}, and of demand parameters β_0 and β_1. Then, consider 100 types of markets according to their firm size. For instance, a vector of market sizes $\{1, 2, ..., 100\}$.

(b) For each market type/size, obtain equilibrium values of the endogenous variables including output per firm, firm's profit, and consumer surplus. For each of these variables, generate a two-way graph with the endogenous variable in vertical axis and market size in the horizontal index.

(c) Now, consider a policy change that increases fixed cost and reduces marginal cost. Obtain two-way graphs of each variable against market size representing the curves both before and after the policy change.

1.6.2 Exercise 2

Write a computer program in your favorite mathematical software that implements the following tasks.

(a) Fix as constants in the program the number of markets, M, time periods in the sample, T, and the values of structural parameters, including the parameters in the distribution of the unobservables and the market size. For instance, you could assume that the four unobservables ε have a join normal distribution with zero mean and a variance-covariance matrix, and that market size is independent of these unobservables and it has a log normal distribution with some mean and variance parameters.

(b) Generate NT random draws from the distribution of the exogenous variables. For each draw of the exogenous variables, obtain the equilibrium values of the endogenous variables. Now, you have generated a panel dataset for $\{P_{mt}, Q_{mt}, N_{mt}, S_{mt}\}$

(c) Use these data to estimate the model by OLS, and also try some of the identification approaches to identify the parameters of the model.

1.6.3 Exercise 3

The purpose of this exercise is to use the estimated model (or the true model) from exercise #2 to evaluate the contribution of different factors to explain the cross-sectional dispersion of endogenous variables such as prices, firm size, or number of firms. Write a computer program that implements the following tasks.

(a) For a particular year of your panel dataset, generate figures for the empirical distribution of the endogenous variables, say price.

(b) Consider the following comparative statics (counterfactual) exercises and obtain the empirical distribution (histogram) for the distribution of prices under each of the following changes: (i) eliminate heterogeneity in market size: set all market sizes equal to the one in the median market; (ii) eliminate market heterogeneity in demand shocks: set all demand shocks equal to zero; (iii) eliminate all the market heterogeneity in marginal costs; and (iv) remove all the market heterogeneity in fixed costs. Generate figures of each of these counterfactual distributions together with the factual distribution.

2. Consumer Demand

2.1 Introduction

The estimation of demand equations is a fundamental component in most empirical applications in IO.[1] It is also important in many other fields in empirical economics. There are important reasons why economists in general, and IO economists in particular, are interested in demand estimation. Knowledge of the demand function, and of the corresponding marginal revenue function, is crucial for the determination of a firm's optimal prices or quantities. In many applications in empirical IO, demand estimation is also a necessary first step to measure market power. In the absence of direct information about firms' costs, the estimation of demand and marginal revenue is key for the identification of marginal costs (using the marginal cost equals marginal revenue condition) and firms' market power. Similarly, the estimation of the degree of substitution between the products of two competing firms is a fundamental factor in evaluating the profitability of a merger between these firms. Demand functions are a representation of consumers' valuation of products. Because we cannot observe consumer utility or satisfaction directly, we obtain consumer preferences by estimating demand equations. As such, the estimation of demand is fundamental in the evaluation of the consumer welfare gains or losses associated with taxes, subsidies, new products, or mergers. Finally, demand estimation can also be used to improve our measures of Cost-of-Living indices (see Hausman, 2003, and Pakes, 2003).[2]

[1] Ackerberg et al. (2007) and Nevo (2011) are survey papers on demand estimation.

[2] For instance, the Boskin commission (Boskin et al., 1997 and 1998) concluded that the US Consumer Price Index (CPI) overstated the change in the cost of living by about 1.1 percentage points per year. CPIs are typically constructed using weights which are obtained from a consumer expenditure survey. For instance, the Laspeyres index for a basket of n goods is $CPI_L = \sum_{i=1}^{n} w_i^0 \left(\frac{P_i^1}{P_i^0}\right)$, where P_i^0 and P_i^1 are the prices of good i at periods 0 and 1, respectively, and w_i^0 is the weight of good i in the total expenditure of a representative consumer at period 0. A source of bias in this index is that it ignores that the weights w_i^0 change over time as the result of changes in relative prices of substitute products, or the introduction of new products between period 0 to period 1. The Boskin Commission identifies the introduction of new goods, quality improvements in existing goods, and changes in relative prices as the main sources of bias

Most products that we find in today's markets are differentiated products: automobiles; smartphones; laptop computers; or supermarket products such as ketchup, soft drinks, breakfast cereals, or laundry detergent. A differentiated product consists of a collection of varieties such that each variety is characterized by some attributes that distinguishes it from the rest. A variety is typically produced by a single manufacturer, but a manufacturer may produce several varieties.

We distinguish two approaches to model demand systems of differentiated products: demand systems in product space, which was the standard approach in EIO until the 1990s, and demand systems in characteristics space. We will see in this chapter that the model in characteristics space has several advantages over the model in product space, which has made it the predominant approach in empirical IO over the last two decades.

2.2 Demand systems in product space

2.2.1 Model

In this model, consumer preferences are defined over products themselves. Consider J different products that we index by $j \in \{1,2,...,J\}$. These J products may include all the product categories that an individual consumer may consume (for instance, food, transportation, clothing, entertainment) and all the varieties of products within each category (for instance, every possible variety of computers, or of automobiles). This means that the number of products J can be of the order of millions. Section 2.2.2 shows how to use multi-stage budgeting to deal with this high dimensionality problem. For this purpose, it is convenient to introduce "product zero" that we denote as the *outside product*, which represents all the other products that are not products 1 to J.

Let q_j denote the quantity of product j that a consumer buys and consumes, and let $(q_0, q_1, ..., q_J)$ be the vector with the purchased quantities of all the products, including the outside good. The price of the outside good is normalized to one, such that q_0 represents the dollar expenditure in goods other than 1 to J. The consumer has a utility function $U(q_0, q_1, ..., q_J)$ defined over the vector of quantities. The consumer's problem consists of choosing the vector $(q_0, q_1, ..., q_J)$ which maximizes her utility subject to her budget constraint.

$$\begin{aligned} \max_{\{q_0,q_1,...,q_J\}} \quad & U(q_0, q_1, ..., q_J) \\ subject\ to: \quad & q_0 + p_1\, q_1 + ... + p_J\, q_J \leq y \end{aligned} \tag{2.1}$$

where p_j is the price of product j, and y is the consumer's disposable income. We can define the Lagrangian problem:

$$\max_{\{q_0,q_1,...,q_J\}} \quad U(q_0, q_1, ..., q_J) + \lambda\,[y - q_0 - p_1\, q_1 - ... - p_J\, q_J] \tag{2.2}$$

The first order conditions are:

$$\begin{aligned} U_j - \lambda\, p_j &= 0 \ \text{ for } j = 0, 1, ..., J; \\ y - q_0 - p_1 q_1 - ... - p_J q_J &= 0 \end{aligned} \tag{2.3}$$

in the CPI as a cost of living index.Hausman (2003) and Pakes (2003) argue that the estimation of demand systems provides a possible solution to these sources of bias in the CPI.

where U_j represents the marginal utility of product j. The demand system is the solution to this optimization problem. We can represent this solution in terms of J functions, one for each product. These are the *Marshallian demand equations*:

$$\begin{array}{rcl} q_0 & = & f_0(p_1, p_2, ..., p_J, y) \\ q_1 & = & f_1(p_1, p_2, ..., p_J, y) \\ ... & & ... \\ q_J & = & f_J(p_1, p_2, ..., p_J, y) \end{array} \tag{2.4}$$

Function f_j provides the optimal consumption of product j as a function of prices and income.

The form of these functions depends on the form of the utility function U. Different utility functions imply different demand systems. Not every system of equations that relates quantities and prices is a demand system. It should come from the solution to the consumer problem for a given utility function. This has two clear implications on a demand system. First, a demand system should satisfy the *adding up condition* $\sum_{j=0}^{J} p_j f_j(p_1, p_2, ..., p_J, y) = y$. And second, it should be homogeneous of degree zero in prices and income: for any scalar $\delta \geq 0$, we have that $f_j(\delta p_1, \delta p_2, ..., \delta p_J, \delta y) = f_j(p_1, p_2, ..., p_J, y)$ for any product j.

A substantial part of the empirical literature on demand deal with finding utility functions that generate demand systems with two important practical features. First, they are simple enough to be estimable using standard econometric methods such as linear regression. And second, they are flexible in the sense of allowing for rich patterns in the elasticities of substitution between products. In general, there is a trade-off between these two features. The following are some examples of models that have been considered in the literature. They are shorted chronologically.

The Linear Expenditure demand system

Consider the Stone-Geary utility function:

$$U = (q_0 - \gamma_0)^{\alpha_0} (q_1 - \gamma_1)^{\alpha_1} ... (q_J - \gamma_J)^{\alpha_J} \tag{2.5}$$

where $\{\alpha_j, \gamma_j : j = 1, 2, ..., J\}$ are parameters. The parameter γ_j can be interpreted as the minimum amount of consumption of good j that a consumer needs to "survive". Parameter α_j represents the intensity of product j in generating utility. More formally, α_j is the elasticity of utility with respect to the amount of product j. Without loss of generality, given the ordinality of the utility function, we consider that $\sum_{i=0}^{J} \alpha_i = 1$. This utility function was first proposed by Geary (1950), and Stone (1954) was the first to estimate the Linear Expenditure System. In the Appendix to this chapter, section 2.5.1, we derive the expression for the demand equations of the Linear Expenditure System. They have the following form:

$$q_j = \gamma_j + \alpha_j \left[\frac{y - P_\gamma}{p_j} \right] \tag{2.6}$$

where P_γ is the aggregate price index $\sum_{i=0}^{J} p_i \gamma_i$.

This system is convenient because of its simplicity. Suppose that we have data on individual purchases and prices over T periods of time ($t = 1, 2, ..., T$): $\{q_{0t}, q_{1t}, ..., q_{Jt}\}$

and $\{p_{1t}, p_{2t}, ..., p_{Jt}\}$.[3] The model implies a system of J linear regressions. For product j:

$$q_{jt} = \gamma_j + \alpha_j \frac{y_t}{p_{jt}} + \beta_{j0} \frac{p_{0t}}{p_{jt}} + ... + \beta_{jJ} \frac{p_{Jt}}{p_{jt}} + \xi_{jt} \tag{2.7}$$

with $\beta_{jk} = -\alpha_j \gamma_k$. Variable ξ_{jt} is an error term that can come, for instance, from measurement error in purchased quantity q_{jt}, or from time variation in the coefficient γ_j. The intercept and slope parameters in these linear regression models can be estimated using instrumental variable methods.

However, the model is also very restrictive. Note that for any $j \neq k$, we have that $\frac{\partial q_j}{\partial p_k} = -\alpha_j \gamma_k / p_j < 0$, such that all the cross-price elasticities are negative. This implies that all the products are complements in consumption. This is not realistic in most applications, particularly when the goods under study are varieties of a differentiated product.

Constant Elasticity of Substitution demand system

Consider the Constant Elasticity of Substitution (CES) utility function:

$$U = \left(\sum_{j=0}^{J} q_j^{\sigma} \right)^{1/\sigma} \tag{2.8}$$

where $\sigma \in [0,1]$ is a parameter that represents the degree of substitution between the $J+1$ products. The marginal utilities are:

$$U_j = q_j^{\sigma-1} \frac{U}{\sum_{i=0}^{J} q_i^{\sigma}} \tag{2.9}$$

For any two pairs of products, j and k, we have that $\frac{\partial^2 U}{\partial q_j \partial q_k} < 0$, such that all the products are substitutes in consumption.

Given the CES utility function, we derive in the Appendix the following expression for the demand equations:

$$q_j = \frac{y}{P_\sigma} \left[\frac{p_j}{P_\sigma} \right]^{-1/(1-\sigma)} \tag{2.10}$$

where P_σ is the following aggregate price index:

$$P_\sigma = \left(\sum_{j=0}^{J} p_j^{-\sigma/(1-\sigma)} \right)^{-(1-\sigma)/\sigma} \tag{2.11}$$

The CES model is also very convenient because of its simplicity. Suppose that we have data of individual purchases and prices over T periods of time. The model implies the following log-linear regression model:

$$\ln\left(\frac{q_{jt}}{y_t} \right) = \beta_0 + \beta_1 \ln(p_{jt}) + \beta_2 \ln(P_{\sigma t}) + \xi_{jt} \tag{2.12}$$

[3] Given information on household income, y_t, the consumption of product zero can be obtained using the budget constraint, $q_{0t} = y_t - \sum_{j=1}^{J} p_j q_j$.

where $\beta_1 = -1/(1-\sigma)$, and $\beta_2 = \sigma/(1-\sigma)$. The error term ξ_{jt} can be interpreted as measurement error in quantities. The construction of the true price index $P_{\sigma t}$ requires knowledge of the parameter σ. To deal with this issue several approaches have been used in the literature: (a) approximating the true price index with a conjecture about σ; (b) controlling for the term $\beta_2 \ln(P_{\sigma t})$ y including time dummies; (c) estimating the model in deviations with respect to the equation for the outside product, $\ln\left(\frac{q_{jt}}{y_t}\right) - \ln\left(\frac{q_{0t}}{y_t}\right) = \beta_1 \ln(p_{jt}) + \xi_{jt} - \xi_{0t}$; and (d) taking into account the structure of the price index as a function of prices and σ and estimating the model using nonlinear least squares.

The demand elasticity β_1 can be estimated using a standard method for linear regression models. For instance, if the number of products in our dataset is large relative to the number of time periods, one can control for the time-effects using time dummies, and β can be estimated using OLS or IV methods.

This model imposes strong restrictions on consumer behavior. In particular, the elasticity of substitution between any pair of products is exactly the same. For any three products, say j, k, and i:

$$Elasticity_{k,j} = \frac{\partial \ln q_k}{\partial \ln p_j} = \frac{-\sigma}{1-\sigma} = \frac{\partial \ln q_i}{\partial \ln p_j} = Elasticity_{i,j}. \tag{2.13}$$

Having the same degree of substitution for all pairs of products can be quite unrealistic in most applications in IO. In fact, there are many industries that have both products that are close substitutes as well as products that are relatively unique. In such industries, we would expect an increase in the price of a product with many close substitutes to generate a substantial reduction in quantity, while this would not be the case if the product was relatively unique (i.e. did not have close substitutes).

Deaton and Muellbauer "Almost Ideal" demand system

The "Almost Ideal" demand system (AIDS) was proposed by Deaton and Muellbauer (1980, 1980). Because of its flexibility, it is the most popular specification in empirical applications where preferences are defined on the product space. The standard derivation of the AIDS does not start from the utility function but from the Expenditure Function of the model. The *expenditure function* of a demand system, $E(u, \mathbf{p})$ is defined as the minimum consumer expenditure required to achieve a level of utility u given the vector of prices $\mathbf{p} = (p_1, p_2, ..., p_J)$.

$$E(u, \mathbf{p}) = \min_{q_0, q_1, ..., q_J} \sum_{j=0}^{J} p_j \, q_j \quad \text{subject to: } U(q_0, q_1, ..., q_J) = u \tag{2.14}$$

Given its definition, the expenditure function is non-decreasing in all its arguments, and it is homogeneous of degree one in prices: for any $\delta \geq 0$, $E(u, \delta\, \mathbf{p}) = \delta\, E(u, \mathbf{p})$. Shephard's Lemma establishes that the derivative of the expenditure function with respect to the price of product j is the Hicksian or compensated demand function, $h_j(u, \mathbf{p})$:

$$q_j = h_j(u, \mathbf{p}) = \frac{\partial E(u, \mathbf{p})}{\partial p_j} \tag{2.15}$$

Similarly, combined with the condition that income is equal to total expenditure, $y = E(u, \mathbf{p})$, Shephard's Lemma implies that the partial derivative of the log-expenditure

function with respect to the log-price of product j is equal to the expenditure share of the product, $w_j \equiv p_j q_j / y$.

$$w_j = \frac{p_j \, q_j}{y} = \frac{\partial \ln E(u, \mathbf{p})}{\partial \ln p_j} \tag{2.16}$$

Therefore, given a expenditure function that is consistent with consumer theory (non-decreasing and homogeneous of degree one), we can derive the demand system. Deaton and Muellbauer propose the following log-expenditure function:

$$\ln E(u, \mathbf{p}) = a(\mathbf{p}) + b(\mathbf{p}) \, u \tag{2.17}$$

with

$$\begin{aligned} a(\mathbf{p}) &= \sum_{j=1}^{J} \alpha_j \, \ln p_j + \frac{1}{2} \sum_{j=1}^{J} \sum_{k=1}^{J} \gamma_{jk}^{*} \, \ln p_j \, \ln p_k \\ b(\mathbf{p}) &= \prod_{j=1}^{J} p_j^{\beta_j} \end{aligned} \tag{2.18}$$

Homogeneity of degree of the expenditure function requires the following restrictions on the parameters:

$$\sum_{j=1}^{J} \alpha_j = 1; \; \sum_{j=1}^{J} \gamma_{jk}^{*} = 0; \; \sum_{k=1}^{J} \gamma_{jk}^{*} = 0; \; \sum_{j=1}^{J} \beta_j = 0. \tag{2.19}$$

Applying Shephard's Lemma to this log-expenditure function, we can derive the following demand system represented in terms of expenditure shares:

$$w_j = \alpha_j + \beta_j \, \left[\ln(y) - \ln(P_{\alpha,\gamma}) \right] + \sum_{k=1}^{J} \gamma_{jk} \, \ln p_k \tag{2.20}$$

where $\gamma_{jk} \equiv (\gamma_{jk}^{*} + \gamma_{kj}^{*})/2$ such that the model implies the symmetry condition $\gamma_{jk} = \gamma_{kj}$; and $P_{\alpha,\gamma}$ is a price index with the following form:

$$\ln P_{\alpha,\gamma} = \sum_{j=1}^{J} \alpha_j \, \ln p_j + \frac{1}{2} \sum_{j=1}^{J} \sum_{k=1}^{J} \gamma_{jk} \, \ln p_j \, \ln p_k \tag{2.21}$$

The number of free parameters in this demand system is $2J + \frac{J(J+1)}{2}$, which increases quadratically with the number of products.

Suppose that we have data on individual purchases, income, and prices over T periods of time. For each product j, we can estimate the regression equation:

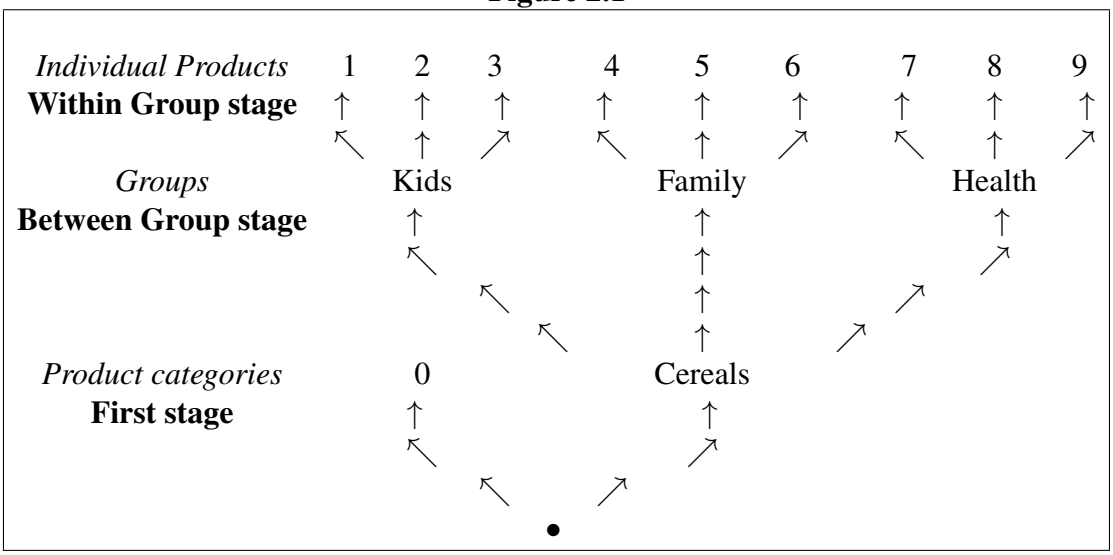
$$w_{jt} = \alpha_j + \beta_j \, \ln(y_t) + \gamma_{j1} \, \ln(p_{1t}) + ... + \gamma_{jJ} \, \ln(p_{Jt}) + \xi_{jt} \tag{2.22}$$

Since the number of parameters increases quadratically with the number of products, the estimation of this model (without restrictions on the parameters) requires that the number of observations T (either time periods or geographic markets) is substantially larger than the number of products J. For differentiated products with many varieties, say $J > 100$ (such as most differentiated products like automobiles, smartphones, cereals, beer, etc), the number of parameters can be of the order of several thousands such that this condition does not hold. Increasing the number of observations by using data from many consumers does not help in the estimation of price elasticities because consumers in the same market face the same prices, that is, prices do not have variation across consumers, only over time and geographic markets.

2.2.2 Multi-stage budgeting

To reduce the number of parameters when J is relatively large, Deaton and Muellbauer propose a multi-stage budgeting approach. Suppose that the $J+1$ products can be classified in G groups or segments. For instance, in the ready-to-eat cereal industry, some empirical studies distinguish three segments: *Kids*, *All family*, and *Health*. The following diagram presents the nested structure of the demand system.

Figure 2.1



Suppose that the utility function is:

$$U = v_0(q_0) + v_1(\widetilde{\mathbf{q}}_1) + ... + v_G(\widetilde{\mathbf{q}}_G) \tag{2.23}$$

where $\widetilde{\mathbf{q}}_g$ is the vector of quantities of product varieties in group g; and $v_g(\widetilde{\mathbf{q}}_g)$ is the sub-utility from group g. Then, the demand system at the lower stage, the **within-group stage**, is:

$$w_{jt} = \alpha_j^{(1)} + \beta_j^{(1)} \ln\left(\frac{e_{gt}}{P_{gt}}\right) + \sum_{k \in \mathscr{J}_g} \gamma_{jk}^{(1)} \ln(p_{kt}) \tag{2.24}$$

where e_{gt} is the expenditure from all the products in group g, and P_{gt} is a price index for group g. According to the model, this price index depends on the parameters of the model in group g. The number of parameters increases quadratically with J_g instead of with J. The demand system at the **group stage** is:

$$\frac{e_{gt}}{e_t} = \alpha_g^{(2)} + \beta_g^{(2)} \ln\left(\frac{e_t}{P_t}\right) + \sum_{g'=1}^{G} \gamma_{g,g'}^{(2)} \ln(P_{gt}) \tag{2.25}$$

where e_t is the total expenditure in the large category (for instance, cereals), and P_t is the price index for the category (for instance, cereals). Finally, at the top-stage, the **demand for the category** is:

$$\frac{e_t}{y_t} = \alpha^{(3)} + \beta^{(3)}[\ln(y_t) - \ln(P_t)] \tag{2.26}$$

This multi-stage budgeting model can reduce substantially the number of parameters. For instance, suppose that a differentiated product category, say cereals, has 50 products

such that the number of parameters in the unrestricted model is $2*50+\frac{50(50+1)}{2}=1,325$. Now, suppose that we can divide the 50 products into 10 groups with 5 products each. This implies that at the within-group stage we have 250 parameters (25 for each group), in the group stage we have 75 parameters, and in the category stage we have 3 parameters, for a total of 328 parameters. Using one year of monthly data over 500 geographic markets, we have 6,000 observations. If these data have enough (exogenous) variation in prices, it seems possible to estimate this restricted system. This is the approach in Hausman (1996) that we describe in more detail in section **??**.

2.2.3 Estimation

In empirical work, the most commonly used demand systems are the Rotterdam Model (Theil, 1975), the Translog Model (Christensen, Jorgenson, and Lau, 1975), and the Almost Ideal Demand System (AIDS) (Deaton and Muellbauer, 1980). Since Deaton and Muellbauer proposed their Almost Ideal Demand System in 1980, this model has been estimated in hundreds of empirical applications. In most of the applications, a "good" is an aggregate product category (for instance, beef meat, or chicken meat). However, there are also some applications for varieties of a differentiated product, such as the one in Hausman (1996) that we examine later in this chapter. In this section we describe the typical application of this class of models.

The typical dataset consists of aggregate market level data for a single market, over T time periods, with information on consumption and prices for a few product categories. For instance, Verbeke and Ward (2001) use monthly data from January 1995 to December 1998 ($T=48$ data points) from a consumer expenditure survey in Belgium. They estimate a demand system for fresh meat products that distinguishes between three product categories: Beef/veal, Pork, and Poultry. We index time by t. For each period t we observe aggregate income y_t, and prices and quantities of the J product categories: $\{y_t, q_{jt}, p_{jt} : t=1,2,...,T;\ j=1,2,...,J\}$. We want to estimate the demand system:

$$w_{jt} = \mathbf{X}_t\ \alpha_j + \beta_j\ \ln(y_t/P_t) + \sum_{k=1}^{J}\gamma_{jk}\ \ln(p_{kt}) + \xi_{jt} \tag{2.27}$$

where $\mathbf{X}_t$ is a vector of exogenous characteristics that may affect demand, for instance, demographic variables. We want to estimate the vector of structural parameters $\theta = \{\alpha_j, \beta_j, \gamma_{jk} : \forall j,k\}$. Typically, this system is estimated by OLS or by Nonlinear Least Squares (NLLS) to incorporate the restriction that $\ln(P_t)$ is equal to $\sum_{j=1}^{J}\left[\mathbf{X}_t\ \alpha_j\right]$ $\ln(p_{jt}) + \frac{1}{2}\sum_{j=1}^{J}\sum_{k=1}^{J}\gamma_{jk}\ln(p_{jt})\ln(p_{kt})$, and the symmetry restrictions on the parameters γ. These estimation methods assume that prices are not correlated with the error terms $\varepsilon's$.

2.2.4 Some limitations

Demand systems in product space suffer of some limitations for the empirical work and questions that we stusy in IO.

(1) Every consumer purchases/consumes each of the J products. The system of demand equations that we have derived above is based on the assumption that the marginal conditions of optimality hold for every product. This means that the optimal bundle for a consumer is an interior solution such that $q_j > 0$ for every product j. This condition is very unrealistic when we consider the demand of differentiated products

within a product category, for instance, the demand of automobiles. In this context, a consumer buys only one unit of a single variety (for instance, one Toyota Corola) or of a few varieties (for instance, one Toyota Corola, and one KIA Sorento minivan). To account for this type of consumer decisions, we need to model the consumer problem as a discrete choice model.

(2) Representative consumer. The representative consumer assumption is a very strong one and it does not hold in practice. The demand of certain goods depends not only on aggregate income but also on the distribution of income and on the distribution of other variables affecting consumers' preferences, for instance, age, education, etc. The propensity to substitute between different products can be also very heterogeneous across consumers. Therefore, ignoring consumer heterogeneity is a very important limitation of the actual applications in this literature.

In principle, demand systems in product space could be applied to household level data. Suppose that we have this type of data. Let us use the subindex h for households. The demand system becomes:

$$w_{jht} = \mathbf{X}_{ht}\ \alpha_j + [\mathbf{X}_{ht}\ \beta_j] \ln(y_{ht}/P_t) + \sum_{k=1}^{J} [\mathbf{X}_{ht}\ \gamma_{jk}] \ln(p_{kt}) + \xi_{jht} \tag{2.28}$$

where $\mathbf{X}_{ht}$ represents a vector of exogenous household characteristics, other than income. Now, α_j, β_j, and γ_{jk} are vectors of parameters with the same dimension as $\mathbf{X}_{ht}$. This model incorporates household observed heterogeneity in a flexible way: in the level of demand, in price elasticities, and in income elasticities.

Note that (typically) prices do not vary across households. Therefore, price elasticities are identified only from the time-series (or market) variation in prices, and not from the cross-sectional variation across households. In this context, household level data is useful to allow for consumer heterogeneity in price responses, but it does not provide additional sample variation to improve the precision in the estimation of price elasticities.

Household level data clearly illustrates the problem of observed zero consumption of some products, that we have mentioned in point (1) above. Some households do not consume all the product categories, even when these categories are quite broad. For instance, vegetarian households do not consume any meat. This class of model predicts that the household consumes a positive amount of every product category. This prediction is typically rejected when using household level data.

(3) The problem of too many parameters. In the standard model, the number of parameters is $2J + \frac{J(J+1)}{2}$, that is, J intercept parameters (α); J income elasticities (γ); and $\frac{J(J+1)}{2}$ free price elasticities (β). The number of parameters increases quadratically with the number of goods. Note also that, in most applications, the sample variation in prices comes only from time series, and the sample size T is relatively small. This feature of the model implies that the number of products, J, should be quite small. For instance, even if J is as small as 5, the number of parameters to estimate is 25. Therefore, with this model and data, it is not possible to estimate demand systems for differentiated products with many varieties. For instance, suppose that we are interested in the estimation of a demand system for different car models, and the number of car models is $J = 100$. Then, the number of parameters in the AIDS model is $5,250$, and

we need many thousands of observations (markets or/and time periods) to estimate this model. This type of data is typically not available.

(4) Finding instruments for prices. Most empirical applications of this class of models have ignored the potential endogeneity of prices.[4] However, it is well known that simultaneity and endogeneity are potentially important issues in any demand estimation. Prices are determined in the equilibrium of the market and depend on all the exogenous variables affecting demand and supply. Therefore, we expect prices to be correlated with the error terms ξ in the demand equations. Correlation between regressors and the error term implies that the OLS method is an inconsistent estimator of the parameters in demand equation. The typical solution to this problem is using instrumental variables. In the context of this model, the researcher needs at least as many instruments as prices, that is, J. The ideal case would be to have information on production costs for each individual good. However, this type of information is rarely available.

(5) Predicting the demand of new goods. In the literature of demand of differentiated products, a class of problem that has received substantial attention is the evaluation or prediction of the demand of a new product. Trajtenberg (1989), Hausman (1996), and Petrin (2002) are some of the prominent applications that deal with this empirical question. In a demand system in product space, estimating the demand of a new good, say good $J+1$, requires estimates of the parameters associated with that good: α_{J+1}, β_{J+1} and $\{\gamma_{J+1,j} : j = 1,2,...,J+1\}$. Of course, this makes it impossible to make counterfactual predictions, that is, predicting the demand of a product that has not yet been introduced in any market. But it also limits the applicability of this model in cases where the new product has been introduced very recently or in very few markets, because we may not have enough data to estimate these parameters.

2.2.5 Dealing with limitations

Hausman (1996) studies the demand for ready-to eat (RTE) cereals in US. This industry has been characterized by the dominant position of six multiproduct firms and by the proliferation of many varieties. During the period 1980-92, the RTE cereal industry was among the most prominent in the introduction of new brands within U.S. industries, with approximately 190 new brands added to the pool of existing 160 brands. Hausman shows that using panel data from multiple geographic markets, together with assumptions on the spatial structure of unobserved demand shocks and costs, it is possible to deal with some of the problems mentioned above within the framework of demand systems in product space. He applies the estimated system to evaluate the welfare gains from the introduction of Apple-Cinnamon Cheerios by General Mills in 1989.

(1) Data. The dataset comes from supermarket scanner data collected by Nielsen company. It covers 137 weeks ($T = 137$) and seven geographic markets ($M = 7$) or standard metropolitan statistical areas (SMSAs), including Boston, Chicago, Detroit, Los Angeles, New York City, Philadelphia, and San Francisco. Though the data includes information from hundreds of brands, the model and the estimation concentrates on 20 brands classified into three segments: adult (7 brands), child (4 brands), and family

[4] An exception is, for instance, Eales and Unnevehr (1993) who find strong evidence on the endogeneity of prices in a system of meat demand in US. They use livestock production costs and technical change indicators as instruments.

(9 brands). Apple-Cinnamon Cheerios are included in the family segment. We index markets by m, time by t, and brands by j, such that the data can be described as $\{p_{jmt}, q_{jmt} : j = 1,2,...,20; m = 1,2,...,7; t = 1,2,...,137\}$. Quantities are measured in physical units. This dataset does not contain information on firms' costs, such as input prices or wholesale prices.

(2) Model. Hausman estimates an Almost-Ideal-Demand-System combined with a nested three-level structure. The nested structure is similar to the one described in the diagram of Figure 2.1. The top level is the overall demand for cereal using a price index for cereal relative to other goods. The middle level of the demand system estimates demand among the three market segments, adult, child, and family, using price indexes for each segment. The bottom level is the choice of brand within a segment. For instance, within the family segment the choice is between the brands Cheerios, Honey-Nut Cheerios, Apple-Cinnamon Cheerios, Corn Flakes, Raisin Bran (Kellogg), Wheat Rice Krispies, Frosted Mini-Wheats, Frosted Wheat Squares, and Raisin Bran (Post). Overall price elasticities are then derived from the estimates in all three segments. The estimation is implemented in reverse order, beginning at the lowest level (within segment). The estimates are then used to construct the next level's price indexes, and to implement the estimation at the next level. At the lowest level, within a segment, the demand system is:

$$s_{jmt} = \alpha^1_{jm} + \alpha^2_t + \beta_j \ \ln(y_{gmt}) + \sum_{k=1}^{J} \gamma_{jk} \ \ln(p_{kmt}) + \xi_{jmt} \tag{2.29}$$

where y_{gmt} is overall expenditure in segment/group g. The terms α^1_{jm} and α^2_t represent product, market and time effects, respectively, which are captured using dummies.

(2) Instruments. Suppose that the supply (pricing equation) is:

$$\ln(p_{jmt}) = \delta_j \ c_{jt} + \tau_{jm} + \kappa_{j1} \ \xi_{1mt} + ... + \kappa_{jJ} \ \xi_{Jmt} \tag{2.30}$$

All the components in the right-hand-side, δ_j, c_{jt}, τ_{jm}, κ's, and ξ's, are unobservable to the researcher. Variable c_{jt} represents a cost at the product level that is common to all the city markets. Variable τ_{jm} is a city-brand fixed effect that captures differences in transportation costs. The terms $\kappa_{j1} \ \xi_{1mt} + ... + \kappa_{jJ} \ \xi_{Jmt}$ captures how the price of product j in market m responds to local demand shocks, $\xi_{1mt}, \xi_{2mt}, ..., \xi_{Jmt}$. The identification assumption is that these demand shocks are not (spatially) correlated across markets:

$$\mathbb{E}(\xi_{jmt} \ \xi_{km't}) = 0 \qquad \text{for any } j,k \text{ and } m' \neq m \tag{2.31}$$

The assumption implies that, after controlling for brand-city fixed effects, all the correlations between prices at different locations come from correlations in costs and not from spatial correlation in demand shocks. Under these assumptions we can use average prices in other local markets, $\overline{P}_{j(-m)t}$, as instruments, where:

$$\overline{P}_{j(-m)t} = \frac{1}{M-1} \sum_{m' \neq m} p_{jm't} \tag{2.32}$$

(3) Evaluating the effects of new goods. Suppose that we define product J as being a "new" product in the market, although it is a product in our sample for which we have

data on prices and quantities, and for which we can estimate all the parameters of the model including α_J^0, $\{\beta_{Jk}\}$ and γ_J. The expenditure function $e(\mathbf{p},u)$ for the Deaton and Muellbauer demand system is:

$$e(\mathbf{p},u) = \sum_{j=1}^{J} \alpha_j \ln(p_j) + \frac{1}{2}\sum_{j=1}^{J}\sum_{k=1}^{J} \gamma_{jk} \ln(p_j)\ln(p_k) + u\prod_{j=1}^{J} p_j^{\beta_j} \tag{2.33}$$

Let $V(\mathbf{p},y)$ be the indirect utility associated with the demand system, that we can easily obtain by solving the demand equations into the utility function. Suppose that we have estimated the demand parameters after the introduction of good J in the market, and let $\hat{\theta}$ be the vector of parameter estimates. We use $\hat{e}(\mathbf{p},u)$ and $\hat{V}(\mathbf{p},y)$ to represent the functions $e(\mathbf{p},u)$ and $V(\mathbf{p},y)$ when we use the parameter estimates $\hat{\theta}$. Similarly, we use $\hat{D}_j(\mathbf{p},y)$ to represent the estimated Marshallian demand of product j.

The concept of *virtual price* plays a key role in Hausman's approach to obtain the value of a new good. The *virtual price* of good J – represented as p_J^* – is the price of this product that makes its demand equal to zero. That is, the virtual price of product J in market m at quarter t is the value p_{Jmt}^* that solves the following equation:

$$\hat{D}_{jmt}(p_{1mt}, p_{2mt}, ..., p_{Jmt}^*) = 0 \tag{2.34}$$

Note that this virtual price depends on the prices of the other products, as well as on other exogenous variables affecting demand.

Given the virtual price p_{Jmt}^*, we have a counterfactual scenario in which consumers do not buy product J in market m in period t. Hausman compares the factual situation with this counterfactual scenario. Let u_{mt} be the utility of the representative consumer in market m at period t with the new product: that is, $u_{mt} = \hat{V}(\mathbf{p}_{mt}, y_{mt})$. By construction, it should be the case that $\hat{e}(\mathbf{p}_{mt}, u_{mt}) = y_{mt}$. To reach the same level of utility u_{mt} without the new product, the representative consumer's expenditure should be $\hat{e}(p_{1mt}, p_{2mt}, ..., p_{Jmt}^*, u_{mt})$. Therefore, we can measure the change in welfare associated to the introduction of the new product using the following *Equivalent Variation* measure:

$$EV_{mt} = \hat{e}(p_{1mt}, p_{2mt}, ..., p_{Jmt}^*, u_{mt}) - y_{mt} \tag{2.35}$$

Hausman considers this measure of consumer welfare.

This approach uses a market with prices and income $(p_{1mt}, p_{2mt}, ..., p_{Jmt}^*, y_{mt})$ as the counterfactual to measure the value of good J in a market with actual prices and income $(p_{1mt}, p_{2mt}, ..., p_{Jmt}, y_{mt})$. However, this choice of counterfactual does not account for the potential effect on prices of the introduction of the new product. In some applications, the welfare gains from these competition effects can be substantial and we are interested in measuring them. To measure these effects we should calculate equilibrium prices before and after the introduction of the new good. This requires the estimation not only of demand parameters but also of firms' marginal costs, as well as an assumption about competition (competitive market, Cournot, Bertrand). Though the Equivalent Variation presented above does not account for competition effects, it has some attractive features. First, it has a clear economic interpretation as the welfare gain *in the absence of competition effects*. Second, since it only depends on demand estimation, it is robust to misspecification of the supply side of the model.

2.3 Demand in characteristics space

2.3.1 Model

The model is based on three basic assumptions. First, a product, say a laptop computer, can be described as a bundle of physical characteristics: for instance, CPU speed, memory, screen size, etc. These characteristics determine a *variety* of the product. Second, consumers have preferences on bundles of characteristics of products, and not on the products per se. And third, a product has J different varieties and each consumer buys at most one variety of the product per period, that is, all the varieties are substitutes in consumption.

We index varieties by $j \in \{1,2,...,J\}$. From an empirical point of view, we can distinguish two sets of product characteristics. Some characteristics are observable and measurable to the researcher. We represent with them using a vector of K attributes $\mathbf{X}_j \equiv (X_{1j}, X_{2j}, ..., X_{Kj})$, where X_{kj} represents the "amount" of attribute k in brand j. For instance, in the case of laptops we could define the variables as follows: X_{1j} represents CPU speed; X_{2j} is RAM memory; X_{3j} is hard disk memory; X_{4j} is weight; X_{5j} is screen size; X_{6j} is a dummy (binary) variable that indicates whether the manufacturer of the CPU processor s Intel or not; etc. Other characteristics are not observable, or at least measurable, to the researcher but they are known and valuable to consumers. There may be many of these unobservable attributes, and we describe these attributes using a vector ξ_j, that contains the "amounts" of the different unobservable attributes of variety j. We index households by $h \in \{1,2,....,H\}$ where H represents the number of households in the market. A household has preferences defined over bundles of attributes. Consider a product with arbitrary attributes $(\mathbf{X}, \xi)$. The utility of consumer h if she consumes that product is $V_h(\mathbf{X}, \xi)$. Importantly, note that the utility function V_h is defined over any possible bundle of attributes $(\mathbf{X}, \xi)$ that may or may not exist in the market. For a product j that exists in the market and has attributes $(\mathbf{X}_j, \xi_j)$, this utility is $V_{hj} = V_h(\mathbf{X}_j, \xi_j)$. The total utility of a consumer has two additive components: the utility from this product, and the utility from other goods: $U_h = u_h(C) + V_h(\mathbf{X}, \xi)$, where C represents the amount of a composite good, and $u_h(C)$ is the utility from the composite good.

Consumers differ in their levels of income, y_h, and in their preferences. Consumer heterogeneity in preferences can be represented in terms of a vector of consumer attributes υ_h that may be completely unobservable to the researcher. Therefore, we can write the utility of consumer h as:

$$U_h = u(C; \upsilon_h) + V(\mathbf{X}, \xi; \upsilon_h) \tag{2.36}$$

We also assume that there is continuum of consumers with measure H, such that υ_h has a well-defined density function f_υ in the market.

Each consumer buys at most one variety of the product (per period). Given her income, y_h, and the vector of product prices $\mathbf{p} = (p_1, p_2, ..., p_J)$, a consumer decides which variety to buy, if any. Let $d_{hj} \in \{0,1\}$ be the indicator of the event "consumer h buys product j". A consumer decision problem is:

$$\begin{array}{ll} \max\limits_{\{d_{h1}, d_{h2}, ..., d_{hJ}\}} & u(C; \upsilon_h) + \sum\limits_{j=1}^{J} d_{hj}\, V(\mathbf{X}_j, \xi_j; \upsilon_h) \\ subject\ to: & C + \sum_{j=1}^{J} d_{jh}\, p_j \leq y_h \\ & d_{hj} \in \{0,1\} \text{ and } \sum_{j=1}^{J} d_{jh} \in \{0,1\} \end{array} \tag{2.37}$$

A consumer chooses between $J+1$ possible choice alternatives: each of the J products and the alternative $j=0$ which represents the choice to not buy any product. The solution to this consumer decision problem provides the consumer-level demand equations $d_j^*(\mathbf{X},\mathbf{p},y_h;\upsilon_h)\in\{0,1\}$ such that:

$$\{d_j^*(\mathbf{X},\mathbf{p},y_h;\upsilon_h)=1\}\Leftrightarrow$$

$$\left\{u(y_h-p_j;\upsilon_h)+V(\mathbf{X}_j,\xi_j;\upsilon_h)>u(y_h-p_k;\upsilon_h)+V(\mathbf{X}_k,\xi_k;\upsilon_h)\text{ for any }k\neq j\right\} \tag{2.38}$$

where $k=0$ represents the alternative of not buying any variety (that is, the outside alternative), that has indirect utility $u(y_h;\upsilon_h)$. Given the demand of individual consumers, $d_j^*(\mathbf{X},\mathbf{p},y_h;\upsilon_h)$, and the joint density function $f(\upsilon_h,y_h)$, we can obtain the aggregate demand functions:

$$q_j(\mathbf{X},\mathbf{p},f)=\int d_j^*(\mathbf{p},y_h;\upsilon_h)\ ,\beta)\ f(\upsilon_h,y_h)\ d\upsilon_h\ d\mathbf{y}_h \tag{2.39}$$

and the market shares $s_j(\mathbf{X},\mathbf{p},f)\equiv\dfrac{q_j(\mathbf{X},\mathbf{p},f)}{H}$.

Now, we provide specific examples of this general model. Each example is based on specific assumptions about the form of the utility function and the probability distribution of consumer heterogeneity. These examples are also important models which are workhorses in the literature on the estimation of demand of differentiated products.

2.3.2 Logit model

Consider the following restrictions on the general model presented above. First, the utility from the outside product is linear and the same for all the consumers: $u(C;\upsilon_h)=\alpha\ C$, where α is a parameter that represents the marginal utility of the composite good C. Second, the utility of purchasing product j is:

$$V(\mathbf{X}_j,\widetilde{\xi}_j,\upsilon_h)=\mathbf{X}_j\beta+\xi_j+\varepsilon_{hj}, \tag{2.40}$$

where ε's are unobservable random variables (for the researcher) which are independently and identically distributed (i.i.d.) over consumers and products with an Extreme Value Type 1 distribution.[5] Then, $U_{hj}=-\alpha\ p_j+\mathbf{X}_j\ \beta+\xi_j+\varepsilon_{hj}$ and the the Extreme Value assumption on the ε variables implies that the market shares have the following closed-form logit structure.

$$s_j=\frac{q_j}{H}=\frac{\exp\{\delta_j\}}{1+\sum_{k=1}^{J}\exp\{\delta_k\}} \tag{2.41}$$

where $\delta_j\equiv-\alpha\ p_j+\mathbf{X}_j\ \beta+\xi_j$ represents the mean utility of buying product j.

The parameter α represents the marginal utility of income and it is measured in utils per dollar. In the vector β, the parameter β_k associated to characteristic X_{jk} (the $k-th$ element of vector $\mathbf{X}_j$) represents the marginal utility of this characteristic and it is measured in utils per unit of X_{jk}. Therefore, for any product attribute k, the ratio of parameters β_k/α is measured in dollars per unit of X_{jk} such that it is a monetary measure of the marginal utility of the attribute.

[5] For more information about the Extreme Value type I distribution see Appendix 11 on Random Utlity discrete choice logit models.

2.3.3 Nested Logit model

As explained below, the logit model imposes strong restrictions on the own and cross price elasticities of products. The Nested Logit model relaxes these restrictions.

Suppose that we partition $J+1$ products (including the outside product) in $G+1$ groups. We index groups of products by $g \in \{0,1,...,G\}$. Let $\mathscr{J}_g$ represent the set of products in group g. The utility function has the same structure as in the Logit model with the only (important) difference that the variables ε_{hj} have the structure of a nested logit model:

$$\varepsilon_{hj} = \lambda\ \varepsilon_{hg}^{(1)} + \varepsilon_{hj}^{(2)} \tag{2.42}$$

where $\varepsilon_{hg}^{(1)}$ and $\varepsilon_{hj}^{(2)}$ are i.i.d. Extreme Value type 1 variables, and λ is a parameter. This model implies the following closed-form expression for the market shares:

$$s_j = \frac{\exp\{\lambda\ I_g\}}{\sum_{g'=0}^{G}\exp\{\lambda\ I_{g'}\}}\ \frac{\exp\{\delta_j\}}{\sum_{k\in\mathscr{J}_g}\exp\{\delta_k\}} \tag{2.43}$$

where I_g is denoted the *inclusive value of group g* and it is defined as follows:

$$I_g \equiv \ln\left(\sum_{j\in\mathscr{J}_g}\exp\{\delta_j\}\right) \tag{2.44}$$

This inclusive value can be interpreted as the expected utility of a consumer who chooses group g knowing the δ values of the products in that group but before knowing the realization of the random variables $\varepsilon_{hj}^{(2)}$. That is,

$$I_g \equiv \mathbb{E}_{\varepsilon^{(2)}}\left(\max_{j\in\mathscr{J}_g}\ \left[\delta_j + \varepsilon_{hj}^{(2)}\right]\right)$$

where $\mathbb{E}_{\varepsilon^{(2)}}(.)$ represents the expectation over the distribution of the random variables $\varepsilon_{hj}^{(2)}$. Because of this interpretation, inclusive values are also denoted as *Emax values*. When the variables $\varepsilon_{hj}^{(2)}$ have a Extreme Value type 1 distribution, this Emax or inclusive value has the simple form presented above as the logarithm of the sum of the exponential of δ's.

The equation for the market shares in the nested Logit model has an intuitive interpretation as the product of between-groups and within-groups market shares. Let $s_g^* \equiv \sum_{j\in\mathscr{J}_g} s_j$ be the aggregate market share of all the products that belong to group g. And let $s_{j|g} \equiv s_j / \sum_{k\in\mathscr{J}_g} s_k$ be the market share of product j *within* its group g. By definition, we have that $s_j = s_g^*\ s_{j|g}$. The nested Logit model implies that within-group market shares have the logit structure $s_{j|g} = \exp\{\delta_j\}/\exp\{I_g\}$, and the group market shares have the logit structure $\exp\{\lambda\ I_g\}/\sum_{g'=0}^{G}\exp\{\lambda\ I_{g'}\}$.

Goldberg and Verboven (2001) estimate a nested logit model for the demand of automobiles in European car markets.

2.3.4 Random Coefficients Logit

Suppose that the utilities $V(\mathbf{X}_j, \widetilde{\xi}_j; \upsilon_h)$ and $u(C;\upsilon_h)$ are linear in parameters, but these parameters are household specific. That is, $U_{hj} = -\alpha_h\ p_j + \mathbf{X}_j\ \beta_h + \xi_j + \varepsilon_{hj}$ where ε's

are still *i.i.d*. Extreme Value Type 1, and

$$\begin{bmatrix} \alpha_h \\ \beta_h \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + v_h \quad \text{with } v_h \sim i.i.d.\ N(0,\Sigma) \tag{2.45}$$

Then, we can write utilities as:

$$U_{hj} = -\alpha\ p_j + \mathbf{X}_j\ \beta + \xi_j + \widetilde{v}_{hj} + \varepsilon_{hj} \tag{2.46}$$

where $\widetilde{v}_{hj} = -v_h^{\alpha}\ p_j + v_h^{\beta_1}\ X_{1j} + ... + v_h^{\beta_K}\ X_{Kj}$ has a heteroskedastic normal distribution. Then, the expression for the market shares is:

$$s_j = \frac{q_j}{H} = \int \frac{\exp\left\{\delta_j + \widetilde{v}_{hj}\right\}}{1 + \sum_{k=1}^{J} \exp\left\{\delta_k + \widetilde{v}_{hk}\right\}}\ f(\widetilde{\mathbf{v}}_h \mid \mathbf{p}, \mathbf{X}, \Sigma)\ d\widetilde{\mathbf{v}}_h \tag{2.47}$$

with $\delta_j \equiv -\alpha\ p_j + \mathbf{X}_j\ \beta + \xi_j$ still representing the mean utility of buying product j.

In general, for any distribution of consumer heterogeneity υ_h, the model implies a mapping between the $J \times 1$ vector of mean utilities $\delta = \{\delta_j : j = 1,2,...,J\}$ and the $J \times 1$ vector of market shares $\mathbf{s} = \{s_j : j = 1,2,...,J\}$:

$$s_j = \sigma_j(\delta \mid \mathbf{p}, \mathbf{X}, \Sigma) \qquad \text{for } j = 1,2,...,J \tag{2.48}$$

or in vector form $\mathbf{s} = \sigma(\delta \mid \mathbf{p}, \mathbf{X}, \Sigma)$.

Berry, Levinsohn, and Pakes (1995) estimate a random coefficients logit model to study the demand of automobiles in the US.

The importance of allowing for random coefficients. In general, the more flexible is the structure of the unobserved consumer heterogeneity, the more flexible and realistic can be the elasticities of substitution between products that the model can generate. The logit model imposes strong, and typically unrealistic, restrictions on demand elasticities. The random coefficients model can generate more flexible elasticities.

In discrete choice models, the *Independence of Irrelevant Alternative* (IIA) is a property of consumer choice that establishes that the ratio between the probabilities that a consumer chooses two alternatives, say j and k, should not be affected by the availability or the attributes of other alternatives:

$$IIA: \frac{\Pr(d_{hj} = 1)}{\Pr(d_{hk} = 1)} \text{depends only on attributes of } j \text{ and } k \tag{2.49}$$

While IIA may be a reasonable assumption when we study the demand of single individual**,** it is quite restrictive when we look at the demand of multiple individuals because these individuals are heterogeneous in their preferences. The logit model implies IIA. In the logit model:

$$\frac{\Pr(d_{hj} = 1)}{\Pr(d_{hk} = 1)} = \frac{s_j}{s_k} = \frac{\exp\left\{-\alpha\ p_j + \mathbf{X}_j\ \beta + \xi_j\right\}}{\exp\left\{-\alpha\ p_k + \mathbf{X}_k\ \beta + \xi_k\right\}} \Rightarrow \text{IIA} \tag{2.50}$$

This property implies a quite restrictive structure for the cross demand elasticities. In the logit model, for $j \neq k$, we have that $\frac{\partial \ln s_j}{\partial \ln p_k} = -\alpha\ p_k\ s_k$, which is the same for any product j. A 1% increase in the price of product k implies the same % increase in the demand of any product other than j. This is very unrealistic.

2.3.5 Berry's Inversion Property

Berry (1994) shows that, under some regularity conditions (more later), the demand system $\mathbf{s} = \sigma(\delta \mid \mathbf{p}, \mathbf{X}, \Sigma)$ is invertible in δ such that there is an inverse function σ^{-1}, and:

$$\delta = \sigma^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma) \tag{2.51}$$

or for a product j, $\delta_j = \sigma_j^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma)$. The form of the inverse mapping σ^{-1} depends on the PDF $f_{\tilde{\nu}}$.

This inversion property has important implications for the estimation of the demand system. Under this inversion, the unobserved product characteristics ξ_j enter additively in the equation $\delta_j = \sigma_j^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma)$. Under this additivity, and the mean independence of the unobservables ξ_j conditional on the exogenous product characteristics $\mathbf{X}$, we can construct moment conditions and obtain GMM estimators of the structural parameters that deal with the endogeneity of prices.

Example: Logit model (Manski, 1983; Berkovec and Rust, 1985)**.** In the logit model, the demand system is $s_j = \exp\{\delta_j\}/D$, where $D \equiv 1 + \sum_{k=1}^{J} \exp\{\delta_k\}$, such that $\ln(s_j) = \delta_j - \ln(D)$. Let s_0 be the market share of the outside good such that, $s_0 = 1 - \sum_{k=1}^{J} s_k$. For the outside good, $s_0 = 1/D$, such that $\ln(s_0) = -\ln(D)$. Combining the equations for $\ln(s_j)$ and $\ln(s_0)$ we have that:

$$\delta_j = \ln(s_j) - \ln(s_0) \tag{2.52}$$

and this equation is the inverse mapping $\sigma_j^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma)$ for the logit model. ■

Example: Nested Logit model. In the nested Logit model, the demand system is $s_j = s_g^* \, s_{j|g}$ where $s_g^* = \exp\{\lambda \, I_g\}/D$ and $s_{j|g} = \exp\{\delta_j\}/\exp\{I_g\}$, such that $\ln(s_j) = \ln(s_g^*) + \ln(s_{j|g})$ with $\ln(s_g^*) = \lambda \, I_g - \ln(D)$ and $\ln(s_{j|g}) = \delta_j - I_g$. For the outside alternative, we have that $\ln(s_0) = -\ln(D)$. Combining these expressions we can obtain that $\ln(s_j) = (\lambda - 1) \, I_g + \ln(s_0) + \delta_j$. And taking into account that $I_g = [\ln(s_g^*) - \ln(s_0)]/\lambda$, we have that:

$$\delta_j = [\ln(s_j) - \ln(s_0)] + \left(\frac{1-\lambda}{\lambda}\right) [\ln(s_g^*) - \ln(s_0)] \tag{2.53}$$

This equation is the inverse mapping σ_j^{-1} for the nested Logit model. ■

We also have a closed-form expression for σ_j^{-1} in the case of the Nested Logit model. However, in general, for the Random Coefficients model we do not have a closed form expression for the inverse mapping σ_j^{-1}. Berry (1994) and Berry, Levinsohn, and Pakes (1995) propose a fixed point algorithm to compute the inverse mapping for the Random Coefficients logit model. They propose the following fixed point mapping: $\delta = F(\delta \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma)$ or $\delta_j = F_j(\delta \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma)$ where:

$$F_j(\delta \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma) \equiv \delta_j + \ln(s_j) - \ln\left(\sigma_j(\delta | \mathbf{p}, \mathbf{X}, \Sigma)\right) \tag{2.54}$$

It is straightforward to see that δ is a fixed point of the mapping $F(\delta \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma)$ if and only if $\delta = \sigma^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma)$. Therefore, finding a solution (fixed point) in δ to the system of equations $\delta = F(\delta \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma)$ is equivalent to finding the inverse function $\sigma^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma)$ at a particular value of $(\mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma)$.

Defnition: Contraction. Let $\mathscr{X}$ be a set in $\mathbb{R}^n$, let $\|.\|$ be the Euclidean distance, and let $f(x)$ be a function from $\mathscr{X}$ into $\mathscr{X}$. We say that $f(x)$ is a *contraction* (with respect to $\mathscr{X}$ and $\|.\|$) if and only if there is a constant $\lambda \in [0,1)$ such that for any pair of values x and x' in $\mathscr{X}$ we have that $\|f(x) - f(x')\| \leq \lambda \|x - x'\|$. ■

Contraction mapping Theorem. If $f : \mathscr{X} \to \mathscr{X}$ is a contraction, then the following results hold. *(A)* there is only one solution in $\mathscr{X}$ to the fixed point problem $x = f(x)$. *(B)* Let x^* be this unique solution such that $x^* = f(x^*)$. For any arbitrary value $x_0 \in \mathscr{X}$ define the sequence $\{x_k : k \geq 1\}$ such that $x_k = f(x_{k-1})$. Then, $\lim_{k\to\infty} x_k = x^*$. ■

Berry (1994) shows that this mapping $F(\delta \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma)$ is a contraction as long as the values of δ are not too small. As established by the *Contraction Mapping Theorem*, this implies that the mapping has a unique fixed point and we can find it by using the *fixed point iteration algorithm*. For this model, the algorithm proceeds as follows.

Fixed Point algorithm

- Start with an initial guess δ^0.

- At iteration $R \geq 1$, we calculate $\sigma_j(\delta^{R-1} | \mathbf{p}, \mathbf{X}, \Sigma)$ for every product j by evaluating the multiple integration expression in equation (2.47) and then we update the vector δ using the updating equation:

$$\delta^R = F_j(\delta^{R-1} \mid \mathbf{s}, \mathbf{p}, \mathbf{X}, \Sigma) = \delta_j^{R-1} + \ln(s_j) - \ln\left(\sigma_j(\delta^{R-1} | \mathbf{p}, \mathbf{X}, \Sigma)\right) \tag{2.55}$$

- Given δ^R, we check for convergence. If $\left\| \delta^R - \delta^{R-1} \right\|$ is smaller than a pre-specified small constant (for instance, 10^{-6}), we stop the algorithm and take δ^R as the solution to the fixed point of the algorithm. Otherwise, we proceed with iteration $R+1$. ■

2.3.6 Dealing with limitations

Discrete choice demand models can deal with some limitations of demand systems in product space.

[1] Representative consumer assumption. The model is micro founded. It takes into account that the shape of demand and price sensitivity is intimately related to consumer heterogeneity in tastes. Therefore, we can estimate demand systems with precision when J is large. In fact, for these models, large J implies more precise estimates.

[2] Too many parameters problem. The number of parameters does not increase with the number of products J but with the number of observable product attributes K.

[3] Instruments for prices. As we describe below, in the regression equation $\sigma^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma) = -\alpha\ p_j + \mathbf{X}_j\ \beta + \xi_j$ we can use the observable exogenous characteristics of other products, $\mathbf{X}_k : k \neq j$, as instruments for price. In the equation for product j, the characteristics of other products, $\{\mathbf{X}_k : k \neq j\}$, are valid instruments for the price of product j. To see this, note that the variables $\{\mathbf{X}_k : k \neq j\}$ are not correlated with the error term ξ_j but they are correlated with the price p_j. The later condition may not be obvious because it depends on an assumption about pricing decisions. Suppose that product prices are the result of price competition between the firms that produce these products. To provide a simple intuition, suppose that there is one firm per product and consider the Logit model of demand. The profit function of firm j is $p_j\ q_j - C_j(q_j) - F_j$ where $C_j(q_j)$ and F_j are the variable and the fixed costs of producing j, respectively. For

the Logit model, $\partial q_j/\partial p_j = -\alpha q_j(1-s_j)$ and the marginal condition of optimality for the price of product j is:

$$p_j = C'_j(q_j) + \frac{1}{\alpha(1-s_j)}$$

Though this is just an implicit equation, it makes it clear that p_j depends (through s_j) on the characteristics of all the products. If $\mathbf{X}_k\beta$ (for $k \neq j$) increases, then s_j will go down, and according to the previous expression the price p_j will also decrease. Therefore, we can estimate the demand parameters by IV using as instruments for prices the characteristics of the other products. We provide further details in the next section.

[4] Problems to predict the demand of new products. Predicting the demand of new products does not require knowing additional parameters. Given the structural parameters β, α, and Σ, we can predict the demand of a new hypothetical product which has never been introduced in the market. Suppose that the new product has observed characteristics $\{x_{J+1}, p_{J+1}\}$ and $\xi_{J+1} = 0$. For the moment, assume also that: (1) incumbent firms do not change their prices after the entry of the new product; and (2) incumbent firms do not exit or introduce new products after the entry of the new product. Then, the demand of the new product is:

$$q_{J+1} = H \int \frac{\exp\{-\alpha\ p_{J+1} + \mathbf{X}_{J+1}\ \beta + \xi_{J+1} + \widetilde{\nu}_{hJ+1}\}}{1 + \sum_{k=1}^{J+1} \exp\{-\alpha\ p_k + \mathbf{X}_k\ \beta + \xi_k + \widetilde{\nu}_{hk}\}}\ f(\widetilde{\mathbf{v}}_h|\mathbf{p},\mathbf{X},\Sigma) \tag{2.56}$$

Note that to obtain this prediction we need also to use the residuals $\{\xi_k\}$ that can be obtained from the estimation of the model. Given any hypothetical new product with characteristics $(X_{J+1},\ p_{J+1},\ \xi_{J+1})$, the model provides the market share of this new product, its demand elasticity, and the effect of introducing this new product on the market share of any pre-existing product.

2.3.7 Estimation

Suppose that the researcher has a dataset from a **single market** at only **one period** but for a product with many varieties: $M = T = 1$ but J is large (for instance, 100 varieties or more). The researcher observes the dataset $\{q_j, \mathbf{X}_j, p_j : j = 1, 2, ..., J\}$. Given these data, the researcher is interested in the estimation of the parameters of the demand system: $\theta = (\alpha, \beta, \Sigma)$. For the moment, we assume that market size H is known to the researcher. But it can be also estimated as a parameter. For the asymptotic properties of the estimators, we consider that $J \to \infty$.

The econometric model is:

$$s_j = \sigma_j(\mathbf{X}, \mathbf{p}, \xi; \theta) \tag{2.57}$$

Unobserved product characteristics ξ are correlated with prices $\mathbf{p}$ (endogeneity). Dealing with endogeneity in nonlinear models where unobservables do not enter additively is complicated. In principle, we would like to avoid using a Maximum Likelihood approach because it requires the specification of how the vector of prices $\mathbf{p}$ depends on the exogenous variables $(\mathbf{X}, \xi)$, and an assumption about the probability distribution of the vector of unobservables ξ. If these assumptions on the supply side of the model are incorrect, the maximum likelihood estimator provides inconsistent estimates of demand

parameters. We would prefer using a method that does not require these additional assumptions.

In this context, an important contribution of Berry (1994) and Berry, Levinsohn, and Pakes (1995) was to show that there is a general class of models with the **invertibility property** described above. This property implies that we can represent the model using a equation where the unobservables ξ enter additively:

$$\sigma_j^{-1}(\mathbf{s} \mid \mathbf{p}, \mathbf{X}, \Sigma) = -\alpha\, p_j + \mathbf{X}_j\, \beta + \xi_j \tag{2.58}$$

Given this representation of the model, we can estimate the structural parameters $\theta = \{\alpha,\ \beta,\ \Sigma\}$ using GMM. The key identification assumption is the mean independence of the unobserved product characteristics and the exogenous product characteristics.

Assumption: $\mathbb{E}\left(\xi_j \mid \mathbf{X}_1, ..., \mathbf{X}_J\right) = 0$.

Generalized Method of Moments (GMM) estimation. Under the previous assumption, we can use the characteristics of other products $(\mathbf{X}_k : k \neq j)$ to construct moment conditions to estimate structural parameters in equation (2.58). For instance, we can use the average characteristics of other products as the vector of instruments, $\frac{1}{J-1}\sum_{k\neq j}\mathbf{X}_k$. It is clear that $\mathbb{E}\left(\frac{1}{J-1}\sum_{k\neq j}\mathbf{X}_k\ \xi_j\right) = 0$, and we can estimate θ using GMM. Suppose that we have a vector of instruments $\mathbf{Z}_j$ (for instance, $\mathbf{Z}_j = [\mathbf{X}_j,\ \frac{1}{J-1}\sum_{k\neq j}\mathbf{X}_k]$) such that the following identification conditions hold:

(ID.1) $\mathbb{E}(\mathbf{Z}_j\ \xi_j) = 0$;

(ID.2) $\dim(\mathbf{Z}_j) \geq \dim(\theta)$;

(ID.3) $\mathbb{E}\left[\left(\frac{\partial \sigma_j^{-1}(\mathbf{s}|\mathbf{p},\mathbf{X},\Sigma)}{\partial \Sigma},\ p_j,\ \mathbf{X}_j\right)' \left(\frac{\partial \sigma_j^{-1}(\mathbf{s}|\mathbf{p},\mathbf{X},\Sigma)}{\partial \Sigma},\ p_j,\ \mathbf{X}_j\right) \mid \mathbf{Z}_j\right]$ is non-singular.

Under conditions (ID.1) to (ID.3), the moment restrictions $\mathbb{E}(\mathbf{Z}_j\ \xi_j) = 0$ can identify the vector of parameters θ.

To obtain the GMM estimator of θ, we replace the population moment restrictions $\mathbb{E}(\mathbf{Z}_j\ \xi_j) = 0$ with their sample counterpart. To do this, we replace the population expectation $\mathbb{E}(.)$ with the sample mean $\frac{1}{J}\sum_{j=1}^{J}(.)$, and the unobservable ξ_j with its expression in terms of observables and parameters of the model. Then, the sample moment conditions becomes:

$$\frac{1}{J}\sum_{j=1}^{J}\mathbf{Z}_j\ \left(\sigma_j^{-1}(\mathbf{s}|\mathbf{p},\mathbf{X},\Sigma) + \alpha\ p_j - \mathbf{X}_j\ \beta\right) = 0 \tag{2.59}$$

If the number of these restrictions (that is, the number of instruments in the vector $\mathbf{Z}_j$) is equal to the number of parameters in θ, then the model is just identified and the GMM estimator is defined as the value of θ that solves exactly this system of sample moment conditions. When the number of restrictions is greater than the number of parameters, the model is over-identified, and the GMM estimator is defined as the value of θ that minimizes a quadratic form of the moment restrictions. Let $m(\theta)$ be function that represents in a compact form the sample moments $\frac{1}{J}\sum_{j=1}^{J}\mathbf{Z}_j\ [\sigma_j^{-1}(\mathbf{s}|\mathbf{p},\mathbf{X},\Sigma) + \alpha\ p_j - \mathbf{X}_j\ \beta]$ as a function of θ. The GMM estimator is defined as:

$$\widehat{\theta} = \arg\min_{\theta}\left[m(\theta)'\ \mathbf{W}\ m(\theta)\right] \tag{2.60}$$

where $\mathbf{W}$ is a weighting matrix.

Choice of instruments

When J is large, a possible concern with the instruments $\frac{1}{J-1}\sum_{k\neq j} X_k$ is that they may have very little sample variability across j. To deal with this problem we can define instruments that take into account some intuitive features on price competition between differentiated products. Product j faces stronger competition if there are other products with similar characteristics. Therefore, we expect that the price of product j declines with the number of its *close neighbors*, where these close neighbors are defined as other products with similar characteristics as product j. To implement this idea, define d^* as the average distance between the observable characteristics of all the products in the market. That is, $d^* = \frac{1}{J(J-1)/2}\sum_{j=1}^{J}\sum_{k>j}\left\|\mathbf{X}_k - \mathbf{X}_j\right\|$. Let $\tau \in (0,1)$ be a small constant such that when the distance between two products is smaller than $\tau\, d^*$ we can say that the two products are very similar (for instance, $\tau = 0.10$). We can define a set of *close neighbors* for product j as:

$$\mathscr{N}_j = \left\{k \neq j : \left\|\mathbf{X}_k - \mathbf{X}_j\right\| \leq \tau d^*\right\} \tag{2.61}$$

Let $\left|\mathscr{N}_j\right|$ represents the number of elements in the set $\mathscr{N}_j$. We can construct the vector of instruments,

$$\mathbf{Z}_j = \left[\mathbf{X}_j,\ \left|\mathscr{N}_j\right|,\ \frac{1}{\left|\mathscr{N}_j\right|}\sum\nolimits_{k\in\mathscr{N}_j}\mathbf{X}_k\right] \tag{2.62}$$

This vector of instruments can have more sample variability than $\frac{1}{J-1}\sum_{k\neq j} X_k$ and it can be also more correlated with p_j.

The vector of instruments $\mathbf{Z}_j$ should have at least as many variables as the number of parameters $\theta = \{\alpha,\ \beta,\ \Sigma\}$. Without further restrictions we have that $\dim(\Sigma) = \frac{K(K+1)}{2}$ where $\dim(X_j) = K$, such that $\dim(\theta) = (K+1) + \frac{K(K+1)}{2}$. Note that the vector of instruments suggested above, $\mathbf{Z}_j = [\mathbf{X}_j, \left|\mathscr{N}_j\right|, \frac{1}{\left|\mathscr{N}_j\right|}\sum_{k\in\mathscr{N}_j}\mathbf{X}_k]$, has only $2K+1$ elements such that the order condition of identification (ID.2) does not hold, that is, $\dim(\mathbf{Z}_j) = 2K+1 < (K+1) + \frac{K(K+1)}{2} = \dim(\theta)$.

Several solutions have been applied to deal with this under-identification problem. A common approach is to impose restrictions on the variance matrix of the random coefficients Σ. The most standard restriction is that Σ is a diagonal matrix (that is, zero correlation between the random coefficients of different product attributes) and there is (at least) one product attribute without random coefficients (i.e. one element in the diagonal of Σ is equal to zero). Under these restrictions, we have that $\dim(\theta) = 2K$ and the order condition of identification holds.

Another approach which has been used in some papers is including additional moments restrictions that come from "micro-moments" or more precisely, market shares for some demographic groups of consumers. This is the approach in Petrin (2002). A third possible approach is to extend the set of instruments beyond $\frac{1}{\left|\mathscr{N}_j\right|}\sum_{k\in\mathscr{N}_j}\mathbf{X}_k$. In this case, one could use the two step method in Newey (1990) to obtain the vector of optimal instruments.

Weak instruments problem

Armstrong (2016) points out a potential inconsistency in this GMM estimator when the number of products is large but the number of markets and firms is small. BLP

instruments affect prices only through price-cost margins. If price-cost margins converge fast enough to a constant as $J \to \infty$, then the GMM-BLP estimator is inconsistent because – asymptotically – the BLP instruments do not have any power to explain prices. This is an extreme case of weak instruments. This is also a potential issue in small samples: the bias and variance of the estimator can be very large in small samples. Armstrong (2016) studies this issue under different data structures.

Suppose that the dataset consists of J products (indexed by j) which belong to N firms (indexed by n) over T markets or time periods (indexed by t), such that we observe prices and quantities p_{jnt} and q_{jnt}. Armstrong studies the predictive power of BLP instruments and the potential inconsistency of the GMM-BLP estimator under different scenarios according to: (a) the form of the demand system, a more specifically whether it is a standard Logit or a random coefficients Logit; and (b) the structure of the panel data, or more precisely, whether the number of products per firm J/N converges to a constant, to zero, or to infinity when J goes to infinity. In the analysis here, we consider that the number of markets T is fixed.

First, consider the case of the standard logit model with single product firms such that $J/N = 1$. Under Bertrand competition, the price equation has the following form (see section 4.3 in chapter 4):

$$p_j = MC_j + \frac{1}{\alpha}\alpha(1 - s_j) \tag{2.63}$$

BLP instruments affect price p_j only through the term $1/(1-s_j)$. If $\sqrt{J}/(1-s_j)$ converges to a constant as $J \to \infty$, then the GMM-BLP estimator is inconsistent: it is asymptotically equivalent to using instrumental variables that are independent of prices and do not have any identification power. This is an extreme case of a problem of weak instruments.

In contrast, when firms are multiproduct the GMM-BLP estimator can be consistent as $J \to \infty$. A multiproduct firm maximizes the joint profit from all its products and obtains price-cost margins which are above those when the products are sold by single-product firms. With this industry data, price-cost margins are larger and converge to a constant more slowly than $\sqrt{J}$. More precisely, as $J \to \infty$, keeping constant the number of firms N, there is a constant $\alpha < 1/2$ such that $J^{\alpha}/(1-s_j)$ converges to a constant. This implies that Logit or random coefficients Logit are consistent as $J \to \infty$.

Armstrong (2016) extends this analysis to the Random Coefficients Logit model. In this model, the GMM-BLP estimator is also inconsistent when the number of markets T and the number of products per firm J/N are fixed. The estimator is consistent when T and N are fixed and firms are asymmetric in their characteristics. Consistency can be also achieved if T goes to infinity and N and J are fixed.

Alternatives to BLP instruments

An alternative to BLP instruments are Hausman-Nevo instruments and Arellano-Bond or Dynamic Panel Data instruments.

Hausman-Nevo instruments. The dataset includes T geographic markets and J products. The T markets belong to R regions where cost shocks are spatially correlated within region, but demand shocks are not. Suppose that the unobservable ξ_{jt} has the

following fixed-effects structure:

$$\xi_{jt} = \xi_j^{(1)} + \xi_t^{(2)} + \xi_{jt}^{(3)} \tag{2.64}$$

and $\xi_{jt}^{(3)}$ is not spatially correlated, that is, for any pair of markets t and t', $\mathbb{E}(\xi_{jt}^{(3)} \xi_{jt'}^{(3)}) = 0$. Under these conditions, we can control for $\xi_j^{(1)}$, and $\xi_t^{(2)}$ using product and market fixed effects, respectively. Furthermore, it is is possible to use prices in other markets to construct valid instrumental variables: that is, variables correlated with price but uncorrelated with the unobservable $\xi_{jt}^{(3)}$. More precisely, let Z_{jt} be the average price in markets in region R (where market t belongs) excluding market t:

$$Z_{jt} = \frac{1}{T_R - 1} \sum_{t' \in \mathscr{T}_R,\, t' \neq t} p_{jt'} \tag{2.65}$$

where T_R is the number of markets in region R, and $\mathscr{T}_R$ is the set of these markets. Since $\xi_{jt}^{(3)}$ is not spatially correlated, we have that $\mathbb{E}(Z_{jt}\ \xi_{jt}^{(3)}) = 0$, and Z_{jt} is correlated with p_{jt} because cost shocks are spatially correlated within the region.

Arellano-Bond or Dynamic Panel Data instruments. Now, consider that the subindex t represents time such that the dataset consists of J products over T periods of time, where T is small and J is large. The demand error term ξ_{jt} has the structure in equation (2.64) and $\xi_{jt}^{(3)}$ is not serially correlated: for any two time periods t and t', $\mathbb{E}(\xi_{jt}^{(3)} \xi_{jt'}^{(3)}) = 0$. Consider the demand equation in first differences, that is, the equation at period t minus the equation at $t-1$:

$$\sigma_j^{-1}(\mathbf{s}_t\ |\mathbf{p}_t, \mathbf{X}_t, \Sigma) - \sigma_j^{-1}(\mathbf{s}_{t-1}\ |\mathbf{p}_{t-1}, \mathbf{X}_{t-1}, \Sigma) = -\alpha\ \Delta p_{jt} + \Delta \mathbf{X}_{jt}\ \beta + \Delta \xi_{jt} \tag{2.66}$$

where $\Delta p_{jt} \equiv p_{jt} - p_{jt-1}$, $\Delta \mathbf{X}_{jt} \equiv \mathbf{X}_{jt} - \mathbf{X}_{jt-1}$, and $\Delta \xi_{jt} \equiv \xi_{jt} - \xi_{jt-1}$. Consider the instruments $\mathbf{Z}_{jt} = \{s_{jt-2},\, p_{jt-2}\}$. Under these assumptions, $\mathbf{Z}_{jt}$ are valid instrumental variables, $\mathbb{E}(\mathbf{Z}_{jt}\ \Delta \xi_{jt}^{(3)}) = 0$. If shocks in marginal costs are serially correlated, then $\mathbf{Z}_{jt}$ is correlated with the price difference Δp_{jt} after controlling for the exogenous regressors $\Delta \mathbf{X}_{jt}$.

Computation of the GMM estimator

Berry, Levinsohn, and Pakes (1995) propose a Nested Fixed Point (NFXP) algorithm to compute the GMM estimator of θ.[6] As indicated by its name, this method can be described in terms of two nested fixed point algorithms: an inner algorithm that consists of fixed point iterations to calculate the values $\sigma_j^{-1}(\mathbf{s}\ |\mathbf{p}, \mathbf{X}, \Sigma)$ for a given value of Σ; and an outer Newton algorithm that minimizes the GMM criterion function with respect to θ.

Let $Q(\theta) = m(\theta)'\ \mathbf{W}\ m(\theta)$ be the GMM criterion function, where $\mathbf{W}$ is a weighting matrix that can give different weights to the moments in the vector $m(\theta)$.[7] The GMM

[6] The term Nested Fixed Point (NFXP) algorithm was coined by Rust (1987) in the context of the estimation dynamic discrete choice structural models. In chapter 7, we describe the algorithm in the context of dynamic discrete choice models.

[7] The GMM estimator is consistent and asymptotically normal for any weighting matrix $\mathbf{W}$ that is semi-positive definite. For instance, the identity matrix is a possible choice. However, there is an optimal weighting matrix that minimizes the variance of the GMM estimator.

estimator is the value $\widehat{\theta}$ that maximizes $Q(\theta)$. The first order conditions of optimality are $\partial Q(\widehat{\theta})/\partial\theta = 0$. Newton's method is based on a first order Taylor's approximation to the condition $\partial Q(\widehat{\theta})/\partial\theta = 0$ around some value θ^0 such that, by the Mean Value Theorem, there exists a scalar $\lambda \in [0,1)$ such that for $\theta^* = (1-\lambda)\theta^0 + \lambda\widehat{\theta}$ we have that:

$$\frac{\partial Q(\widehat{\theta})}{\partial\theta} = \frac{\partial Q(\theta^*)}{\partial\theta} + \frac{\partial^2 Q(\theta^*)}{\partial\theta\partial\theta'}[\widehat{\theta} - \theta^0] \tag{2.67}$$

Therefore, we have that $\partial Q(\theta^*)/\partial\theta + \partial^2 Q(\theta^*)/\partial\theta\partial\theta' \, [\widehat{\theta} - \theta^0] = 0$, and solving for $\widehat{\theta}$, we get:

$$\widehat{\theta} = \theta^0 - \left[\frac{\partial^2 Q(\theta^*)}{\partial\theta\partial\theta'}\right]^{-1}\left[\frac{\partial Q(\theta^*)}{\partial\theta}\right] \tag{2.68}$$

If we knew the value θ^*, then we could obtain the estimator $\widehat{\theta}$ using this expression. However, note that $\theta^* = (1-\lambda)\theta^0 + \lambda\widehat{\theta}$ such that it depends on $\widehat{\theta}$ itself. We have a "chicken and egg" problem. To deal with this problem, Newton's method proposes an iterative procedure.

Newton's algorithm. We start with an initial candidate for estimator, θ^0. At every iteration $R \geq 1$, we update the value of θ, from θ^{R-1} to θ^R, using the following formula:

$$\theta^R = \theta^{R-1} - \left[\frac{\partial^2 Q(\theta^{R-1})}{\partial\theta\partial\theta'}\right]^{-1}\left[\frac{\partial Q(\theta^{R-1})}{\partial\theta}\right] \tag{2.69}$$

iven θ^R and θ^{R-1}, we check for convergence. If $\left\|\theta^R - \theta^{R-1}\right\|$ is smaller than a pre-specified small constant (for instance, 10^{-6}), we stop the algorithm and take θ^R as the estimator $\widehat{\theta}$. Otherwise, we proceed with iteration $R+1$. ■

The Nested Fixed Point algorithm makes it explicit that the evaluation of the criterion function $Q(\theta)$ at any value of θ, and of its first and second derivatives, requires the solution of another fixed point problem to evaluate the inverse mapping $\sigma_j^{-1}(\mathbf{s}|\ \mathbf{p},\mathbf{X},\Sigma)$. Therefore, there are two fixed point algorithms which are nested: the outer algorithm minimizes function $Q(\theta)$ using Newton's method; and the inner algorithm that obtains a fixed point for δ.

Nested Fixed Point algorithm. We start with an initial guess $\theta^0 = (\alpha^0, \beta^0, \Sigma^0)$. At every iteration $R \geq 1$ of the outer (Newton) algorithm, we take Σ^{R-1} and apply the Fixed Point described above (equation (2.55)) to compute the inverse mapping $\sigma_j^{-1}(\mathbf{s}|\ \mathbf{p},\mathbf{X},\Sigma^{R-1})$. We also apply the same Fixed Point algorithm to calculate numerically the gradient vector $\partial\sigma_j^{-1}(\mathbf{s}|\ \mathbf{p},\mathbf{X},\Sigma^{R-1})/\partial\Sigma$ and the Hessian matrix $\partial^2\sigma_j^{-1}(\mathbf{s}|\ \mathbf{p},\mathbf{X},\Sigma^{R-1})/\partial\Sigma\partial\Sigma'$. Given these objects, we can obtain the gradient vector $\frac{\partial Q(\theta^{R-1})}{\partial\theta}$ and the Hessian matrix $\frac{\partial^2 Q(\theta^{R-1})}{\partial\theta\partial\theta'}$. Then, we apply one Newton iteration as described in equation (2.69).

Given θ^R and θ^{R-1}, we check for convergence. If $\left\|\theta^R - \theta^{R-1}\right\|$ is smaller than a pre-specified small constant (for instance, 10^{-6}), we stop the algorithm and take θ^R as the estimator $\widehat{\theta}$. Otherwise, we proceed with iteration $R+1$. ■

The Nested Fixed Point algorithm may be computationally intensive because it requires the repeated solution of the fixed point problem that calculates the inverse

mapping $\sigma_j^{-1}(\mathbf{s}|\ \mathbf{p},\mathbf{X},\Sigma)$, which itself requires Monte Carlo simulation methods to approximate the multiple dimension integrals that define the market shares. Some alternative algorithms have been proposed to reduce the number of times that the inner algorithm is called to computed the inverse mapping. Dubé, Fox, and Su (2012) propose the MPEC algorithm. Lee and Seo (2015) propose a Nested Pseudo Likelihood method in the same spirit as Aguirregabiria and Mira (2002).

2.3.8 Nonparametric identification

Empirical applications of discrete choice models of demand make different parametric assumptions such as the normal distribution of the random coefficients and the additive separability of observable and unobservable product characteristics in the utility function. Berry and Haile (2014) show that these parametric restrictions are not essential for the identification of this type of demand system.

Let v_{hj} be the utility of consumer h for purchasing product j. Define $v_h = (v_{h1}, ..., v_{hJ})$ that has CDF $F_v(v_h|\mathbf{X},\mathbf{p},\xi)$, where $(\mathbf{X},\mathbf{p},\xi)$ are the vectors of characteristics of all the products. In this general model, the interest is in the nonparametric identification of the distribution function $F_v(v_h|\mathbf{X},\mathbf{p},\xi)$. The following assumption plays a key role in the identification results by Berry and Haile (2014).

Assumption BH-1. Unobserved product characteristics, ξ_j, enter in the distribution of consumers' preferences v_h through the term $X_{1j}+\xi_j$, where X_{1j} is one of the observable product attributes.

$$F_v(v_h \mid \mathbf{X},\mathbf{p},\xi) = F_v(v_h \mid \mathbf{X}_{(-1)},\mathbf{p},\ \mathbf{X}_1+\xi) \tag{2.70}$$

where $\mathbf{X}_{(-1)}$ represents the observable product characteristics other than X_1, and $\mathbf{X}_1+\xi$ represents the vector $(X_{11}+\xi_1,\ ...,\ X_{1J}+\xi_J)$.

Assumption BH-1 implies that the marginal rate of substitution between the observable characteristic X_{1j} and the unobservable ξ_j is constant. The restriction that it is equal to one is without lost of generality. Under this assumption, it is clear that:

$$s_j = \Pr\left(j = \arg\max_k v_{ik} \mid \mathbf{X}_{(-1)},\mathbf{p},\ \mathbf{X}_1+\xi\right) = \sigma_j\left(\mathbf{X}_{(-1)},\mathbf{p},\ \mathbf{X}_1+\xi\right) \tag{2.71}$$

For notational convenience, we use ξ_j^* to represent $X_{j1}+\xi_j$ and the vector ξ^* to represent $\mathbf{X}_1+\xi$ such that we can write the market share function as $\sigma_j\left(\mathbf{X}_{(-1)},\mathbf{p},\ \xi^*\right)$.

Assumption BH-2. The mapping $\mathbf{s} = \sigma_j\left(\mathbf{X}_{(-1)},\mathbf{p},\ \xi^*\right)$ is invertible in ξ^* such that we have, $\xi_j^* = X_{j1}+\xi_j = \sigma_j^{-1}\left(\mathbf{s} \mid \mathbf{X}_{(-1)},\mathbf{p}\right)$.

What are the economic conditions that imply this inversion property? **Connected substitutes.** The assumption of **Connected substitutes** can be described in terms of two conditions.

Condition (i). All goods are weak gross substitutes, that is, for any $k \neq j$, $\sigma_j\left(\mathbf{X}_{(-1)},\mathbf{p},\ \xi^*\right)$ is weakly decreasing in ξ_k^*. A sufficient condition is that, as in the parametric model, higher values of ξ_j^* raise the utility of good j without affecting the utilities of other goods.

Condition (ii). "Connected strict substitution". Starting from any inside good, there is a chain of substitution [that is, σ_j is strictly decreasing in ξ_k^*] leading to the outside good.

Connected strict substitution requires only that there is not a subset of products that substitute only among themselves, that is, all the goods must belong in one demand system.

Suppose that we have data from T markets, indexed by t. We can write the inverse demand system as:

$$X_{jt}^{(1)} = \sigma_j^{-1}\left(\mathbf{s}_t \mid \mathbf{X}_{(-1)t}, \mathbf{p}_t\right) - \xi_{jt} \tag{2.72}$$

Let $\mathbf{Z}_t$ be a vector of instruments [we explain below how to obtain these instruments] and suppose that: (a) $\mathbb{E}[\xi_{jt} \mid \mathbf{Z}_t] = 0$; (b) [completeness] if $\mathbb{E}[B(\mathbf{s}_t, \mathbf{X}_{(-1)t}, \mathbf{p}_t) \mid \mathbf{Z}_t] = 0$, then $B(\mathbf{s}_t, \mathbf{X}_{(-1)t}, \mathbf{p}_t) = 0$ *almost surely*. **Important:** Completeness requires that $\dim(\mathbf{Z}_t) \geq \dim(\mathbf{s}_t, \mathbf{X}_{(-1)t}, \mathbf{p}_t)$, that is, instruments for all $2J$ endogenous variables $(\mathbf{s}_t, \mathbf{p}_t)$

Under these conditions, all the inverse functions σ_j^{-1} are nonparametrically identified. Then, ξ_{jt} is identified and we can again invert σ_j^{-1} to identify the demand system σ.

Sources of instruments. Note that we need not only J instruments for prices but also J instruments for for market shares $\mathbf{s}_t$. Instruments for $\mathbf{s}_t$ must affect quantities not only through prices. For instance, supply/marginal cost shifters or Hausman-Nevo are IVs for prices but they are not useful for $\mathbf{s}_t$ because they affect quantities only through prices. The vector $\mathbf{X}_{1t}$ is a natural candidate as IV for $\mathbf{s}_t$**.** By the implicit function theorem, $\dfrac{\partial \sigma^{-1}(\mathbf{s}_t, \mathbf{p}_t)}{\partial \mathbf{s}_t'} = \left[\dfrac{\partial \sigma(\delta_t, \mathbf{p}_t)}{\partial \delta_t'}\right]$. Identifying the effects of $\mathbf{s}_t$ on σ^{-1} is equivalent to identifying the effects of ξ^* on market shares σ. The vector $\mathbf{X}_{1t}$ directly shifts the indices ξ^* , so these are natural instruments for market shares in the inverse demand function.

Identification of Utility [Welfare analysis]. Without further restrictions, identification of the system of demand equations σ does not imply identification of the distribution of random utilities F_v. In general, to identify changes in consumer welfare we need F_v.

Assumption BH-3 [Quasi-linear preferences]. $v_{hj} = \mu_{hj} - p_j$ where the variables μ_{hj} are independent of p_j conditional on $(\xi^*, \mathbf{X}_{(-1)})$.

Under Assumption 3, the distribution F_v is identified from the demand system σ.

2.4 Valuation of product innovations

Product innovation is ubiquitous in most industries, and a key strategy for differentiation. During the last decades we have witnessed a large increase in the number of varieties of different products. Evaluating consumer value of new products, and of quality improvements in existing products, has received substantial attention in the context of: improving Cost of Living Indexes (COLI); costs and benefits of firms' product differentiation; and social value of innovations.

The common approach consists of the following steps: estimating a demand system of differentiated products; using the estimated system to obtain consumers' indirect utility functions (or surplus functions); and comparing consumers' utilities with and without the new product. Typically, one of the two scenarios (with or without the new product) is a counterfactual.

2.4.1 Hausman on cereals

Hausman (1996) presents an application of demand in product space to an industry with many varieties: ready-to eat (RTE) cereals in US. This industry has been characterized by the proliferation of many varieties. We have described the Hausman's data, model, and estimation method in section 2.5 above. Here we describe Hausman's evaluation of the welfare effects of the introduction of a new brand, Cinnamon Cheerios.

Hausman uses the estimated demand system to evaluate the value of a new variety that was introduced during this period: apple-cinamon cheerios (*ACC*). He first obtains the value of the price *ACC* that makes the demand of this product equal to zero. He obtains a virtual price of \$7.14 (double the actual observed price \$3.5). Given this price, he calculates the consumer surplus (alternatively the CV or the EV).

He obtains estimated welfare gains of \$32,268 per city and weekly average with a standard error of \$3,384. Aggregated at the level of US and annually, the consumer-welfare gain is \$78.1 million (or \$0.31 per person per year) which is a sizable amount of consumer's surplus.

Valuation of new products. Consider an individual with preference parameters $(\alpha_h, \beta_h, \varepsilon_h)$ facing a set of products $\mathscr{J}$ with vector of prices $\mathbf{p}$. The indirect utility function is defined as (income effects are assumed away because linearity):

$$v(\mathbf{p}, \alpha_h, \beta_h, \varepsilon_h) = \max_{j \in \mathscr{J}} \left[-\alpha_h p_j + \mathbf{x}_j \beta_h + \xi_j + \varepsilon_{hj} \right] \tag{2.73}$$

To measure aggregate consumer welfare, Hausman uses the money-metric welfare function in McFadden (1974) and Small and Rosen (1981). As indicated by its name, an attractive feature of this welfare measure is that its units are monetary units. To obtain this money metric, we divide utility by the marginal utility of income. The money-metric welfare for consumer h is defined as $\frac{1}{\alpha_h} v(\mathbf{p}, \alpha_h, \beta_h, \varepsilon_h)$. The money metric at the aggregate market level is $W(\mathbf{p}) = \int \frac{1}{\alpha_h} v(\mathbf{p}, \alpha_h, \beta_h, \varepsilon_h) \, dF(\alpha_h, \beta_h, \varepsilon_h)$. For the random coefficients logit model:

$$W(\mathbf{p}) = \int \frac{1}{\alpha_h} \ln \left[\sum_{j=0}^{J} \exp\left\{ -\alpha_h p_j + \mathbf{x}_j \beta_h + \xi_j \right\} \right] dF(\alpha_h, \beta_h) \tag{2.74}$$

We can include $\mathbf{x}$ and $\mathscr{J}$ as explicit arguments of the welfare function: $W(\mathbf{p}, \mathbf{x}, \mathscr{J})$. We can use $W$to measure the welfare effects of a change in: Prices, $W(\mathbf{p}^1, \mathbf{x}, \mathscr{J}) - W(\mathbf{p}^0, \mathbf{x}, \mathscr{J})$; Products characteristics: $W(\mathbf{p}, \mathbf{x}^1, \mathscr{J}) - W(\mathbf{p}, \mathbf{x}^0, \mathscr{J})$; Set of products: $W(\mathbf{p}, \mathbf{x}, \mathscr{J}^1) - W(\mathbf{p}, \mathbf{x}, \mathscr{J}^0)$.

Some limitations.

[1] Problems to evaluate **radical innovations**. That is, innovations that introduce a product with new characteristics that previous products did not have in any amount.

[2] With **logit errors**, there is very limited "crowding" of products such that the welfare function goes infinity when $J \to \infty$ even if all J have exactly the same attributes X and ξ. See section 2.4.4 below on the Logit model and the value of new products.

[3] **Outside alternative**. Unobserved "qualities" ξ_{jt} are relative to the outside alternative. For instance, if there exist quality improvements in the outside alternative, then this approach underestimates the welfare improvements in this industry.

2.4.2 Trajtenberg (1989)

Trajtenberg (1989) on computed tomography scanners. The computed tomography (CT) scanner is considered a key innovation in imaging diagnosis in medicine during the 1970s. The first was installed in the US in 1973, and soon after 20 firms entered in this market with different varieties, General Electric being the leader. Clients are hospitals. Three characteristics are key to scanner quality: scan time, image quality, and reconstruction time.

Data. The data contains 55 products and covers the period 1973 to 1981. Observable product characteristics are price, scan speed, resolution, and reconstruction speed. The quantity variable is sales in the US market. The data includes also information on the identity and attributes of the buying hospital. It is hospital-year level data and the dependent variable is the product choice of hospital h at year t.

Model. The model is a nested logit where scanners are divided in two groups depending on the part of the body for which the scanner is designed to scan. Then, the groups are "head scanners" and "body scanners". The utility function is quadratic in the three product attributes (other than price).

Estimation results. The estimation method does not account for potential endogeneity of prices. The estimated elasticity of substitution between the two groups is very close to zero. That is, it seems that head scanners and body scanners are very different products and there is almost zero substitution between these two groups. The estimated coefficient for price – parameter α – is significantly positive (correct sign) for head scanners but it is negative (wrong sign) for body scanners. The wrong sign of this parameter estimate is likely an implication of the endogeneity of price: that is, the positive correlation between price and unobserved product quality.

Welfare effects. The counterfactual experiment consists of eliminating all CT scanner products, keeping only the outside product. The estimated welfare effect of CT scanners during this period is \$16 million of 1982. Using data on firms' R&D investment, Trajtenberg obtains a social rate of return of 270%. That is, every dollar of investment in the R&D of CT scanners generate 2.7 dollars in return. This is a very substantial rate of return.

2.4.3 Petrin (2002) on minivans

The aim of this study was to evaluate the consumer welfare gains from the introduction of a new type of car, the minivan. Estimation of a BLP demand system of automobiles. Combining market level and micro moments. Observing average family size conditional on the purchase of a minivan and asking the model to match this average helps to identify parameters that capture consumer taste for the characteristics of minivans.

In 1984, Chrysler introduced its own minivan: the Dodge caravan. It was an immediate success. GM and Ford responded by quickly introducing their own minivans in 1985. By 1998, there were 6 firms selling a total of 13 different minivans, with Chrysler being the leader with a market share of 44% within the minivan market segment.

Data. It is a product-year panel dataset during the period 1981-1993 and with $J =$ 2407 products. Variables in the dataset include quantity sold in the US market, price,

acceleration, dimensions, drive type, fuel efficiency, and indicators for luxury car, SUV, minivans, and full-size vans.

The dataset also includes consumer level information from the Consumer expenditure survey (CEX). The CEX links demographics of purchasers of new vehicles to the type of vehicles they purchase. In the CEX, we observe 2,660 new vehicle purchases over the period and sample. This micro-level data is used to estimate the probabilities of new vehicle purchases for different income groups. Observed purchases of minivans (120), station wagons (63), SUVs (131), and full-size vans (23). Used to estimate average family size and age of purchasers of each of these vehicle types.

FAMILY VEHICLE SALES AS A PERCENTAGE OF TOTAL VEHICLE SALES: U.S. AUTOMOBILE MARKET, 1981–93

Year	Minivans (1)	Station Wagons (2)	Sport-Utilities (3)	Full-Size Vans (4)	Minivans and Station Wagons (5)	U.S. Auto Sales (Millions) (6)
1981	.00	10.51	.58	.82	10.51	7.58
1982	.00	10.27	.79	1.17	10.27	7.05
1983	.00	10.32	3.51	1.04	10.32	8.48
1984	1.58	8.90	5.51	1.20	10.48	10.66
1985	2.32	7.33	6.11	1.05	9.65	11.87
1986	3.63	6.70	5.73	.85	10.43	12.21
1987	4.86	6.47	6.44	.73	11.33	11.21
1988	5.97	5.14	7.18	.69	11.11	11.76
1989	6.45	4.13	7.47	.61	10.58	11.06
1990	7.95	3.59	7.78	.27	11.54	10.51
1991	8.29	3.05	7.80	.29	11.34	9.75
1992	8.77	3.07	9.33	.39	11.84	10.12
1993	9.93	3.02	11.66	.29	12.95	10.71

Figure 2.1: Petrin (2002) Market shares by type of automobile

Estimates. Tables 2.2 to 2.4 present estimates of demand parameters separated in three groups: price coefficients, marginal utilities of product characteristics, and random coefficients.

Price effects of the introduction of minivans. Petrin used the estimated model to implement the counterfactual experiment of eliminating minivan cars from consumers' choice set. This experiment takes into account that in the counterfactual scenario without minivans the equilibrium prices of all the products will change. Table 2.6 presents equilibrium prices with and without minivans. The introduction of minivans (particularly, Dodge caravan) had an important negative effect on the prices of many substitutes that were top-selling vehicles in the large-sedan and wagon segments of the market. There were also some price increases due to cannibalization of own products.

Welfare effects. The preferred estimates are those of the model with random coefficients, using BLP instruments, and using micro moments. Based on these estimates, the mean

TABLE 4
PARAMETER ESTIMATES FOR THE DEMAND-SIDE EQUATION

Variable	OLS Logit (1)	Instrumental Variable Logit (2)	Random Coefficients (3)	Random Coefficients and Microdata (4)
		A. Price Coefficients (α's)		
α_1	.07	.13	4.92	7.52
	(.01)**	(.01)**	(9.78)	(1.24)**
α_2			11.89	31.13
			(21.41)	(4.07)**
α_3			37.92	34.49
			(18.64)**	(2.56)**

Figure 2.2: Petrin (2002) Parameter estimates. Price coefficients

per capita Compensated Variation of introducing minivans is \$1247. This is a very substantial welfare gain. Petrin compares this estimated welfare gain with the ones using other models and estimates of the model: OLS logit; IV logit; and IV BLP without micro moments. These other models and methods imply estimated welfare gains that are substantially smaller than the preferred model. This is mainly because these methods under-estimate the marginal utility of income parameters.

Petrin also provides a decomposition of the welfare gains in the contribution of product characteristics x_j and ξ_j, and of the logit errors ε_{hj}. For the preferred model, the mean per capita welfare gain of \$1247 is decomposed into a contribution of \$851 from product characteristics, and a contribution of \$396 from the logit errors. The other models and methods imply very implausible contributions from the logit errors.

2.4.4 Logit and new products

The Logit errors can have unrealistic implications on the evaluation of welfare gains. Because of these errors, welfare increases unboundedly (though concavely) with J. To illustrate this, consider the simpler case where all the products are identical except for the logit errors. In this case, the aggregate welfare function is $W = \ln(\sum_{j=0}^{J} \exp\{\delta\}) = \delta \ln(J+1)$, which is an increasing and concave function of the number of products J. Though the BLP or Random Coefficients-Logit model limits the influence of the logit errors, this model is still subject to this problem.

Ackerberg and Rysman (2005) propose a simple modification of the logit model that can contribute to correct for this problem. Consider a variation of the BLP model where the dispersion of the logit errors depends on the number of products in the market. For $j > 0$, $U_{hj} = -\alpha_h p_j + \mathbf{x}_j \beta_h + \xi_j + \sigma(J)\ \varepsilon_{hj}$. The parameter $\sigma(J)$ is strictly decreasing in J and it goes to 0 as J goes to ∞. As J increases, the differentiation from the $\varepsilon's$

RANDOM COEFFICIENT PARAMETER ESTIMATES

	RANDOM COEFFICIENTS (γ's)	
VARIABLE	Uses No Microdata (1)	Uses CEX Microdata (2)
Constant	1.46	3.23
	(.87)*	(.72)**
Horsepower/weight	.10	4.43
	(14.15)	(1.60)**
Size	.14	.46
	(8.60)	(1.07)
Air conditioning standard	.95	.01
	(.55)*	(.78)
Miles/dollar	.04	2.58
	(1.22)	(.14)**
Front wheel drive	1.61	4.42
	(.78)**	(.79)**
γ_{mi}	.97	.57
	(2.62)	(.10)**
γ_{sw}	3.43	.28
	(5.39)	(.09)**
γ_{su}	.59	.31
	(2.84)	(.09)**
γ_{fw}	4.24	.42
	(32.23)	(.21)**

Figure 2.3: Petrin (2002) Parameter estimates. Random coefficients

becomes less and less important. Function $\sigma(J)$ can be parameterized and its parameters can be estimated together with the rest of the model. Though Ackerberg and Rysman consider this approach, they favor a similar approach that is simpler to implement. They consider the model:

$$U_{hj} = -\alpha_h p_j + \mathbf{x}_j \beta_h + \xi_j + f(J,\gamma) + \varepsilon_{hj} \tag{2.75}$$

where $f(J,\gamma)$ is a decreasing function of J parameterized by γ. For instance, $f(J,\gamma) = \gamma \ln(J)$. It can be also extended to a nested logit version. For group g: $f_g(J,\gamma) = \gamma_g \ln(J_g)$. The reason for the specification $f(J,\gamma)$ instead of $\sigma(J,\gamma)$ is simplicity in estimation.

2.4.5 Product complementarity

The class of discrete choice demand models we have considered so far rules out complementarity between products. This is an important limitation in some relevant contexts. For instance, this is a significant limitation in the evaluation of the merger between two firms producing complements, such as Pepsico and Frito-Lay, or in the evaluation of the welfare effects of new products that may complement with existing products. This is also a significant limitation when considering industries with both substitution and complementarity effects. Examples include radio stations that play recorded music, movies based on a book novel and the book itself, or, arguably, Uber and taxis. Gentzkow (2007) extends the McFadenn / BLP framework to allow for complementarity, and studies the demand and welfare effect of online newspapers.

Model

Consumers can choose bundles of products. We start with a simple example. There are two products A and B. The set of possible choices for a consumer is $\{0, A, B, AB\}$. The

PARAMETER ESTIMATES FOR THE COST SIDE
Dependent Variable: Estimated (Log of) Marginal Cost

Variable (τ's)	Parameter Estimate	Standard Error
Constant	1.50	.08
ln(horse power/weight)	.84	.03
ln(weight)	1.28	.04
ln(MPG)	.23	.04
Air conditioning standard	.24	.01
Front wheel drive	.01	.01
Trend	−.01	.01
Japan	.12	.01
Japan × trend	−.01	.01
Europe	.47	.03
Europe × trend	−.01	.01
ln(q)	−.05	.01

Figure 2.4: Petrin (2002) Parameter estimates. Marginal Cost

utilities of these choice alternatives are 0, u_A, u_B, and $u_{AB} = u_A + u_B + \Gamma$, respectively. The parameter Γ measures the degree of demand complementarity between products A and B. The indirect utilities u_A and u_A have the following form: $u_A = \beta_A - \alpha p_A$ and $u_B = \beta_B - \alpha p_B$, where p_A and p_B are prices. Let $\mathbb{P}_j = \Pr(u_j = \max\{0, u_A, u_B, u_{AB}\})$ be the probability or proportion of consumers that choose alternative j. Importantly, in contrast to the discrete choice demand models considered above, in this model $\mathbb{P}_A$ and $\mathbb{P}_B$ are not the market shares of products A and B, respectively. To obtain the market shares of these products we should take into account the share of consumers buying bundle AB. Let s_A and s_B be the market shares of products A and B, respectively. We have that:

$$\begin{aligned} s_A &= \mathbb{P}_A + \mathbb{P}_{AB} \\ s_B &= \mathbb{P}_B + \mathbb{P}_{AB} \end{aligned} \tag{2.76}$$

The substitutability or complementarity between products A and B depends on the cross-price derivatives in the demand of these products. Products A and B are substitutes if $\partial s_A / \partial p_B > 0$, and they are complements if $\partial s_A / \partial p_B < 0$. Complements / substitutes is closely related to the sign of Γ. Figure 2.7 illustrates this relationship.

We represent u_A on the horizontal axis and u_B on the the vertical axis. We consider three cases: case 1 with $\Gamma = 0$; case 2 with $\Gamma > 0$; and case 3 with $\Gamma < 0$. For each case, we partition the space in four regions where each region represents the values (u_A, u_B) for which an alternative is the optimal choice. Consider the effect of a small increase in the price of product B. Because it is a marginal increase, it affects only those consumers who are in the frontier of the choice sets. More precisely, the increase in p_B implies that all the frontiers shift vertically and upward. That is, to keep the same choice as with previous prices the utility u_B should be larger.

EQUILIBRIUM PRICES WITH AND WITHOUT THE MINIVAN, 1984:
1982–84 CPI-ADJUSTED DOLLARS

	PRICE			
	With Minivan	Without Minivan	ΔPRICE	% ΔPRICE
	A. Largest Price Decreases on Entry			
GM Oldsmobile Toronado (large sedan)	15,502	15,643	−141	.90
GM Buick Riviera (large sedan)	15,379	15,519	−139	.89
GM Buick Electra (large sedan)	12,843	12,978	−135	1.04
GM Chevrolet Celebrity (station wagon)	8,304	8,431	−127	1.51
Ford Cadillac Eldorado (large sedan)	19,578	19,704	−126	.64
Ford Cadillac Seville (large sedan)	21,625	21,749	−125	.57
GM Pontiac 6000 (station wagon)	9,273	9,397	−123	1.31
GM Oldsmobile Ciera (station wagon)	9,591	9,714	−123	1.27
GM Buick Century (station wagon)	8,935	9,056	−121	1.34
GM Oldsmobile Firenza (station wagon)	7,595	7,699	−104	1.35
	B. Largest Price Increases on Entry			
Chrysler LeBaron (station wagon)	9,869	9,572	297	3.10
Volkswagen Quattro (station wagon)	13,263	13,079	184	1.41
Chrysler (Dodge) Aries K (station wagon)	7,829	7,659	170	2.22
AMC Eagle (station wagon)	10,178	10,069	109	1.08

Figure 2.5: Petrin (2002) Prices with and without minivans

In case 1 with $\Gamma = 0$, this implies that P_{AB} declines and P_A increases but they do it by the same absolute magnitude such that $s_A = P_A + P_{AB}$ does not change. Therefore, with $\Gamma = 0$, we have that $\partial s_A / \partial p_B = 0$ and products A and B are neither substitutes nor complements.

In case 2 with $\Gamma > 0$, we can distinguish two different types of marginal consumers: those located in a point like m and those in a point like o in Figure 2.7 panel 2. For consumers in point m, an increase in p_B makes them switch from choosing the bundle AB to choosing A. As in case 1, this change does not affect the demand of product A. However, we have now also the consumers in point o. These consumers switch from buying the bundle AB to buying nothing. This implies a reduction in P_{AB} without an increase in P_A such that it has a negative effect on the demand of product A. Therefore, with $\Gamma > 0$, we have that $\partial s_A / \partial p_B < 0$ and products A and B are complements in demand.

In case 3 with $\Gamma < 0$, we can also distinguish two different types of marginal consumers: those located in a point like m and those in a point like o in Figure 2.7 panel 3. Similarly to the previous two cases, for consumers in point m, an increase in p_B does not have any effect on the demand of product A. For consumers in point o, an increase in p_B makes them switch from buying product B to buying product A. This implies an increase in the demand of product A. Therefore, with $\Gamma < 0$, we have that $\partial s_A / \partial p_B > 0$ and products A and B are substitutes in demand.

Suppose that: $u_{hA} = \beta_A - \alpha\ p_A + v_{hA}$; and $u_{hB} = \beta_B - \alpha\ p_B + v_{hB}$. Allowing for correlation between unobservables v_{hA} and v_{hB} is very important. Observing that frequent online readers are also frequent print readers might be evidence that the products in question are complementary, or it might reflect the correlation between unobservable tastes for goods. Suppose that (v_{hA}, v_{hB}) are standard normals with correlation ρ. The

AVERAGE COMPENSATING VARIATION CONDITIONAL ON MINIVAN PURCHASE, 1984: 1982–84 CPI-ADJUSTED DOLLARS

	OLS Logit	Instrumental Variable Logit	Random Coefficients	Random Coefficients and Microdata
Compensating variation:				
Median	9,573	5,130	1,217	783
Mean	13,652	7,414	3,171	1,247
Welfare change from difference in:				
Observed characteristics ($\delta_j+\mu_{ij}$)	−81,469	−44,249	−820	851
Logit Error (ϵ_{ij})	95,121	51,663	3,991	396
Income of minivan purchasers:				
Estimate from model	23,728	23,728	99,018	36,091
Difference from actual (CEX)	−15,748	−15,748	59,542	−3,385

Figure 2.6: Petrin (2002) Consumer welfare effects of minivans

parameters of the model are: β_A, β_B, α, ρ, Γ. The researcher (with consumer level data) observes prices and bundles market shares: $\mathbb{P}_A$, $\mathbb{P}_B$, $\mathbb{P}_{AB}$.

Identification

Even with micro-level data with information on shares $\mathbb{P}_A$, $\mathbb{P}_B$, $\mathbb{P}_{AB}$, the parameters (β_A, β_B, α, ρ, Γ) are not identified. Even if α is known, we have 3 data points and 4 parameters. Without further restrictions, a high value of P_{AB} can be explained by either high Γ or high ρ. We want to distinguish between these two interpretations because they have different economic and policy implications. Gentzkow considers two sources of identification: (1) Exclusion restrictions; and (2) Panel data and restrictions on the structure of the unobservables.

(a) Exclusion restrictions. Suppose that there is an exogenous consumer characteristic (or vector) z that enters in consumer valuation of product A but not of product B: $\beta_A(z)$, but β_B does not depend on z. For instance, if B is a print newspaper and A is its online version, z could be Internet access at work (at home could be more endogenous). Suppose z is binary for simplicity. Now, the data $[\mathbb{P}_A(z), \mathbb{P}_B(z), \mathbb{P}_{AB}(z)\text{: } z \in \{0,1\}]$ can identify $\beta_A(0)$, $\beta_A(1)$, β_B, Γ, and ρ. Intuition: if $\Gamma > 0$ (complementarity), then $z = 1$ should increase $P_A(z)$ and $P_{AB}(z)$. Otherwise, if $\Gamma = 0$, then $z = 1$ should increase $P_A(z)$ but not $P_{AB}(z)$.

(2) Panel Data. Suppose that we observe consumer choices at different periods of time, and suppose that: $v_{jht} = \eta_{jh} + \varepsilon_{jht}$. The time-invariant effects η_{Ah} and η_{Bh} are correlated with each other; but ε_{Aht} and ε_{Aht} are independent and i.i.d. over h,t. Preference parameters are assumed to be time invariant. Suppose that $T = 2$. We have 8 possible choice histories, 7 probabilities, and 4 parameters: β_A, β_B, Γ, and ρ. Identification intuition: if $\Gamma > 0$, changes over time in demand should be correlated

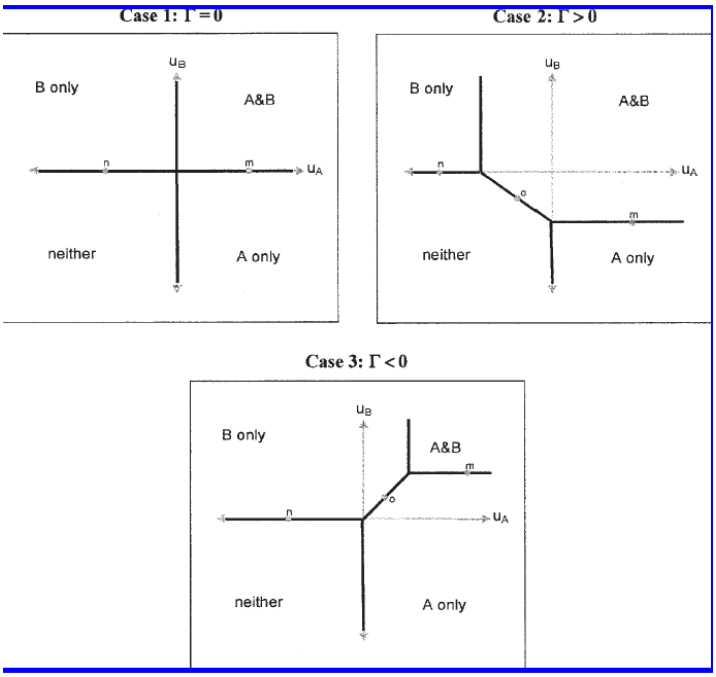


Figure 2.7: Gentzkow (2007) Choice regions and effect of Γ

between the two goods. If $\Gamma = 0$, changes over time should be uncorrelated between goods.

Data

Survey: 16,179 individuals in Washington DC, March-2000 and Feb. 2003. Information on individual and household characteristics, and readership of: print local newspapers read over last week; major local online newspapers over last week. Two main local print newspapers: Times and Post. One main online newspaper: post.com. Three products: Times, Post, and post.com. Outside alternative being all the other local papers.

Empirical results

Estimation results from reduced-form OLS regressions and from a structural model without heterogeneity suggest that the print and online editions of the Post are strong complements. According to those estimates, the addition of the post.com to the market increases profits from the Post print edition by \$10.5 million per year. However, **properly accounting for consumer heterogeneity changes the conclusions substantially**. Estimates of the model with both observed and unobserved heterogeneity show that the print and online editions are **significant substitutes**. Figure 2.8 presents estimates of the Γ parameters.

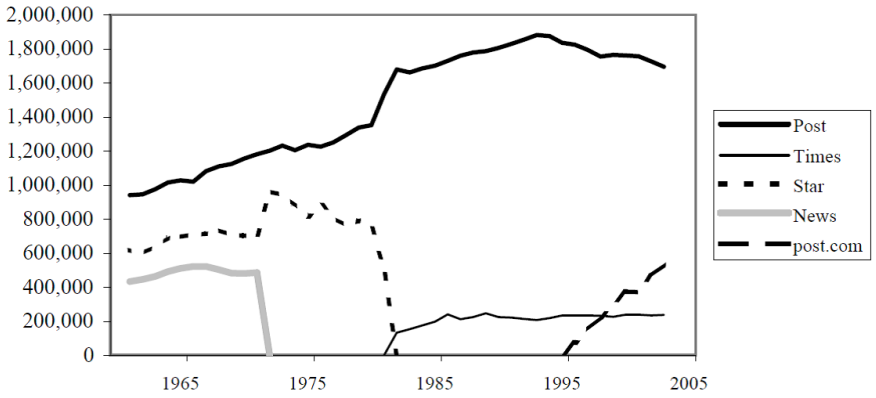


Figure 2.8: Gentzkow (2007) Time series of readers

Table 2.9 presents estimates of the effect of the online edition on the print edition. Raising the price of the Post by \$0.10 would increase post.com readership by about 2%. Removing the post.com from the market entirely would increase readership of the Post by 27,000 readers per day, or 1.5%. The estimated \$33.2 million of revenue generated by the post.com comes at a cost of about \$5.5 million in lost Post readership. For consumers, the online edition generated a per-reader surplus of \$0.30 per day, implying a total welfare gain of \$45 million per year. Reduced-form OLS regressions and a structural model without heterogeneity suggest that the print and online editions of the Post are strong complements.

TABLE 6—PARAMETER ESTIMATES FROM FULL MODEL: OTHER

Interaction terms		*Excluded variables (coefficient in utility of post.com)*	
Post-post.com	−1.285**	Internet at work	1.357**
	(0.2307)		(0.180)
Post-Times	0.0809	Fast connection	0.146
	(0.2479)		(0.193)
post.com-*Times*	−1.231**	Use for education-related	0.361
	(0.4832)		(0.212)
Nonlinear parameters		Use for work	0.582**
τ	6.846**		(0.222)
	(0.5027)		
γ	0.0454**		
	(0.0179)		

Figure 2.9: Gentzkow (2007) Estimates of Γ parameters

2.5 Appendix

2.5.1 Derivation of demand systems

The Linear Expenditure System

The utility function has the Stone-Geary form:

$$U = (q_0 - \gamma_0)^{\alpha_0} (q_1 - \gamma_1)^{\alpha_1} \dots (q_J - \gamma_J)^{\alpha_J} \tag{2.77}$$

The marginal utility of product j is $U_j = \alpha_j \dfrac{U}{q_j - \gamma_j}$. Therefore, the marginal condition of optimality $U_j - \lambda p_j = 0$ implies that $\alpha_j \dfrac{U}{q_j - \gamma_j} = \lambda p_j$, or equivalently,

$$p_j\, q_j = \alpha_j \frac{U}{\lambda} + p_j\, \gamma_j \tag{2.78}$$

Adding up this expression over the $J+1$ products and using the budget constraint and the restriction $\sum_{j=0}^{J} \alpha_j = 1$, we have that $y = \frac{U}{\lambda} + \sum_{j=0}^{J} p_j\, \gamma_j$ such that:

$$\frac{U}{\lambda} = y - \sum_{j=0}^{J} p_j \gamma_j \tag{2.79}$$

Plugging this expression into equation $p_j\, q_j = \alpha_j \frac{U}{\lambda} + p_j\, \gamma_j$, we obtain the equations of the Linear Expenditure System:

$$q_j = \gamma_j + \alpha_j \left[\frac{y - P_\gamma}{p_j} \right] \tag{2.80}$$

where P_γ is the aggregate price index $\sum_{i=0}^{J} p_i\, \gamma_i$.

TABLE 8—IMPACT OF THE ONLINE EDITION ON DEMAND FOR PRINT

Case 1: Full model	
Cross-price derivative	8,358
	(1,436)
Change in print readership	−26,822
	(4,483)
Change in print profits	−$ 5,466,846
	(913,699)
Case 2: Model with observable characteristics only	
Cross-price derivative	−8,421
	(752)
Change in print readership	25,655
	(2,270)
Change in print profits	$ 5,229,009
	(462,771)
Case 3: Model with no heterogeneity	
Cross-price derivative	−16,143
	(702)
Change in print readership	51,897
	(2,254)
Change in print profits	$10,577,720
	(459,464)

Figure 2.10: Gentzkow (2007) Effect of Online on Print

Constant Elasticity of Substitution demand system

The utility function is:

$$U = \left(\sum_{j=0}^{J} q_j^{\sigma} \right)^{1/\sigma} \tag{2.81}$$

The marginal utility is $U_j = \dfrac{q_j^{\sigma-1}\, U}{\sum_{i=0}^{J} q_i^{\sigma}}$ such that the marginal condition of optimality for product j is $\dfrac{q_j^{\sigma-1}\, U}{\sum_{i=0}^{J} [\alpha_i\, q_i]^{\sigma}} - \lambda\ p_j = 0$. We can re-write this condition as:

$$\frac{q_j^{\sigma-1}}{\sum_{i=0}^{J} [\alpha_i\, q_i]^{\sigma}} \frac{U}{\lambda} = p_j\, q_j \tag{2.82}$$

Adding the expression over the $J+1$ products, we have that:

$$y = \sum_{j=0}^{J} p_j\, q_j = \frac{U}{\lambda} \tag{2.83}$$

That is, $\dfrac{U}{\lambda} = y$. Plugging this result into the marginal condition for product j above, and taking into account that $\sum_{i=0}^{J} q_i^{\sigma} = U^{\sigma}$, we have that:

$$\frac{q_j^{\sigma}}{U^{\sigma}} y = p_j\, q_j \tag{2.84}$$

This equation can be re-written as:

$$q_j = \left[\frac{y}{p_j}\right]^{1/(1-\sigma)} \left[\frac{1}{U^\sigma}\right]^{1/(1-\sigma)} \tag{2.85}$$

Plugging this expression in the definition of the utility function, we can get:

$$U = \left(\sum_{j=0}^{J} \left[\frac{y}{p_j}\right]^{\sigma/(1-\sigma)}\right)^{1/\sigma} \left[\frac{1}{U^\sigma}\right]^{1/(1-\sigma)} \tag{2.86}$$

Solving for U, we have:

$$U = \left(\sum_{j=0}^{J} \left[\frac{y}{p_j}\right]^{\sigma/(1-\sigma)}\right)^{(1-\sigma)/\sigma} = \frac{y}{P_\sigma} \tag{2.87}$$

where P_σ is the price index:

$$P_\sigma = \left(\sum_{j=0}^{J} \left[p_j\right]^{-\sigma/(1-\sigma)}\right)^{-(1-\sigma)/\sigma} \tag{2.88}$$

Finally, plugging these results into the expression $q_j = \left[\frac{y}{p_j}\right]^{1/(1-\sigma)} \left[\frac{1}{U^\sigma}\right]^{1/(1-\sigma)}$, we get the CES demand equations:

$$q_j = \frac{y}{P_\sigma} \left[\frac{p_j}{P_\sigma}\right]^{-1/(1-\sigma)} \tag{2.89}$$

2.6 Exercises

2.6.1 Exercise 1

To answer the questions in this exercise you need to use the dataset `verboven_cars.dta` Use this dataset to implement the estimations describe below. Please, provide the STATA code that you use to obtain the results. For all the models that you estimate below, impose the following conditions:

- For market size (number of consumers), use Population/4, that is, `pop/4`

- Use prices measured in euros (`eurpr`).

- For the product characteristics in the demand system, include the characteristics: `hp, li, wi, cy, le,` and `he.`

- Include also as explanatory variables the market characteristics: ln(pop) and log(gdp).

- In all the OLS estimations include fixed effects for market (`ma`), year (`ye`), and brand (`brd`).

- Include the price in logarithms, that is, ln(`eurpr`).

- Allow the coefficient for log-price to be different for different markets (countries). That is, include as explanatory variables the log price, but also the log price interacting (multiplying) each of the market (country) dummies except one country dummy (say the dummy for Germany) that you use as a benchmark.

Question 1.1 Obtain the OLS-Fixed effects estimator of the Standard logit model. Interpret the results.

Question 1.2 Test the null hypothesis that all countries have the same price coefficient.

Question 1.3 Based on the estimated model, obtain the average price elasticity of demand for each country evaluated at the mean values of prices and market shares for that country.

2.6.2 Exercise 2

The STATA datafile `eco2901_problemset_01_2012_airlines_data.dta` contains a panel dataset of the US airline industry in 2004. A market is a *route* or directional city-pair, for instance, round-trip Boston to Chicago. A product is the combination of route (m), airline (f), and the indicator of stop flight or nonstop flight. For instance, a round-trip Boston to Chicago, non-stop, with American Airlines is an example of product. Products compete with each other at the market (route) level. Therefore, the set of products in market m consists of all the airlines with service in that route either with nonstop or with stop flights. The dataset contains $2,950$ routes, 4 quarters, and 11 airlines (where the airline "Others" is a combination of multiple small airlines). The following table includes the list of variables in the dataset and a brief description.

Variable name		Description
route_city	:	Route: Origin city to Destination City
route_id	:	Route: Identification number
airline	:	Airline: Name (Code)
direct	:	Dummy of Non-stop flights
quarter	:	Quarter of year 2004
pop04_origin	:	Population Origin city, 2004 (in thousands)
pop04_dest	:	Population Destination city, 2004 (in thousands)
price	:	Average price: route, airline, stop/nonstop, quarter (in dollars)
passengers	:	Number of passengers: route, airline, stop/nonstop, quarter
avg_miles	:	Average miles flown for route, airline, stop/nonstop, quarter
HUB_origin	:	Hub size of airline at origin (in million passengers)
HUB_dest	:	Hub size of airline at destination (in million passengers)

In all the models of demand that we estimate below, we include time-dummies and the following vector of product characteristics:

{ price, direct dummy, avg_miles, HUB_origin, HUB_dest, airline dummies

In some estimations we also include market (route) fixed effects. For the construction of market shares, we use as measure of market size (total number of consumers) the average population in the origin and destination cities, in number of people, that is, 1000*(pop04_origin + pop04_dest)/2.

Question 2.1. Estimate a Standard Logit model of demand: (a) by OLS without route fixed effects; (b) by OLS with route fixed effects. Interpret the results. What is the average consumer willingness to pay (in dollars) for a nonstop flight (relative to a stop flight), ceteris paribus? What is the average consumer willingness to pay for one million more people of hub size in the origin airport, ceteris paribus? What is the average consumer willingness to pay for Continental relative to American Airlines, ceteris paribus? Based on the estimated model, obtain the average elasticity of demand for Southwest products. Compare it with the average elasticity of demand for American Airline products.

Question 2.2. Consider a Nested Logit model where the first nest consists of the choice between groups "Stop", "Nonstop", and "Outside alternative", and the second nest consists in the choice of airline. Estimate this Nested Logit model of demand: (a) by OLS without route fixed effects; (b) by OLS with route fixed effects. Interpret the results. Answer the same questions as in Question 2.1.

Question 2.3. Consider the Nested Logit model in Question 2.2. Propose and implement an IV estimator that deals with the potential endogeneity of prices. Justify your choice of instruments, for instance, BLP, or Hausman-Nevo, or Arellano-Bond, ... Interpret the results. Compare them with the ones from Question 2.2.

Question 2.4. Given your favorite estimation of the demand system, calculate price-cost margins for every observation in the sample. Use these price cost margins to estimate a marginal cost function in terms of all the product characteristics, except price. Assume constant marginal costs. Include also route fixed effects. Interpret the results.

Question 2.5. Consider the route Boston to San Francisco ("BOS to SFO") in the fourth quarter of 2004. There are 13 active products in this route-quarter, and 5 of them are non-stop products. The number of active airlines is 8: with both stop and non-stop flights, America West (HP), American Airlines (AA), Continental (CO), US Airways (US), and United (UA); and with only stop flights, Delta (DL), Northwest (NW), and "Others". Consider the "hypothetical" merger (in 2004) between Delta and Northwest. The new airline, say DL-NW, has airline fixed effects, in demand and costs, equal to the average of the fixed effects of the merging companies DL and NW. As for the characteristics of the new airline in this route: `avg_miles` is equal to the minimum of `avg_miles` of the two merging companies; `HUB_origin = 45`; `HUB_dest = 36`; and the new airline still only provides stop flights in this route.

(a) Using the estimated model, obtain airlines profits in this route-quarter before the hypothetical merger.
(b) Calculate equilibrium prices, number of passengers, and profits , in this route-quarter after the merger. Comment the results.
(c) Suppose that, as the result of the merger, the new airline decides also to operate non-stop flights in this route. Calculate equilibrium prices, number of passengers, and profits , in this route-quarter after the merger. Comment the results.

3. Production Functions

3.1 Introduction

Production functions (PF) are important primitive components of many economic models. The estimation of PFs plays a key role in the empirical analysis of issues such as productivity dispersion and misallocation, the contribution of different factors to economic growth, skill-biased technological change, estimation of economies of scale and economies of scope, evaluation of the effects of new technologies, learning-by-doing, or the quantification of production externalities, among many others.

In empirical IO, the estimation of production functions can be used to obtain firms' costs. Cost functions play an important role in any empirical study of industry competition. As explained in chapter 1, data on production costs at the firm-market-product level is rare. For this reason cost functions are often estimated in an indirect way, using first order conditions of optimality for profit maximization (see chapter 4). However, cross-sectional or panel datasets with firm-level information on output and inputs of the production process are more commonly available. Given this information, it is possible to estimate the industry production function and use it to obtain firms' cost functions.

There are multiple issues that should be taken into account in the estimation of production functions.

(a) Measurement issues. There are important issues in the measurement of inputs, such as differences in the quality of labor, or the measurement error that results from the construction of the capital stock using a perpetual inventory method. There are also issues in the measurement of output. For instance, the problem of observing revenue instead of output in physical units.

(b) Specification assumptions. The choice of functional form for the production function is an important modelling decision, especially when the model includes different types of labor and capital inputs that may be complements or substitutes.

(c) Simultaneity / endogeneity. This is a key econometric issue in the estimation of production functions. Observed inputs (for instance, labor and capital) can be correlated with unobserved inputs or productivity shocks (for instance, managerial ability, quality of land, materials, capacity utilization). This correlation introduces biases in some

estimators of PF parameters.

(d) Multicollinearity between observed inputs is also a relevant issue in some empirical applications. The high correlation between observed labor and capital can seriously reduce the precision in the estimation of PF parameters.

(e) Endogenous exit. In panel datasets, firm exit from the sample is not exogenous and it is correlated with firm size. Smaller firms are more likely to exit compared to larger firms. Endogenous exit can introduce selection-bias in some estimators of PF parameters.

In this chapter, we concentrate on the problems of simultaneity, multicollinearity, and endogenous exit, and on different solutions that have been proposed to deal with these issues. For the sake of simplicity, we discuss these issues in the context of a Cobb-Douglas PF. However, the arguments and results can be extended to more general specifications of PFs. In principle, some of the estimation approaches can be generalized to estimate nonparametric specifications of PF. Griliches and Mairesse (1998), Bond and Van Reenen (2007), and Ackerberg et al. (2007) include surveys of this literature.

3.2 Model and data

3.2.1 Model

Basic framework

A Production Function (PF) is a description of a production technology that relates the physical output of a production process to the physical inputs or factors of production. A general representation is:

$$Y = F(X_1, X_2, ..., X_J, A) \tag{3.1}$$

where Y is a measure of firm output, X_1, X_2, .., and X_J are measures of J firm inputs, and A represents the firm's technological efficiency. The marginal productivity of input j is $MP_j = \partial F / \partial X_j$.

Given the production function $Y = F(X_1, X_2, ..., X_J, A)$ and input prices $(W_1, W_2, ..., W_J)$, the cost function $C(Y)$ is defined as the minimum cost of producing the amount of output Y:

$$\begin{aligned} C(Y) \quad = \quad & \min_{\{X_1, X_2, ..., X_J\}} W_1 X_1 + W_2 X_2 + ... + W_J X_J \\ & \text{subject to: } Y \geq F(X_1, X_2, ..., X_J, A) \end{aligned} \tag{3.2}$$

The marginal conditions of optimality imply that for every input j:

$$W_j \; - \lambda \; F_j(X_1, X_2, ..., X_J, A) \; = 0, \tag{3.3}$$

where $F_j(X_1, X_2, ..., X_J, A)$ is the marginal productivity of input j, and λ is the Lagrange multiplier of the restriction.

Cobb-Douglas production and cost functions

A very common specification is the Cobb-Douglas PF (Cobb and Douglas, 1928):

$$Y = L^{\alpha_L} \; K^{\alpha_K} \; A \tag{3.4}$$

where L and K represent labor and capital inputs, respectively, and α_L and α_K are technological parameters that are assumed the same for all the firms in the market and industry under study. This Cobb-Douglas PF can be generalized to include more inputs, for instance, $Y = L^{\alpha_L} K^{\alpha_K} R^{\alpha_R} E^{\alpha_E} A$, where R represents R&D and E is energy inputs. We can also distinguish different types of labor (for instance, blue collar and white collar labor) and capital (for instance, equipment, buildings, and information technology). For the Cobb-Douglas PF, the productivity term A is denoted the *Total Factor Productivity* (TFP). The marginal productivity of input j is $MP_j = \alpha_j \frac{Y}{X_j}$. All the inputs are complements in production, that is, the marginal productivity of any input j increases with the amount of any other input k:

$$\frac{\partial MP_j}{\partial X_k} = \frac{\alpha_j}{X_j}\frac{\alpha_k}{X_k} Y > 0 \tag{3.5}$$

Note that this is not necessarily the case for other production functions such as the Constant Elasticity of Substitution (CES) or the Translog

More generally, we can consider a Cobb-Douglas PF with J inputs: $Y = X_1^{\alpha_1} ... X_1^{\alpha_J} A$. Given this PF and input prices W_j, we can obtain the expression for the corresponding cost function. The marginal condition of optimality for input j implies $W_j - \lambda \alpha_j (Y/X_j) = 0$, or equivalently:

$$W_j X_j = \lambda \alpha_j Y \tag{3.6}$$

Therefore, the cost is equal to $\sum_{j=1}^{J} W_j X_j = \lambda \alpha Y$, where the parameter α is defined as $\alpha \equiv \sum_{j=1}^{J} \alpha_j$. Note that α represents the returns to scale in the production function: constant if $\alpha = 1$, decreasing if $\alpha < 1$, and increasing if $\alpha > 1$. To obtain the expression of the cost function, we still need to obtain the (endogenous) value of the Lagrange multiplier λ. For this, we substitute the marginal conditions $X_j = \lambda \alpha_j Y/W_j$ into the production function:

$$Y = A \left(\frac{\lambda \alpha_1 Y}{W_1}\right)^{\alpha_1} \left(\frac{\lambda \alpha_2 Y}{W_2}\right)^{\alpha_2} ... \left(\frac{\lambda \alpha_J Y}{W_J}\right)^{\alpha_J} \tag{3.7}$$

Using this expression to solve for the Lagrange multiplier, we get

$$\lambda = \left(\frac{W_1}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha}} \left(\frac{W_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha}} ... \left(\frac{W_J}{\alpha_J}\right)^{\frac{\alpha_J}{\alpha}} Y^{\frac{1-\alpha}{\alpha}} A^{\frac{-1}{\alpha}}. \tag{3.8}$$

And plugging this multiplier into the expression $\lambda \alpha Y$ for the cost, we obtain the cost function:

$$C(Y) = \alpha \left(\frac{Y}{A}\right)^{\frac{1}{\alpha}} \left(\frac{W_1}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha}} \left(\frac{W_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha}} ... \left(\frac{W_J}{\alpha_J}\right)^{\frac{\alpha_J}{\alpha}} \tag{3.9}$$

Looking at the Cobb-Douglas cost function in equation (3.9) we can identify some interesting properties. First, the returns to scale parameter α determines the shape of the cost as a function of output. More specifically, the sign of the second derivative $C''(Y)$ is equal to the sign of $\frac{1}{\alpha} - 1$. If $\alpha = 1$ (*constant returns to scale*, CRS), we have $C''(Y) = 0$ such that the cost function is linear in output. If $\alpha < 1$ (*decreasing returns to scale*, DRS), we have $C''(Y) > 0$ and the cost function is strictly convex in output. Finally, if $\alpha > 1$ (*increasing returns to scale*, IRS), we have $C''(Y) < 0$ such that the cost function is concave in output.

Production functions and the linear regression model

An attractive feature of the Cobb-Douglas PF from the point of view of estimation is that it is linear in logarithms:

$$y = \alpha_L \, \ell + \alpha_K \, k + \omega \tag{3.10}$$

where y is the logarithm of output, ℓ is the logarithm of labor, k is the logarithm of physical capital, and ω is the logarithm of TFP. The simplicity of the Cobb-Douglas PF also comes with some limitations. One of its drawbacks is that it implies an elasticity of substitution between labor and capital (or between any two inputs) that is always equal to one. This implies that all technological changes are neutral for the demand of inputs. For this reason, the Cobb-Douglas PF cannot be used to study topics such as skill-biased technological change. For empirical studies where it is important to have a flexible form for the elasticity of substitution between inputs, the translog PF has been a popular specification:

$$Y = L^{[\alpha_{L0} + \alpha_{LL}\ell + \alpha_{LK}k]} \, K^{[\alpha_{K0} + \alpha_{KL}\ell + \alpha_{KK}k]} \, A \tag{3.11}$$

which in logarithms becomes,

$$y = \alpha_{L0} \, \ell + \alpha_{K0} \, k + \alpha_{LL} \, \ell^2 + \alpha_{KK} \, k^2 + (\alpha_{LK} + \alpha_{KL}) \, \ell \, k + \omega \tag{3.12}$$

3.2.2 Data

The most common type of data that has been used for the estimation of PFs consists of panel data of firms or plants with annual frequency and information on: (i) a measure of output, for instance, units produced, revenue, or value added; (ii) a measure of labor input, such as number of workers; (iii) a measure of capital input. Some datasets also include measures of other inputs such as materials, energy, or R&D, and information on input prices, typically at the industry level but sometimes at the firm level. For the US, the most commonly used datasets in the estimation of PFs are Compustat, and the Longitudinal Research Database from US Census Bureau. In Europe, some countries' Central Banks (for instance, Bank of Italy, Bank of Spain) collect firm level panel data with rich information on output, inputs, and prices.

For the rest of this chapter we consider that the researcher observes a panel dataset of N firms, indexed by i, over several periods of time, indexed by t, with the following information:

$$\textbf{Data} \; = \; \{y_{it}, \ell_{it}, k_{it}, w_{it}, r_{it} : \; i = 1, 2, ...N; t = 1, 2, ..., T_i\} \tag{3.13}$$

where y, ℓ, and k have been defined above, and w and r represent the logarithms of the price of labor and the price of capital for the firm, respectively. T_i is the number of periods that the researcher observes firm i.

Throughout this chapter, we consider that all the observed variables are in mean deviations. Therefore, we omit constant terms in all the equations.

3.3 Econometric issues

We are interested in the estimation of the parameters α_L and α_K in the Cobb-Douglas PF (in logs):

$$y_{it} = \alpha_L \, \ell_{it} + \alpha_K \, k_{it} + \omega_{it} + e_{it} \tag{3.14}$$

ω_{it} represents inputs that are known to the firm when it makes its capital and labor decisions, but are unobserved to the econometrician. These include managerial ability, quality of land, materials, etc. We refer to ω_{it} as the logarithm of *total factor productivity (log-TFP),* or *unobserved productivity,* or *productivity shock.* e_{it} represents measurement error in output, or any shock affecting output that is unknown to the firm when it chooses its capital and labor. We assume that the error term e_{it} is independent of inputs and of the productivity shock. We use y_{it}^e to represent the "true" expected value of output for the firm, $y_{it}^e \equiv y_{it} - e_{it}$.

3.3.1 Simultaneity problem

The simultaneity problem in the estimation of a PF establishes that if the unobserved productivity ω_{it} is known to the firm when it decides the amount of inputs to use in production, (k_{it}, ℓ_{it}), then these observed inputs should be correlated with the unobservable ω_{it} and the OLS estimator of α_L and α_K will be biased and inconsistent. This problem was pointed out in the seminal paper by Marschak and Andrews (1944).

Example 3.1. Suppose that firms in our sample operate in the same markets for output and inputs. These markets are competitive. Output and inputs are homogeneous products across firms. For simplicity, consider a PF with only one input, say labor: $Y = L^{\alpha_L} \exp\{\omega + e\}$. The first order condition of optimality for the demand of labor implies that the expected marginal productivity should be equal to the price of labor W_L: that is, $\alpha_L Y^e/L = W_L$, where $Y^e = Y/\exp\{e\}$, because the firm's profit maximization problem does not depend on the measurement error or/and non-anticipated shocks in e_{it}. Note that the price of labor W_L is the same for all the firms because, by assumption, they operate in the same competitive output and input markets. Then, the model can be described in terms of two equations: the production function and the marginal condition of optimality in the demand for labor. In logarithms, and in deviations with respect to mean values (no constant terms), these two equations are:[1]

$$\begin{aligned} y_{it} &= \alpha_L \ell_{it} + \omega_{it} + e_{it} \\ y_{it} - \ell_{it} &= e_{it} \end{aligned} \tag{3.15}$$

The reduced form equations of this structural model are:

$$\begin{aligned} y_{it} &= \frac{\omega_{it}}{1-\alpha_L} + e_{it} \\ \ell_{it} &= \frac{\omega_{it}}{1-\alpha_L} \end{aligned} \tag{3.16}$$

Given these expressions for the reduced form equations, it is straightforward to obtain the bias in the OLS estimation of the PF. The OLS estimator of α_L in this simple regression model is a consistent estimator of $Cov(y_{it}, \ell_{it})/Var(\ell_{it})$. But the reduced form equations, together with the condition $Cov(\omega_{it}, e_{it}) = 0$, imply that the covariance between log-output and log-labor should be equal to the variance of log-labor: $Cov(y_{it}, \ell_{it}) = Var(\ell_{it})$.

[1] The firm's profit maximization problem depends on output $\exp\{y_i^e\}$ without the measurement error e_i.

Therefore, under the conditions of this model, the OLS estimator of α_L converges in probability to 1 regardless of the true value of α_L. Even in the hypothetical case that labor has very low productivity and α_L is close to zero, the OLS estimator still converges in probability to 1. It is clear that – at least in this case – ignoring the endogeneity of inputs can generate a serious bias in the estimation of the PF parameters. ■

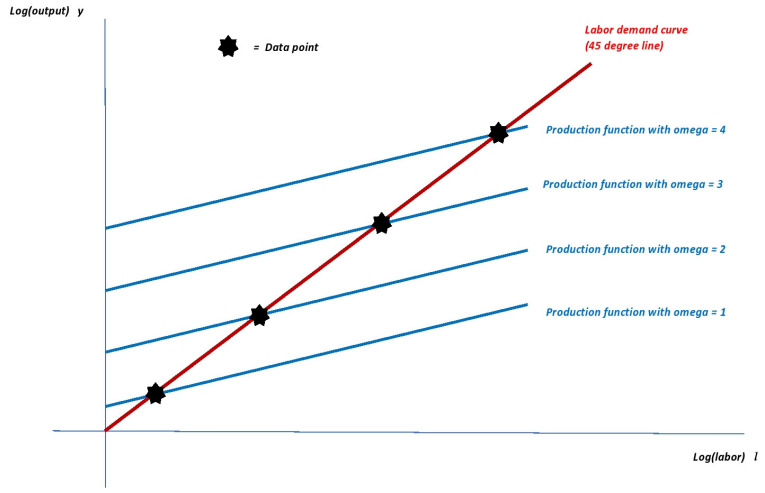


Figure 3.1: Production function and labor demand

Example 3.2: Consider similar conditions as in Example 1, but now firms produce differentiated products and use differentiated labor inputs. In particular, the price of labor R_{it} is an exogenous variable that has variation across firms and over time. Suppose that a firm is a price taker in the market for its labor input, and the price of this input, R_{it}, is independent of the firm's productivity shock, ω_{it}. In this version of the model the system of structural equations is very similar to the one in (3.15), with the only difference being that the labor demand equation now includes the logarithm of the price of labor – denoted by r_{it} — such that we have $y_{it} - \ell_{it} = r_{it} + e_{it}$. The reduced form equations for this model are:

$$\begin{aligned} y_{it} &= \frac{\omega_{it} - r_{it}}{1 - \alpha_L} + r_{it} + e_{it} \\ \ell_{it} &= \frac{\omega_{it} - r_{it}}{1 - \alpha_L} \end{aligned} \tag{3.17}$$

Again, we can use these reduced form equations to obtain the asymptotic bias in the estimation of α_L if we ignore the endogeneity of labor in the estimation of the PF. The OLS estimator of α_L converges in probability to $Cov(y_{it}, \ell_{it})/Var(\ell_{it})$, and in this case

this implies the following expression for the bias:

$$Bias\left(\hat{\alpha}_L^{OLS}\right) = \frac{1-\alpha_L}{1+\sigma_r^2/\sigma_\omega^2} \tag{3.18}$$

where σ_ω^2 and σ_r^2 represent the variances of log-TFP and of the logarithm of labor price, respectively. This bias – of the OLS estimator of α_L – is always upward because the firm's labor demand is always positively correlated with the firm's log-TFP. The ratio between the variance of log-labor-price and the variance of log-TFP, $\sigma_r^2/\sigma_\omega^2$, plays a key role in the determination of the magnitude of this bias. Sample variability in input prices, if it is not correlated with the productivity shock, induces exogenous variability in the labor input. This exogenous sample variability in labor reduces the bias of the OLS estimator. The bias of the OLS estimator declines monotonically with the variance ratio $\sigma_r^2/\sigma_\omega^2$. Nevertheless, the bias can be very significant if the exogenous variability in input prices is not much larger than the variability in unobserved productivity. ■

3.3.2 Endogenous exit

Exit and selection problem

Panel datasets of firms or establishements can contain a significant number of firms/plants that exit from the market. Exiting firms are not randomly chosen from the population of operating firms. For instance, existing firms are typically smaller than surviving firms.

Let V_{it}^1 be the value of firm i at period t if the owners decide to stay active in the market. This value is the expected present value of future profits. Let V_{it}^0 be the value of the assets of firm i if the owners choose to exit from the market at period t. This value includes the scrap value of the assets minus exit costs such as indemnifications to workers and clients. These two values depend on the "installed" inputs of the firm an don the current value of TFP. That is, $V_{it}^1 = V^1(\ell_{it-1}, k_{it}, \omega_{it})$ and be the value of the firm at period staying in the market, $V_{it}^0 = V^0(\ell_{it-1}, k_{it}, \omega_{it})$. Let d_{it} be the indicator of the event "firm i stays in the market at the end of period t". The firm's owners maximize present value. Then, the optimal exit/stay decision is:

$$d_{it} = 1\left\{ V^1(\ell_{it-1}, k_{it}, \omega_{it}) - V^0(\ell_{it-1}, k_{it}, \omega_{it}) \geq 0 \right\} \tag{3.19}$$

where $1\{S\}$ is the indicator function, such that $1\{S\} = 1$ if statement S is true, and $1\{S\} = 0$ otherwise. Under standard conditions, the difference between the value of being in the market and the value of being out, $V^1(\ell_{it-1}, k_{it}, \omega_{it}) - V^0(\ell_{it-1}, k_{it}, \omega_{it})$, is a strictly increasing in all its arguments, that is, all the inputs are more productive in the current firm/industry than in the best alternative use. Therefore, the function is invertible with respect to the productivity shock ω_{it} and we can write the optimal exit/stay decision as a single-threshold condition:

$$d_{it} = 1\left\{ \omega_{it} \geq v(\ell_{it-1}, k_{it}) \right\} \tag{3.20}$$

where the threshold function $v(.,.)$ is strictly decreasing in all its arguments.

Consider the PF $y_{it} = \alpha_L\, \ell_{it} + \alpha_K\, k_{it} + \omega_{it} + e_{it}$. In the estimation of this PF, we use the sample of firms that survived at period t: that is, $d_{it} = 1$. Therefore, the error term in the estimation of the PF is $\omega_{it}^{d=1} + e_{it}$, where:

$$\omega_{it}^{d=1} \equiv \{\omega_{it} \mid d_{it} = 1\} = \left\{\omega_{it} \mid \omega_{it} \geq v(\ell_{i,t-1}, k_{it})\right\} \tag{3.21}$$

where the notation $\{x|S\}$ represents the random variable x conditional on event S. Even if the productivity shock ω_{it} is independent of the state variables $(\ell_{i,t-1}, k_{it})$, the self-selected productivity shock $\omega_{it}^{d=1}$ will not be mean-independent of $(\ell_{i,t-1}, k_{it})$. That is,

$$\begin{aligned}\mathbb{E}\left(\omega_{it}^{d=1} \mid \ell_{i,t-1}, k_{it}\right) &= \mathbb{E}\left(\omega_{it} \mid \ell_{i,t-1}, k_{it}, d_{it}=1\right) \\ &= \mathbb{E}\left(\omega_{it} \mid \ell_{i,t-1}, k_{it}, \omega_{it} \geq v\left(\ell_{i,t-1}, k_{it}\right)\right) \\ &= \lambda\left(\ell_{i,t-1}, k_{it}\right)\end{aligned} \tag{3.22}$$

$\lambda\left(\ell_{i,t-1}, k_{it}\right)$ is the selection term. Therefore, the PF can be written as:

$$y_{it} = \alpha_L\, \ell_{it} + \alpha_K\, k_{it} + \lambda\left(\ell_{i,t-1}, k_{it}\right) + \tilde{\omega}_{it} + e_{it} \tag{3.23}$$

where $\tilde{\omega}_{it} \equiv \{\omega_{it}^{d=1} - \lambda\left(\ell_{i,t-1}, k_{it}\right)\}$ is, by construction, mean-independent of $(\ell_{i,t-1}, k_{it})$.

Ignoring the selection term $\lambda\left(\ell_{i,t-1}, k_{it}\right)$ introduces bias in our estimates of the PF parameters. The selection term is an increasing function of the threshold $v\left(\ell_{i,t-1}, k_{it}\right)$, and therefore it is decreasing in $\ell_{i,t-1}$ and k_{it}. Both ℓ_{it} and k_{it} are negatively correlated with the selection term, but the correlation with the capital stock tends to be larger because the value of a firm depends more strongly on its capital stock than on its "stock" of labor. Therefore, this selection problem tends to bias downward the estimate of the capital coefficient.

To provide an intuitive interpretation of this bias, first consider the case of very large firms. Firms with a large capital stock are very likely to survive, even if the firm receives a bad productivity shock. Therefore, for large firms, endogenous exit induces little censoring in the distribution of productivity shocks. Consider now the case of very small firms. Firms with a small capital stock have a large probability of exiting, even if their productivity shocks are not too negative. For small firms, exit induces a very significant left-censoring in the distribution of productivity, that is, we only observe small firms with good productivity shocks and therefore with high levels of output. If we ignore this selection, we will conclude that firms with large capital stocks are not much more productive than firms with small capital stocks. But that conclusion is partly spurious because we do not observe many firms with low capital stocks that would have produced low levels of output if they had stayed.

The relationship between firm size and firm growth

This type of selection problem has been also analyzed by researchers interested in the relationship between firm growth and firm size. This relationship has relevant policy implications. Mansfield (1962), Evans (1987), and Hall (1987) are seminal papers in this literature.

Consider the regression equation:

$$\Delta s_{it} = \alpha + \beta\, s_{i,t-1} + \varepsilon_{it} \tag{3.24}$$

where s_{it} represents the logarithm of a measure of firm size, for instance, the logarithm of capital stock, or the logarithm of the number of workers.

The so called *Gibrat's law* – sometimes described as the *rule of proportionate growth* – is a hypothesis establishing that the rate of growth of a firm is independent of its size.

This "law" was postulated by gibrat (1931) – see the survey by Sutton (1997). Using equation (3.24), we can enunciate Gibrat's hypothesis as the model with $\beta = 0$.

Suppose that the exit decision at period t depends on firm size, $s_{i,t-1}$, and on a shock ε_{it}. More specifically,

$$d_{it} = 1\left\{ \varepsilon_{it} \geq v\left(s_{i,t-1}\right) \right\} \tag{3.25}$$

where $v(.)$ is a decreasing function, that is, smaller firms are more likely to exit. In a regression of Δs_{it} on $s_{i,t-1}$, we can use only observations from surviving firms. Therefore, the regression of Δs_{it} on $s_{i,t-1}$ can be represented using the equation $\Delta s_{it} = \alpha + \beta s_{i,t-1} + \varepsilon_{it}^{d=1}$, where $\varepsilon_{it}^{d=1} \equiv \{\varepsilon_{it} | d_{it} = 1\} = \{\varepsilon_{it} | \varepsilon_{it} \geq v\left(s_{i,t-1}\right)\}$. Thus,

$$\Delta s_{it} = \quad \alpha + \beta s_{i,t-1} + \lambda\left(s_{i,t-1}\right) + \tilde{\varepsilon}_{it} \tag{3.26}$$

where $\lambda\left(s_{i,t-1}\right) \equiv \mathbb{E}(\varepsilon_{it} | \varepsilon_{it} \geq v\left(s_{i,t-1}\right))$, and $\tilde{\varepsilon}_{it} \equiv \{\varepsilon_{it}^{d=1} - \lambda\left(\ell_{i,t-1}, k_{it}\right)\}$ which, by construction, is mean-independent of firm size at $t-1$. The selection term $\lambda\left(s_{i,t-1}\right)$ is an increasing function of the threshold $v\left(s_{i,t-1}\right)$, and therefore it is decreasing in firm size. If the selection term is ignored in the regression of Δs_{it} on $s_{i,t-1}$, then the OLS estimator of β will be downward biased. That is, it seems that smaller firms grow faster just because small firms that would like to grow slowly have exited the industry and they are not observed in the sample.

Mansfield (1962) already pointed out to the possibility of a selection bias due to endogenous exit. He uses panel data from three US industries, steel, petroleum, and tires, over several periods. He tests the null hypothesis of $\beta = 0$, that is, Gibrat's law. Using only the subsample of surviving firms, he can reject Gibrat's Law in 7 of the 10 samples. Including also exiting firms and using the imputed values $\Delta s_{it} = -1$ for these firms, he rejects Gibrat's Law for only for 4 of the 10 samples. An important limitation of Mansfield's approach is that including exiting firms using the imputed values $\Delta s_{it} = -1$ does not correct completely for the selection bias. But Mansfield's paper was written more than a decade before James Heckman's seminal contributions on sample selection in econometrics – Heckman (1974, 1976, 1979). Hall (1987) and Evans (1987) dealt with the selection problem using Heckman's two-step estimator. Both authors find that ignoring endogenous exit induces significant downward bias in β. These two studies find that after controlling for endogenous selection à la Heckman, the estimate of β is significantly smaller than zero such that they reject Gibrat's law. A limitation of their approach is that their models do not have any exclusion restriction and identification is based on functional form assumptions: the assumptions of normal distribution of the error term, and linear (causal) relationship between firm size and firm growth.

3.4 Estimation methods

3.4.1 Input prices as instruments

If input prices, r_i, are observable and uncorrelated with log-TFP ω_i, then we can use these variables as instruments in the estimation of the PF. However, this approach has several important limitations. First, input prices are not always observable in some datasets, or they are only observable at the aggregate level but not at the firm level. Second, if firms in our sample use homogeneous inputs, and operate in the same output and input markets, we should not expect to find any significant cross-sectional variation in input prices. This

is a problem because there may not be enough time-series variation for identification, or it can be confounded with any aggregate effect in the error term. Instead, suppose that firms in our sample operate in different input markets, and the researcher observes significant cross-sectional variation in input prices. In this context, a third problem is that this cross-sectional variation in input prices is likely to be endogenous: the different markets where firms operate can be different in the average unobserved productivity of firms, and therefore $cov(\omega_i, r_i) \neq 0$. That is, input prices will not be valid instruments.

3.4.2 Panel data: Fixed-effects

Suppose that we have firm level panel data with information on output, capital and labor for N firms during T time periods. The Cobb-Douglas PF is:

$$y_{it} = \alpha_L \, \ell_{it} + \alpha_K \, k_{it} + \omega_{it} + e_{it} \tag{3.27}$$

Mundlak (1961) and Mundlak and Hoch (1965) are pioneer studies in using panel data for the estimation of production functions. They consider the estimation of a production function of an agricultural product. They postulate the following assumptions:

Assumption PD-1: ω_{it} has the following variance-components structure: $\omega_{it} = \eta_i + \delta_t + u_{it}$. The term η_i is a time-invariant, firm-specific effect that may be interpreted as the quality of a fixed input such as managerial ability, or land quality. δ_t is an aggregate shock affecting all firms. And u_{it} is an firm idiosyncratic shock.

Assumption PD-2: The amount of inputs depend on some other exogenous time-varying variables, such that $var\left(\ell_{it} - \bar{\ell}_i\right) > 0$ and $var\left(k_{it} - \bar{k}_i\right) > 0$, where $\bar{\ell}_i \equiv T^{-1} \sum_{t=1}^{T} \ell_{it}$, and $\bar{k}_i \equiv T^{-1} \sum_{t=1}^{T} k_{it}$.

Assumption PD-3: u_{it} is not serially correlated.

Assumption PD-4: The idiosyncratic shock u_{it} is realized after the firm decides the amount of inputs to employ at period t. In the context of an agricultural PF, this shock may be interpreted as weather, or another random and unpredictable shock.

The Within-Groups estimator (WGE) or fixed-effects estimator of the PF is simply the OLS estimator applied to the Within-Groups transformation of the model. The equation that describes the within-groups transformation can be obtained by taking the difference between equation $y_{it} = \alpha_L \, \ell_{it} + \alpha_K \, k_{it} + \omega_{it} + e_{it}$ and this equation averaged at the firm level, that is $\bar{y}_i = \alpha_L \bar{\ell}_i + \alpha_K \bar{k}_i + \bar{\omega}_i + \bar{e}_i$. The within-groups equation is:

$$(y_{it} - \bar{y}_i) = \alpha_L \left(\ell_{it} - \bar{\ell}_i\right) + \alpha_K \left(k_{it} - \bar{k}_i\right) + (\omega_{it} - \bar{\omega}_i) + (e_{it} - \bar{e}_i) \tag{3.28}$$

Under assumptions (PD-1) to (PD-4), the WGE is consistent. Under these assumptions, the only endogenous component of the error term is the fixed effect η_i. The transitory shocks u_{it} and e_{it} do not induce any endogeneity problem. The WG transformation removes the fixed effect η_i.

It is important to point out that, for short panels (that is, T fixed), the consistency of the WGE requires the regressors $x_{it} \equiv (\ell_{it}, k_{it})$ to be strictly exogenous. That is, for any (t, s):

$$cov(x_{it}, u_{is}) = cov(x_{it}, e_{is}) = 0 \tag{3.29}$$

Otherwise, the WG-transformed regressors $\left(\ell_{it}-\bar{\ell}_i\right)$ and $\left(k_{it}-\bar{k}_i\right)$ would be correlated with the error $(\omega_{it}-\bar{\omega}_i)$. This is why Assumptions (PD-3) and (PD-4) are necessary for the consistency of the OLS estimator.

However, it is very common to find that the WGE estimator provides very small estimates of α_L and α_K (see Griliches and Mairesse, 1998). There are at least two possible reasons that can explain this empirical regularity. First, though assumptions (PD-2) and (PD-3) may be plausible for estimating PFs of agricultural firms, they are unrealistic for other industries, such as manufacturing. And second, the bias induced by measurement-error in the regressors can be exacerbated by the WG transformation. To see this, consider the model with only one input, such as capital, and suppose that it has measurement error. We observe k_{it}^* where $k_{it}^* = k_{it} + e_{it}^k$, and e_{it}^k represents measurement error in capital and it satisfies the classical assumptions on measurement error.[2] The *noise-to-signal ratio* is the ratio of variances $Var(e^k)/Var(k)$. In the estimation of the PF in levels we have that:

$$Bias(\hat{\alpha}_L^{OLS}) = \frac{Cov(k,\eta)}{Var(k)+Var(e^k)} - \frac{\alpha_L\, Var(e^k)}{Var(k)+Var(e^k)} \tag{3.30}$$

If the *noise-to-signal ratio* $Var(e^k)/Var(k)$ is small, then the (downward) bias introduced by the measurement error is negligible in the estimation in levels. In the estimation in first differences (similar to WGE, in fact equivalent when $T=2$), we have that:

$$Bias(\hat{\alpha}_L^{WGE}) = -\frac{\alpha_L\, Var(\Delta e^k)}{Var(\Delta k)+Var(\Delta e^k)} \tag{3.31}$$

Suppose that k_{it} is very persistent (that is, $Var(k)$ is much larger than $Var(\Delta k)$) and that e_{it}^k is not serially correlated (that is, $Var(\Delta e^k) = 2 * Var(e^k)$). Under these conditions, the *noise-to-signal ratio* for capital in first differences, $Var(\Delta e^k)/Var(\Delta k)$, can be large even when the ratio $Var(e^k)/Var(k)$ is quite small. Therefore, the WGE may be significantly downward biased.

3.4.3 Dynamic panel data: GMM

In the WGE described in the previous section, the assumption of strictly exogenous regressors is very unrealistic. However, we can relax that assumption and estimate the PF using the GMM method proposed by Arellano and Bond (1991). Consider the PF in first differences:

$$\Delta y_{it} = \alpha_L\, \Delta \ell_{it} + \alpha_K\, \Delta k_{it} + \Delta \delta_t + \Delta u_{it} + \Delta e_{it} \tag{3.32}$$

We maintain assumptions (PD-1), (PD-2), and (PD-3), but we remove assumption (PD-3). Instead, we consider the following assumption.

Assumption PD-5: A firm's demands for labor and capital are dynamic. More formally, the demand equations for labor and capital are $\ell_{it} = f_L(\ell_{i,t-1}, k_{i,t-1}, \omega_{it})$ and $k_{it} = f_K(\ell_{i,t-1}, k_{i,t-1}, \omega_{it})$, respectively, where either $\ell_{i,t-1}$ or $k_{i,t-1}$, or both, have non-zero partial derivatives in f_L and f_K.

[2] Classical measurement error is independent of the true value, independently and identically distributed over observations, and with zero mean.

There are multiple reasons why the demand for capital or and labor are dynamic – that is, depend on the amount of labor and capital at previous period. Hiring and firing cost for labor, irreversibility of capital investments, installation costs, time-to-build, and other forms of adjustment costs are the most common arguments for the existence of dynamics in the demand of these inputs.

Under the conditions in Assumption PD-5, the lagged variables $\{\ell_{i,t-j}, k_{i,t-j}, y_{i,t-j} : j \geq 2\}$ are valid instruments in the PF equation in first differences. Identification comes from the combination of two assumptions: (1) serial correlation of inputs; and (2) no serial correlation in productivity shocks $\{u_{it}\}$. The presence of adjustment costs implies that the marginal cost of labor or capital depends on the firm's amount of the input at previous period. This implies that this shadow price varies across firms even if firms face the same input prices. This variability in shadow prices can be used to identify PF parameters. The assumption of no serial correlation in $\{u_{it}\}$ is key, but it can be tested (see Arellano and Bond ,1991).

This GMM in first-differences approach has also its own limitations. In some applications, it is common to find unrealistically small estimates of α_L and α_K and large standard errors (see Blundell and Bond ,2000). Overidentifying restrictions are typically rejected. Furthermore, the i.i.d. assumption on u_{it} is typically rejected, and this implies that $\{x_{i,t-2}, y_{i,t-2}\}$ are not valid instruments. It is well-known that the Arellano-Bond GMM estimator may suffer from a weak-instruments problem when the serial correlation of the regressors in first differences is weak (see Arellano and Bover ,1995, and Blundell and Bond ,1998). First difference transformation also eliminates the cross-sectional variation in inputs and it is subject to the problem of measurement error in inputs.

The weak-instruments problem deserves further explanation. For simplicity, consider the model with only one input, x_{it}. We are interested in the estimation of the PF:

$$y_{it} = \alpha\, x_{it} + \eta_i + u_{it} + e_{it} \tag{3.33}$$

where u_{it} and e_{it} are not serially correlated. Consider the following dynamic reduced form equation for the input x_{it}:

$$x_{it} = \delta\, x_{i,t-1} + \lambda_1\, \eta_i + \lambda_2\, u_{it} \tag{3.34}$$

where δ, λ_1, and λ_2 are reduced form parameters, and $\delta \in [0,1]$ captures the existence of adjustment costs. The PF in first differences is:

$$\Delta y_{it} = \alpha\, \Delta x_{it} + \Delta u_{it} + \Delta e_{it} \tag{3.35}$$

For simplicity, consider that the number of periods in the panel is $T = 3$. In this context, Arellano-Bond GMM estimator is equivalent to a simple instrumental variables estimator where the instrument is $x_{i,t-2}$. This IV estimator is:

$$\hat{\alpha}_N = \frac{\sum_{i=1}^{N} x_{i,t-2}\, \Delta y_{it}}{\sum_{i=1}^{N} x_{i,t-2}\, \Delta x_{it}} \tag{3.36}$$

Therefore, under the previous assumptions, $\hat{\alpha}_N$ identifies α if the R-square in the auxiliary regression of Δx_{it} on $x_{i,t-2}$ is not zero.

By definition, the R-square coefficient in the auxiliary regression of Δx_{it} on $x_{i,t-2}$ is such that:

$$p\lim R^2 = \frac{Cov\left(\Delta x_{it}, x_{i,t-2}\right)^2}{Var\left(\Delta x_{it}\right)\ Var\left(x_{i,t-2}\right)} = \frac{\left(\gamma_2 - \gamma_1\right)^2}{2\left(\gamma_0 - \gamma_1\right)\gamma_0} \tag{3.37}$$

where $\gamma_j \equiv Cov\left(x_{it}, x_{i,t-j}\right)$ is the autocovariance of order j of $\{x_{it}\}$. Taking into account that $x_{it} = \frac{\lambda_1\ \eta_i}{1-\delta} + \lambda_2(u_{it} + \delta\ u_{i,t-1} + \delta^2\ u_{i,t-2} + ...)$, we can derive the following expressions for the autocovariances:

$$\begin{aligned} \gamma_0 &= \frac{\lambda_1^2\ \sigma_\eta^2}{(1-\delta)^2} + \frac{\lambda_2^2\ \sigma_u^2}{1-\delta^2} \\ \gamma_1 &= \frac{\lambda_1^2\ \sigma_\eta^2}{(1-\delta)^2} + \delta\ \frac{\lambda_2^2\ \sigma_u^2}{1-\delta^2} \\ \gamma_2 &= \frac{\lambda_1^2\ \sigma_\eta^2}{(1-\delta)^2} + \delta^2\ \frac{\lambda_2^2\ \sigma_u^2}{1-\delta^2} \end{aligned} \tag{3.38}$$

Therefore, $\gamma_0 - \gamma_1 = (\lambda_2^2\sigma_u^2)/(1+\delta)$ and $\gamma_1 - \gamma_2 = \delta(\lambda_2^2\sigma_u^2)/(1+\delta)$. The R-square is:

$$\begin{aligned} R^2 &= \frac{\left(\delta\ \frac{\lambda_2^2\sigma_u^2}{1+\delta}\right)^2}{2\left(\frac{\lambda_2^2\sigma_u^2}{1+\delta}\right)\left(\frac{\lambda_1^2\ \sigma_\eta^2}{(1-\delta)^2} + \frac{\lambda_2^2\ \sigma_u^2}{1-\delta^2}\right)} \\ &= \frac{\delta^2\left(1-\delta\right)^2}{2\left(1-\delta+(1+\delta)\rho\right)} \end{aligned} \tag{3.39}$$

with $\rho \equiv \lambda_1^2\sigma_\eta^2/\lambda_2^2\sigma_u^2 \geq 0$. We have a problem of weak instruments and poor identification if this R-square coefficient is very small.

It is simple to verify that this R-square is small both when adjustment costs are small (that is, δ is close to zero) and when adjustment costs are large (that is, δ is close to one). When using this IV estimator, large adjustments costs are bad news for identification because, with delta close to one, the first difference Δx_{it} is almost iid and it is not correlated with lagged input (or output) values. What is the maximum possible value of this R-square? It is clear that this R-square is a decreasing function of ρ. Therefore, the maximum R-square occurs for $\lambda_1^2\sigma_\eta^2 = \rho = 0$ – that is, no fixed effects in the input demand. Under this condition, we have that $R^2 = \delta^2\left(1-\delta\right)/2$, and the maximum value of this R-square is $R^2 = 0.074$ which occurs when $\delta = 2/3$. This upper bound for the R-square is over-optimistic because it is based on the assumption of no fixed effects. For instance, suppose that $\lambda_1^2\sigma_\eta^2 = \lambda_2^2\sigma_u^2$ such that $\rho = 1$. In this case, we have that $R^2 = \delta^2\left(1-\delta\right)^2/4$ and the maximum value of this R-square is $R^2 = 0.016$, which occurs when $\delta = 1/2$.

Arellano and Bover (1995) and Blundell and Bond (1998) have proposed GMM estimators that deal with this weak-instrument problem. Suppose that at some period $t_i^* \leq 0$ (that is, before the first period in the sample, $t = 1$) the shocks u_{it}^* and e_{it} were

zero, and input and output were equal to their firm-specific, steady-state mean values:

$$\begin{aligned} x_{it_i^*} &= \frac{\lambda_1 \eta_i}{1-\delta} \\ y_{it_i^*} &= \alpha \frac{\lambda_1 \eta_i}{1-\delta} + \eta_i \end{aligned} \tag{3.40}$$

Then, it is straightforward to show that for any period t in the sample:

$$\begin{aligned} x_{it} &= x_{it_i^*} + \lambda_2 \left(u_{it} + \delta u_{it-1} + \delta^2 u_{it-2} + ... \right) \\ y_{it} &= y_{it_i^*} + u_{it} + \alpha\lambda_2 \left(u_{it} + \delta u_{it-1} + \delta^2 u_{it-2} + ... \right) \end{aligned} \tag{3.41}$$

These expressions imply that input and output in first differences depend on the history of the i.i.d. shock $\{u_{it}\}$ between periods t_i^* and t, but they do not depend on the fixed effect η_i. Therefore, $cov(\Delta x_{it}, \eta_i) = cov(\Delta y_{it}, \eta_i) = 0$ and lagged first differences are valid instruments in the equation in levels. That is, for $j > 0$:

$$\begin{aligned} \mathbb{E}\left(\Delta x_{it-j} \left[\eta_i + u_{it} + e_{it} \right] \right) = 0 \quad &\Rightarrow \quad \mathbb{E}\left(\Delta x_{it-j} \left[y_{it} - \alpha x_{it} \right] \right) = 0 \\ \mathbb{E}\left(\Delta y_{it-j} \left[\eta_i + u_{it} + e_{it} \right] \right) = 0 \quad &\Rightarrow \quad \mathbb{E}\left(\Delta y_{it-j} \left[y_{it} - \alpha x_{it} \right] \right) = 0 \end{aligned} \tag{3.42}$$

These moment conditions can be combined with the "standard" Arellano-Bond moment conditions to obtain a more efficient GMM estimator. The Arellano-Bond moment conditions are, for $j > 1$:

$$\begin{aligned} \mathbb{E}\left(x_{it-j} \left[\Delta u_{it} + \Delta e_{it} \right] \right) = 0 \quad &\Rightarrow \quad \mathbb{E}\left(x_{it-j} \left[\Delta y_{it} - \alpha \Delta x_{it} \right] \right) = 0 \\ \mathbb{E}\left(y_{it-j} \left[\Delta u_{it} + \Delta e_{it} \right] \right) = 0 \quad &\Rightarrow \quad \mathbb{E}\left(y_{it-j} \left[\Delta y_{it} - \alpha \Delta x_{it} \right] \right) = 0 \end{aligned} \tag{3.43}$$

Based on Monte Carlo experiments and on actual data of UK firms, Blundell and Bond (2000) have obtained very promising results using this GMM estimator. Alonso-Borrego and Sanchez (2001) have obtained similar results using Spanish data. The reason why this estimator works better than Arellano-Bond GMM is that the second set of moment conditions exploit cross-sectional variability in output and input. This has two implications. First, instruments are informative even when adjustment costs are larger and δ is close to one. And second, the problem of large measurement error in the regressors in first-differences is reduced.

Bond and Söderbom (2005) present a very interesting Monte Carlo experiment to study the actual identification power of adjustment costs in inputs. The authors consider a model with a Cobb-Douglas PF and quadratic adjustment cost with both deterministic and stochastic components. They solve numerically the firm's dynamic programming problem, simulate data on inputs and output using the optimal decision rules, and use the Blundell-Bond GMM method to estimate PF parameters. The main results of their experiments are the following. When adjustment costs have only deterministic components, the identification is weak if adjustment costs are too low, or too high, or too similar between the two inputs. With stochastic adjustment costs, identification results improve considerably. Given these results, one might be tempted to "claim victory": if

the true model is such that there are stochastic shocks (independent of productivity) in the costs of adjusting inputs, then the panel data GMM approach can identify with precision the PF parameters. However, as Bond and Soderbom explain, there is also a negative interpretation of this result. Deterministic adjustment costs have little identification power in the estimation of PFs. The existence of shocks in adjustment costs that are independent of productivity seems to be a strong identification condition. If these shocks are not present in the "true model", the apparent identification using the GMM approach could be spurious because the identification would be due to the misspecification of the model. As we will see in the next section, we obtain a similar conclusion when using a control function approach.

Table 3.1: Blundell and Bond (2000); Estimation Results

509 manufacturing firms; 1982-89				
Parameter	**OLS-Levels**	**WG**	**AB-GMM**	**SYS-GMM**
β_L	0.538	0.488	0.515	0.479
	(0.025)	(0.030)	(0.099)	(0.098)
β_K	0.266	0.199	0.225	0.492
	(0.032)	(0.033)	(0.126)	(0.074)
ρ	0.964	0.512	0.448	0.565
	(0.006)	(0.022)	(0.073)	(0.078)
Sargan (p-value)	-	-	0.073	0.032
m2	-	-	-0.69	-0.35
Constant RS (p-v)	0.000	0.000	0.006	0.641

3.4.4 Control function methods

Consider a system of simultaneous equations where some unobservables can enter in more than one structural equation. Under some conditions, we can use one of the equations to solve for an unobservable and represent it as a function of observable variables and parameters. Then, we can plug this function into another equation where this unobservable enters, such that we "control for" this unobservable by including observables. This is a particular example of a control function approach and it can be used to deal with endogeneity problems.

More generally, a control function method is an econometric procedure to correct for endogeneity problems by exploiting the structure that the model imposes on its error terms. In general, this approach implies different restrictions than the instrumental variables approach. Heckman and Robb (1985) introduced this term, though the concept had been used before in some empirical applications. An attractive feature of the control function approach is that it can provide consistent estimates of structural parameters in models where unobservables are not additively separable. In those models, instrumental variable estimators are typically inconsistent or at least do not consistently estimate the average causal effect over the whole population.

Olley and Pakes method

In a seminal paper, Olley and Pakes (1996) propose a control function approach to estimate PFs. Levinsohn and Petrin (2003) have extended this method.

Consider the Cobb-Douglas PF in the context of the following model of simultaneous equations:

$$\begin{aligned} (PF) \quad & y_{it} = \alpha_L \, \ell_{it} + \alpha_K \, k_{it} + \omega_{it} + e_{it} \\ (LD) \quad & \ell_{it} = f_L \left(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it} \right) \\ (ID) \quad & i_{it} = f_K \left(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it} \right) \end{aligned} \tag{3.44}$$

where equations (LD) and (ID) represent the firms' optimal decision rules for labor and capital investment, respectively, in a dynamic decision model with state variables $(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it})$. The vector r_{it} represents input prices. Under certain conditions on this system of equations, we can estimate consistently α_L and α_K using a control function method.

Olley and Pakes consider the following assumptions:

Assumption OP-1: $f_K \left(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it} \right)$ is invertible in ω_{it}.

Assumption OP-2: There is no cross-sectional variation in input prices. For every firm i, $r_{it} = r_t$.

Assumption OP-3: ω_{it} follows a first order Markov process. That is, at any period $t \geq 0$, the transition probability $\Pr(\omega_{it} \mid \omega_{it-1}, ..., \omega_{i0})$ is equal to $\Pr(\omega_{it} | \omega_{it-1})$.

Assumption OP-4: Time-to-build physical capital. Investment i_{it} is chosen at period t but it is not productive until period $t+1$. And $k_{it+1} = (1-\delta) k_{it} + i_{it}$.

In the Olley and Pakes model, the labor input is assumed to be a static input such that lagged labor, $\ell_{i,t-1}$, is not an explanatory variable in the labor demand function f_L. This is a strong assumption as there may be substantial adjustments costs in hiring and firing workers. Most importantly, this assumption is not necessary for the Olley-Pakes method to provide a consistent estimator of the production function parameters. Therefore, we present here a version of the Olley-Pakes method where both labor and capital are dynamic inputs.

Assumption OP-2 implies that the only unobservable variable in the investment equation that has cross-sectional variation across firms is the productivity shock ω_{it}. This restriction is crucial for the OP method and for the related Levinshon-Petrin method. This imposes restrictions on the underlying model of market competition and inputs demands. This assumption implicitly establishes that firms operate in the same input markets, they do not have any monopsony power in these markets, and there are not internal labor markets within firms. Since a firm's input demand depends also on output price (or on the exogenous demand variables affecting product demand), assumption OP-2 also implies that firms operate in the same output market with either homogeneous goods or completely symmetric product differentiation. Note that these economic restrictions can be relaxed if the researcher has data on inputs prices at the firm level, that is, if r_{it} is observable.

The method proceeds in two-steps. The first step estimates α_L using a control function approach, and it relies on assumptions *(OP-1)* and *(OP-2).* This first step is

the same with and without endogenous exit. The second step estimates α_K and it is based on assumptions *(OP-3)* and *(OP-4)*. The Olley-Pakes method deals both with the simultaneity problem and with the selection problem due to endogenous exit.

Step 1: Estimation of α_L. Under assumptions *(OP-1)* and *(OP-2)*, we can invert the investment function to obtain a firm's TFP: that is, $\omega_{it} = f_K^{-1}(\ell_{i,t-1}, k_{it}, i_{it}, r_t)$. Solving this equation into the PF we have:

$$\begin{aligned} y_{it} &= \alpha_L\, \ell_{it} + \alpha_K\, k_{it} + f_L^{-1}(\ell_{i,t-1}, k_{it}, i_{it}, r_t) + e_{it} \\ &= \alpha_L\, \ell_{it} + \phi_t(\ell_{i,t-1}, k_{it}, i_{it}) + e_{it} \end{aligned} \tag{3.45}$$

where $\phi_t(\ell_{i,t-1}, k_{it}, i_{it}) \equiv \alpha_K\, k_{it} + f_L^{-1}(\ell_{i,t-1}, k_{it}, i_{it}, r_t)$. Without a parametric assumption on the investment equation f_K, equation (3.45) is a *partially linear model*.[3]

The parameter α_L and the functions ϕ_1, ϕ_2, ..., ϕ_T can be estimated using semi-parametric methods. Olley and Pakes use polynomial series approximations for the nonparametric functions ϕ_t. Alternatively, one can use the method in Robinson (1988).

This method is a control function method. Instead of instrumenting the endogenous regressors, we include additional regressors that capture the endogenous part of the error term (that is, proxy for the productivity shock). By including a flexible function in $(\ell_{i,t-1}, k_{it}, i_{it})$, we control for the unobservable ω_{it}. Therefore, α_L is identified if given $(\ell_{i,t-1}, k_{it}, i_{it})$ there is enough cross-sectional variation left in ℓ_{it}.

The key conditions for the identification of α_L are: (a) the invertibility of the labor demand function $f_L(\ell_{i,t-1}, k_{it}, \omega_{it}, r_t)$ with respect to ω_{it}; (b) $r_{it} = r_t$, that is, no cross-sectional variability in unobservables, other than ω_{it}, affecting investment; and (c) given $(\ell_{i,t-1}, k_{it}, i_{it}, r_t)$, current labor ℓ_{it} still has enough sample variability. Assumption (c) is key, and it forms the basis for Ackerberg, Caves, and Frazer (2015) criticism (and extension) of the Olley-Pakes approach.

Example 3.3. Consider the Olley-Pakes model but with a parametric specification of the optimal investment equation (ID). More specifically, the inverse function f_K^{-1} has the following linear form:

$$\omega_{it} = \gamma_1\, i_{it} + \gamma_2\, \ell_{i,t-1} + \gamma_3\, k_{it} + r_{it} \tag{3.46}$$

Solving this equation into the PF, we have that:

$$y_{it} = \alpha_L\, \ell_{it} + (\alpha_K + \gamma_3)\, k_{it} + \gamma_1\, i_{it} + \gamma_2\, \ell_{i,t-1} + (r_{it} + e_{it}) \tag{3.47}$$

Note that current labor ℓ_{it} is correlated with current input prices r_{it}. That is the reason why we need Assumption OP-2, that is, $r_{it} = r_t$. Given that assumption we can control for the unobserved r_t by including time-dummies. Furthermore, to identify α_L with enough precision, there should not be high collinearity between current labor ℓ_{it} and the other regressors $(k_{it}, i_{it}, \ell_{i,t-1})$. ■

[3] The *partially linear model* is a regression model with two sets of regressors. One set of regressors enters linearly according to the linear index $x\beta$, and the other set of regressorts enters in a nonparametric function $\phi(z)$. That is, the regression model is $y = x\beta + g(z) + u$. The partially linear model is a class of semiparametric model that has received substantial attention in econometrics. See Li and Racine (2007).

In this first step, the control function approach deals also with the selection problem due to endogenous exit. This is because the control function controls for the value of the unobserved productivity ω_{it} such that there is not a selection problem associated with this nobservable.

Step 2: Estimation of α_K. For the sake of clarity, we first describe a version of the method that does not deal with the selection problem. We will discuss later the approach to deal with endogenous exit.

Given the estimate of α_L in step 1, the estimation of α_K is based on Assumptions *(OP-3)* and *(OP-4)*, that is, the Markov structure of the productivity shock, and the assumption of time-to-build productive capital. Since ω_{it} is first order Markov, we can write:

$$\omega_{it} = \mathbb{E}[\omega_{it} \mid \omega_{i,t-1}] + \xi_{it} = h(\omega_{i,t-1}) + \xi_{it} \tag{3.48}$$

where ξ_{it} is an innovation which is mean independent of any information at $t-1$ or before. Function $h(.)$ is unknown to the researcher and it has nonparametric form. Define $\phi_{it} \equiv \phi_t(\ell_{i,t-1}, k_{it}, i_{it})$, and remember that $\phi_t(\ell_{i,t-1}, k_{it}, i_{it}) = \alpha_K\, k_{it} + \omega_{it}$. Therefore, we have that:

$$\begin{aligned} \phi_{it} &= \alpha_K\, k_{it} + h(\omega_{i,t-1}) + \xi_{it} \\ &= \alpha_K\, k_{it} + h(\phi_{i,t-1} - \alpha_K\, k_{i,t-1}) + \xi_{it} \end{aligned} \tag{3.49}$$

Though we do not know the true value of ϕ_{it}, we have consistent estimates of these values from step 1: that is, $\hat{\phi}_{it} = y_{it} - \hat{\alpha}_L\, \ell_{it}$.[4]

If function $h(.)$ is nonparametrically specified, equation (3.49) is a partially linear model. However, it is not a standard partially linear model because the argument in function $h(.)$ is not observable. That is, though $\phi_{i,t-1}$ and $k_{i,t-1}$ are observable to the researcher (after the first step), the argument $\phi_{i,t-1} - \alpha_K k_{i,t-1}$ is unobservable because parameter α_K is unknown.

To estimate function $h(.)$ and parameter α_K, Olley and Pakes propose a recursive method. For the sake of illustration, suppose that we consider a quadratic function for $h(.)$: that is, $h(\omega) = \pi_1 \omega + \pi_2 \omega^2$. We start with an initial value for the parameter α_K, say $\hat{\alpha}_K^0$. Given this value, we construct the regressor $\hat{\omega}_{it}^0 = \hat{\phi}_{it} - \hat{\alpha}_K^0 k_{it}$, and estimate parameters (α_K, π_1, π_2) by applying OLS to the regression equation $\hat{\phi}_{it} = \alpha_K k_{it} + \pi_1 \hat{\omega}_{it-1}^0 + \pi_2 (\hat{\omega}_{it-1}^0)^2 + \xi_{it}$. Let $\hat{\alpha}_K^1$ be the OLS estimate of α_K. Then, we construct new values $\hat{\omega}_{it}^1 = \hat{\phi}_{it} - \hat{\alpha}_K^1 k_{it}$ and estimate again α_K, π_1, and π_2 by OLS. We apply this method repeatedly until convergence: that is, until the distance between the estimates of α_K in the last two iterations is smaller than a small constant: until $|\hat{\alpha}_K^n - \hat{\alpha}_K^{n-1}| < 10^{-6}$.

An alternative to this recursive procedure is the following Minimum Distance method. Again for concreteness, suppose that the specification of function $h(\omega)$ is quadratic. We have the regression model:

$$\hat{\phi}_{it} = \beta_1\, k_{it} + \beta_2\, \hat{\phi}_{i,t-1} + \beta_3\, \hat{\phi}_{i,t-1}^2 + \beta_4\, k_{i,t-1} + \beta_5\, k_{i,t-1}^2 + \beta_6\, \hat{\phi}_{i,t-1} k_{i,t-1} + \xi_{it} \tag{3.50}$$

where, according to the model, the parameters β in this regression satisfy the following restrictions: $\beta_1 = \alpha_K$; $\beta_2 = \pi_1$; $\beta_3 = \pi_2$; $\beta_4 = -\pi_1 \alpha_K$; $\beta_5 = \pi_2 \alpha_K^2$; and $\beta_6 = -2\pi_2 \alpha_K$.

[4] In fact, $\hat{\phi}_{it}$ is an estimator of $\phi_{it} + e_{it}$, but this does not have any incidence on the consistency of the estimator.

We can estimate the six β parameters by OLS. Then, in a second step, we use the OLS estimate of β and its variance-covariance matrix to estimate (α_K,π_1,π_2) by minimum distance imposing the six restrictions that relate the vector β with (α_K,π_1,π_2). More precisely, this minimum distance estimator is:

$$(\widehat{\alpha}_K,\widehat{\pi}_1,\widehat{\pi}_2) = \arg \min_{(\alpha_K,\pi_1,\pi_2)} \left[\widehat{\beta} - f(\alpha_K,\pi_1,\pi_2)\right]' \left[\widehat{V}(\widehat{\beta})\right]^{-1} \left[\widehat{\beta} - f(\alpha_K,\pi_1,\pi_2)\right] \tag{3.51}$$

where: $\widehat{\beta}$ is the column vector of OLS estimates; $\widehat{V}(\widehat{\beta})$ is its estimated variance matrix; and $f(\alpha_K,\pi_1,\pi_2)$ is the column vector with the functions $(\alpha_K,\ \pi_1,\ \pi_2,\ -\pi_1\alpha_K,\ \pi_2\alpha_K^2,\ -2\pi_2\alpha_K)$.

Example 3.4: Suppose that ω_{it} follows the AR(1) process $\omega_{it} = \rho\ \omega_{i,t-1} + \xi_{it}$, where $\rho \in [0,1)$ is a parameter. Then, $h(\omega_{i,t-1}) = \rho\,\omega_{i,t-1} = \rho(\phi_{i,t-1} - \alpha_K\ k_{i,t-1})$, and we can write:

$$\phi_{it} = \beta_1\ k_{it} + \beta_2\ \phi_{i,t-1} + \beta_3\ k_{i,t-1} + \xi_{it} \tag{3.52}$$

where $\beta_1 = \alpha_K$, $\beta_2 = \rho$, and $\beta_3 = -\rho\,\alpha_K$. In this regression, parameters α_K and ρ are over-identified. There is a testable over-identifying restriction: $\beta_3 = -\beta_1\beta_2$. ■

Time-to build is a key assumption for the consistency of this method. If new investment at period t is productive in the same period t, then we have that: $\phi_{it} = \alpha_K\ k_{i,t+1} + h(\phi_{i,t-1} - \alpha_K\ k_{it}) + \xi_{it}$. Now, the regressor $k_{i,t+1}$ depends on investment at period t and therefore it is correlated with the innovation in productivity ξ_{it}.

Empirical application. Olley and Pakes (1996) study the US telecommunication equipment industry during the period 1974-1987. During this period, the industry experienced substantial technological change and deregulation. There were elimination of barriers to entry. The 1984 Consent Decree was a antitrust decision to divest the industry leader, AT&T. There was substantial entry/exit of plants in the industry.

The authors use annual firm level data on output, capital, labor, and investment from the US Census of manufacturers. They estimate the production function for this industry. Table 3.2 presents their estimates using different estimation methods: OLS, Within-Groups, and the Olley-Pakes method described above. We can see that going from the OLS balanced panel to OLS full sample almost doubles β_K and reduces β_L by 20%. This result provides supportive evidence on the importance of selection bias due to endogenous exit. Controlling for simultaneity further increases β_K and reduces β_L.

Levinsohn and Petrin method

Levinsohn and Petrin (2003) propose an alternative control function method. A main difference between the models and methods by OP and the ones by Levinsohn and Petrin (LP) is that the latter use a control function for the unobserved productivity that comes from inverting the demand materials, instead of inverting the investment equation as in OP method. There are two main motivations for using this alternative control function. First, investment can be responsive only to persistent shocks in TFP; materials is responsive to every shock in TFP. Second, in some datasets there is a substantial fraction of observations with zero investment. At $i_{it} = 0$ (corner solution / extensive margin) there is not invertibility between i_{it} and ω_{it}. This has two implications: loss of efficiency because of the smaller number of observations, and, after estimation of

TABLE VI

ALTERNATIVE ESTIMATES OF PRODUCTION FUNCTION PARAMETERS[a]
(STANDARD ERRORS IN PARENTHESES)

Sample:	Balanced Panel		Full Sample[c,d]						
								Nonparametric F_{ω}	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Estimation Procedure	Total	Within	Total	Within	OLS	Only P	Only h	Series	Kernel
Labor	.851	.728	.693	.629	.628			.608	
	(.039)	(.049)	(.019)	(.026)	(.020)			(.027)	
Capital	.173	.067	.304	.150	.219	.355	.339	.342	.355
	(.034)	(.049)	(.018)	(.026)	(.018)	(.02)	(.03)	(.035)	(.058)
Age	.002	−.006	−.0046	−.008	−.001	−.003	.000	−.001	.010
	(.003)	(.016)	(.0026)	(.017)	(.002)	(.002)	(.004)	(.004)	(.013)
Time	.024	.042	.016	.026	.012	.034	.011	.044	.020
	(.006)	(.017)	(.004)	(.017)	(.004)	(.005)	(.01)	(.019)	(.046)
Investment	—	—	—	—	.13	—	—	—	—
					(.01)				
Other Variables	—	—	—	—	—	Powers of P	Powers of h	Full Polynomial in P and h	Kernel in P and h
# Obs.[b]	896	896	2592	2592	2592	1758	1758	1758	1758

Figure 3.2: Olley and Pakes (1996): Production Function Estimation

the model, no possibility of recovering the value of TFP for observations with zero investment.

LP consider a Cobb-Douglas production function in terms of labor, capital, and intermediate inputs (materials):

$$y_{it} = \alpha_L \, \ell_{it} + \alpha_K \, k_{it} + \alpha_M \, m_{it} + \omega_{it} + e_{it} \tag{3.53}$$

The investment equation is replaced with the intermediate input demand:

$$m_{it} = f_M \left(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it} \right) \tag{3.54}$$

Note that this demand for intermediate inputs is static in the sense that the lagged value m_{it-1} is not an argument in this demand function.

TABLE XI

DECOMPOSITION OF PRODUCTIVITY[a]
(EQUATION (16))

Year	p_t	$\bar{p}_t$	$\Sigma_i \Delta s_{it} \Delta p_{it}$	$\rho(p_t, k_t)$
1974	1.00	0.90	0.01	−0.07
1975	0.72	0.66	0.06	−0.11
1976	0.77	0.69	0.07	−0.12
1977	0.75	0.72	0.03	−0.09
1978	0.92	0.80	0.12	−0.05
1979	0.95	0.84	0.12	−0.05
1980	1.12	0.84	0.28	−0.02
1981	1.11	0.76	0.35	0.02
1982	1.08	0.77	0.31	−0.01
1983	0.84	0.76	0.08	−0.07
1984	0.90	0.83	0.07	−0.09
1985	0.99	0.72	0.26	0.02
1986	0.92	0.72	0.20	0.03
1987	0.97	0.66	0.32	0.10

Figure 3.3: Olley and Pakes (1996): Productivity estimates

Levinsohn and Petrin maintain assumptions OP-2 to OP-4, but replace the assumption of invertibility of the investment function in OP-1 with the following assumption of invertibility of the demand for intermediate inputs:

Assumption LP-1: $f_M(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it})$ is invertible in ω_{it}.

Similarly to the Olley-Pakes method, the key identification restriction in Levinsohn-Petrin method is that the only unobservable variable in the intermediate input demand equation that has cross-sectional variation across firms is the productivity shock ω_{it}. This is *assumption OP-2:* there is no cross-sectional variation in input prices such that $r_{it} = r_t$ for every firm i.

The LP method also proceeds in two steps. The first step consists of the least squares estimation of the parameter α_L and the nonparametric functions $\{\phi_t : t = 1, 2, ..., T\}$ in

the semiparametric regression equation:

$$y_{it} \quad = \quad \alpha_L \, \ell_{it} + \phi_t(\ell_{i,t-1}, k_{it}, m_{it}) + e_{it} \tag{3.55}$$

where $\phi_t(\ell_{i,t-1}, k_{it}, m_{it}) = \alpha_K \, k_{it} + f_M^{-1}(\ell_{i,t-1}, k_{it}, m_{it}, r_t)$ and f_M^{-1} represents the inverse function of the demand for intermediate inputs with respect to productivity.

The second step is also in the spirit of OP's second step, but it is substantially different because it requires instrumental variables or GMM estimation. More specifically, the estimates of α_L and ϕ_t are plugged-in, such that we have the regression equation:

$$\phi_{it} \quad = \quad \alpha_K \, k_{it} + \alpha_M \, m_{it} + h(\phi_{i,t-1} - \alpha_K \, k_{i,t-1} - \alpha_M \, m_{i,t-1}) + \xi_{it} \tag{3.56}$$

The main difference with respect to the OP method is that now the regressor m_{it} is correlated with the error term ξ_{it}. LP propose two alternative approaches to deal with this endogeneity problem. The first approach – described as "unrestricted method" – consists in applying instrumental variables, using lagged values to instrument m_{it} [see GNR (2013) criticism]. The second approach – described as "restricted method" – consists in using the first order condition for profit maximization with respect to materials. Under the assumptions that materials is an static input the firm is a price taker, the first oder condition implies that parameters β_M is equal to the ratio between the firm's cost of materials and its revenue.

Example 3.5: As in equation 3.4 above, suppose that ω_{it} follows the AR(1) process $\omega_{it} = \rho \; \omega_{i,t-1} + \xi_{it}$. Then, $h(\omega_{i,t-1}) = \rho \, \omega_{i,t-1} = \rho(\phi_{i,t-1} - \alpha_K \, k_{i,t-1} - \alpha_M \, m_{i,t-1})$, and we have that:

$$\phi_{it} = \beta_1 \, k_{it} + \beta_2 \, m_{it} + \beta_3 \, \phi_{i,t-1} + \beta_4 \, k_{i,t-1} + \beta_5 \, m_{i,t-1} + \xi_{it} \tag{3.57}$$

where: $\beta_1 = \alpha_K$, $\beta_2 = \alpha_M$, $\beta_3 = \rho$, $\beta_4 = -\rho \, \alpha_K$, and $\beta_5 = -\rho \, \alpha_M$. We have only three free parameters – α_K, α_M, and ρ – and the model implies four moment conditions: $\mathbb{E}(k_{it} \; \xi_{it}) = 0$; $\mathbb{E}(\phi_{i,t-1} \; \xi_{it}) = 0$; $\mathbb{E}(k_{i,t-1} \; \xi_{it}) = 0$; and $\mathbb{E}(m_{i,t-1} \; \xi_{it}) = 0$. These four moment conditions over-identify the three parameters. ■

Empirical application. LP use plant-level data from 8 different Chilean manufacturing industries during the period 1979-1985.

Ackerberg-Caves-Frazer critique

This critique applies both to Olley-Pakes and Levinsohn-Petrin methods. For the sake of concreteness, we focus here on Olley-Pakes method.

Under Assumptions *(OP-1)* and *(OP-2)*, we can invert the investment equation to obtain the productivity shock $\omega_{it} = f_K^{-1}(\ell_{i,t-1}, k_{it}, i_{it}, r_t)$. Then, we can solve the expression into the labor demand equation, $\ell_{it} = f_L(\ell_{i,t-1}, k_{it}, \omega_{it}, r_t)$, to obtain the following relationship:

$$\ell_{it} = f_L\left(\ell_{i,t-1}, \; k_{it}, \; f_K^{-1}(\ell_{i,t-1}, k_{it}, i_{it}, r_t), \; r_t\right) = G_t\left(\ell_{i,t-1}, k_{it}, i_{it}\right) \tag{3.58}$$

This expression shows an important implication of Assumptions *(OP-1)* and *(OP-2)*. For any cross-section t, there should be a deterministic relationship between employment at period t and the observable state variables $(\ell_{i,t-1}, k_{it}, i_{it})$. In other words, once we condition on the observable variables $(\ell_{i,t-1}, k_{it}, i_{it})$, employment at period t should

TABLE 3
Average Nominal Revenue Shares (Percentage), 1979-85

Industry	Unskilled	Skilled	Materials	Fuels	Electricity
Metals	15.2	8.3	44.9	1.6	1.7
Textiles	13.8	6.0	48.2	1.0	1.6
Food Products	12.1	3.5	60.3	2.1	1.3
Beverages	11.3	6.8	45.6	1.8	1.5
Other Chemicals	18.9	10.1	37.8	1.7	0.7
Printing & Pub.	19.8	10.7	40.1	0.5	1.3
Wood Products	20.6	5.3	47.0	3.0	2.4
Apparel	14.0	4.9	52.4	0.9	0.3

Figure 3.4: Levinsohn and Petrin (2003): Input shares

not have any cross-sectional variability. It should be constant. This implies that in the regression in step 1, $y_{it} = \alpha_L \ \ell_{it} + \phi_t(\ell_{i,t-1}, k_{it}, i_{it}) + e_{it}$, it should not be possible to identify α_L because the regressor ℓ_{it} does not have any sample variability that is independent of the other regressors $(\ell_{i,t-1}, k_{it}, i_{it})$.

Example 3.6: The problem can be simply illustrated using linear functions for the optimal investment and labor demand. Suppose that the inverse function f_K^{-1} is $\omega_{it} = \gamma_1 \ i_{it} + \gamma_2 \ \ell_{i,t-1} + \gamma_3 \ k_{it} + \gamma_4 r_t$; and the labor demand equation is $\ell_{it} = \delta_1 \ell_{i,t-1} + \delta_2 k_{it} + \delta_3 \omega_{it} + \delta_4 r_t$. Then, solving the inverse function f_K^{-1} into the production function, we get:

$$y_{it} = \alpha_L \ \ell_{it} + (\alpha_K + \gamma_3) \ k_{it} + \gamma_1 \ i_{it} + \gamma_2 \ \ell_{i,t-1} + (\gamma_4 r_t + e_{it}) \tag{3.59}$$

TABLE 2
Percent of Usable Observations, 1979-85

Industry	Investment	Fuels	Materials	Electricity
Metals	44.8	63.1	99.9	96.5
Textiles	41.2	51.2	99.9	97.0
Food Products	42.7	78.0	99.8	88.3
Beverages	44.0	73.9	99.8	94.1
Other Chemicals	65.3	78.4	100	96.5
Printing & Pub.	39.0	46.4	99.9	96.8
Wood Products	35.9	59.3	99.7	93.8
Apparel	35.2	34.5	99.9	97.2

Figure 3.5: Levinsohn and Petrin (2003): Frequency of nonzeroes

And solving the inverse function f_K^{-1} into the labor demand, we have that:

$$\ell_{it} = (\delta_1 + \delta_3\gamma_2)\ell_{i,t-1} + (\delta_2 + \delta_3\gamma_3)k_{it} + \delta_3\gamma_1 i_{it} + (\delta_4 + \delta_3\gamma_4)r_t \tag{3.60}$$

Equation (3.60) shows that, using one year of data (say year t) such that r_t is constant over this cross-sectional sample, there is perfect collinearity between ℓ_{it} and $(\ell_{i,t-1}, k_{it}, i_{it})$. This perfect multi-collinearity implies that it should not be possible to estimate α_L in equation (3.59). In most datasets, we find that this is not the case. That is, we find that ℓ_{it} has cross-sectional variation that is independent of $(\ell_{i,t-1}, k_{it}, i_{it})$. The presence of this independent variation contradicts the model. According to equation (3.60), a simple and plausible way to explain this independent variation is that input prices r_{it} have cross-sectional variation. However, this variation in input prices introduces an endogeneith problem in the estimation of equation (3.59) because the unobservable r_{it} is

TABLE 4
Unrestricted and Restricted Parameter Estimates for 8 Industries
(Bootstrapped Standard Errors in Parentheses)

	Industry (ISIC Code)							
Input	311	381	321	331	352	322	342	313
Unskilled labor								
	0.138	0.164	0.138	0.206	0.137	0.163	0.192	0.087
	(0.010)	(0.032)	(0.027)	(0.035)	(0.039)	(0.044)	(0.048)	(0.082)
Skilled labor								
	0.053	0.185	0.139	0.136	0.254	0.125	0.161	0.164
	(0.008)	(0.017)	(0.030)	(0.032)	(0.036)	(0.038)	(0.036)	(0.087)
Materials								
	0.703	0.587	0.679	0.617	0.567	0.621	0.483	0.626
	(0.013)	(0.017)	(0.019)	(0.022)	(0.045)	(0.020)	(0.028)	(0.075)
Fuels								
	0.023	0.024	0.041	0.018	0.004	0.0162	0.053	0.087
	(0.004)	(0.008)	(0.012)	(0.018)	(0.020)	(0.016)	(0.014)	(0.027)
Capital								
unrestricted	0.13	0.09	0.08	0.18	0.17	0.10	0.21	0.08
	(0.032)	(0.027)	(0.054)	(0.029)	(0.034)	(0.024)	(0.042)	(0.050)
restricted	0.14	0.09	0.06	0.11	0.15	0.09	0.21	0.07
	(0.011)	(0.02)	(0.019)	(0.025)	(0.034)	(0.039)	(0.045)	(0.11)
Electricity								
unrestricted	0.038	0.020	0.017	0.032	0.017	0.022	0.020	0.012
	(0.021)	(0.010)	(0.024)	(0.028)	(0.032)	(0.014)	(0.024)	(0.022)
restricted	0.011	0.015	0.014	0.021	0.005	0.008	0.011	0.012
No. Obs.	6051	1394	1129	1032	758	674	507	465

Figure 3.6: Levinsohn and Petrin (2003): PF estimate

part of the error term. That is, though there is apparent identification, it seems that this identification is spurious. ■

After pointing out this important problem in the Olley-Pakes model and method, Ackerberg, Caves, and Frazer discuss additional conditions in the model under which the Olley-Pakes estimator is consistent – that is, conditions under which there is no perfect collinearity problem, and the control function approach still solves the endogeneity problem.

For identification, we need some source of exogenous variability in labor demand that is independent of productivity and does not affect capital investment. Ackerberg-Caves-Frazer discuss several possible arguments/assumptions that incorporate this kind of exogenous variability in the model.

Consider a model with the same Cobb-Douglas PF as in the OP model but with the

following specification of labor demand and optimal capital investment:

$$\begin{aligned} (LD') \quad & \ell_{it} = f_L\left(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it}^L\right) \\ (ID') \quad & i_{it} = f_K\left(\ell_{i,t-1}, k_{it}, \omega_{it}, r_{it}^K\right) \end{aligned} \tag{3.61}$$

Ackerberg-Caves-Frazer propose to maintain Assumptions (OP-1), (OP-3), and (OP-4), and to replace Assumption (OP-2) by the following assumption:

Assumption ACF: Unobserved input prices r_{it}^L and r_{it}^K are such that conditional on (t, i_{it}, $\ell_{i,t-1}$, k_{it}): (a) r_{it}^L has cross-sectional variation, that is, $var(r_{it}^L \,|t, i_{it}, \ell_{i,t-1}, k_{it}) > 0$; and (b) r_{it}^L and r_{it}^K are independently distributed.

There are different possible interpretations of Assumption ACF. The following list of conditions (a) to (d) is a group of economic assumptions that generate Assumption ACF: (a) the capital market is perfectly competitive and the price of capital is the same for every firm ($r_{it}^K = r_t^K$); (b) there are internal labor markets such that the price of labor has cross-sectional variation; (c) the realization of the cost of labor r_{it}^L occurs after the investment decision takes place, and therefore r_{it}^L does not affect investment; and (d) the idiosyncratic labor cost shock r_{it}^L is not serially correlated such that lagged values of this shock are not state variables for the optimal investment decision. Aguirregabiria and Alonso-Borrego (2014) consider similar assumptions for the estimation of a production function with physical capital, permanent employment, and temporary employment.

Other identifying conditions: Quasi-fixed inputs

Consider a Cobb-Douglas PF with labor and capital as the only inputs. Suppose that the OP assumptions hold such that ℓ_{it} is perfectly collinear with $\phi_t(\ell_{i,t-1}, k_{it}, i_{it})$. If both capital and labor are quasi-fixed inputs, then it is possible to use a control function method in the spirit of OP or LP to identify/estimate β_L and β_K. Or in other words, this model has moment conditions that identify β_L and β_K (Wooldridge, 2009).

In the first step we have:

$$\begin{aligned} y_{it} &= \beta_L\, \ell_{it} + \phi_t(\ell_{i,t-1}, k_{it}, i_{it}) + e_{it} \\ &= \beta_L\, g_t(\ell_{i,t-1}, k_{it}, i_{it}) + \phi_t(\ell_{i,t-1}, k_{it}, i_{it}) + e_{it} \\ &= \psi_t(\ell_{i,t-1}, k_{it}, i_{it}) + e_{it} \end{aligned}$$

In this first step, we estimate $\psi_t(\ell_{i,t-1}, k_{it}, i_{it})$ nonparametrically. In the second step, given ψ_{it}, and taking into account that $\psi_{it} = \beta_L\, \ell_{it} + \beta_K\, k_{it} + \omega_{it}$, and $\omega_{it} = h(\omega_{i,t-1}) + \xi_{it}$, we have that:

$$\psi_{it} = \beta_L\, \ell_{it} + \beta_K\, k_{it} + h(\psi_{it} - \beta_L\, \ell_{it-1} + \beta_K\, k_{it-1}) + \xi_{it}$$

In this second step, ℓ_{it} is correlated with ξ_{it}, but $(k_{it}, \psi_{it}, \ell_{it-1},\, k_{it-1})$ are not, and (ℓ_{it-2}, k_{it-2}) can be used to instrument ℓ_{it}. This approach is in the same spirit as the Dynamic Panel Data (DPD) methods of Arellano-Bond and Blundell-Bond. This approach cannot be applied if some inputs (for instance, materials) are perfectly flexible. The PF coefficient parameter of the flexible inputs cannot be identified from the moment conditions in the second step.

Other identifying conditions: F.O.C. for flexible inputs

Klette and Griliches (1996), Doraszelski and Jaumandreu (2013), and Gandhi, Navarro, and Rivers (2017) propose combining conditions from the PF with conditions from the demand of variable inputs. This approach requires the price of the variable input to be observable to the researcher, though this price may not have cross-sectional variation across firms.

Note that in the LP method, the function that relates m_{it} with the state variables is just the condition "VMP of materials equal to price of materials". The parameters in this condition are the same as in the PF. This approach takes these restrictions into account.

For the CD-PF, with materials as flexible input, we have that:

$$\begin{aligned} (PF) \quad & y_{it} = \beta_L \, \ell_{it} + \beta_K \, k_{it} + \beta_M \, m_{it} + \omega_{it} + e_{it} \\ (FOC) \quad & p_t - p_t^M = \ln(\beta_M) + \beta_L \, \ell_{it} + \beta_K \, k_{it} + (\beta_M - 1) m_{it} + \omega_{it} \end{aligned} \tag{3.62}$$

The difference between these two equations is:

$$\ln(s_{it}^M) \equiv \ln\left(\frac{P_t^M M_{it}}{P_t Y_{it}}\right) = \ln(\beta_M) + e_{it}$$

where s_{it}^M is the ratio between materials expenditure and revenue. The parameter(s) of the flexible inputs are identified from the expenditure-share equations. The parameter(s) of the quasi-fixed inputs are identified using the dynamic conditions described above.

Gandhi, Navarro, and Rivers (2017) show that this approach can be extended in two important ways: (1) to a nonparametric specification of the production function: $y_{it} = f(\ell_{it}, k_{it}, m_{it}) + \omega_{it} + e_{it}$; and (2) to a model with monopolistic competition – instead of perfect competition – with an isoelastic product demand. Their approach to get extension (2) relies on an important assumption: there is not any bias or missing parameter in the marginal cost of the flexible input. For instance, suppose that the marginal cost of material were $MC_{Mt} = P_t^M \, \tau$, then our estimate of β_M would actually estimate $\beta_M \, \tau$.

3.4.5 Endogenous exit

Semiparametric selection models

The estimator in Olley and Pakes (1996) controls for selection bias due to endogenous exit of firms. Before describing their approach, it can be helpful to describe some general features of semiparametric selection models.

Consider a selection model with outcome equation,

$$y_i = \begin{cases} x_i \, \beta + \varepsilon_i & \text{if} \quad d_i = 1 \\ \\ \text{unobserved} & \text{if} \quad d_i = 0 \end{cases} \tag{3.63}$$

and selection equation

$$d_i = \begin{cases} 1 & \text{if} \quad h(z_i) - u_i \geq 0 \\ \\ 0 & \text{if} \quad h(z_i) - u_i < 0 \end{cases} \tag{3.64}$$

where x_i and z_i are exogenous regressors; (u_i, ε_i) are unobservable variables independently distributed of (x_i, z_i); and $h(.)$ is a real-valued function. We are interested in the consistent estimation of the vector of parameters β. We would like to have an estimator that does not rely on parametric assumptions on the function h or on the distribution of the unobservables.

The outcome equation can be represented as a regression equation: $y_i = x_i\,\beta + \varepsilon_i^{d=1}$, where $\varepsilon_i^{d=1} \equiv \{\varepsilon_i | d_i = 1\} = \{\varepsilon_i | u_i \leq h(z_i)\}$. Or similarly,

$$y_i = x_i\beta + \mathbb{E}(\varepsilon_i^{d=1} | x_i, z_i) + \tilde{\varepsilon}_i \tag{3.65}$$

where $\mathbb{E}(\varepsilon_i^{d=1} | x_i, z_i)$ is the selection term. The new error term, $\tilde{\varepsilon}_i$, is equal to $\varepsilon_i^{d=1} - \mathbb{E}(\varepsilon_i^{d=1} | x_i, z_i)$ and, by construction, it has mean zero and it is mean-independent of (x_i, z_i). The selection term is equal to $\mathbb{E}(\varepsilon_i \mid x_i, z_i, u_i \leq h(z_i))$. Given that u_i and ε_i are independent of (x_i, z_i), it is simple to show that the selection term depends on the regressors only through the function $h(z_i)$: that is, $\mathbb{E}(\varepsilon_i \mid x_i, z_i, u_i \leq h(z_i)) = g(h(z_i))$. The form of the function g depends on the distribution of the unobservables, and it is unknown if we adopt a nonparametric specification of that distribution. Therefore, we have the following partially linear model: $y_i = x_i\beta + g(h(z_i)) + \tilde{\varepsilon}_i$.

Define the *propensity score* P_i as:

$$P_i \equiv \Pr(d_i = 1 \mid z_i) = F_u(h(z_i)) \tag{3.66}$$

where F_u is the CDF of u. Note that $P_i = \mathbb{E}(d_i \mid z_i)$, and therefore we can estimate propensity scores nonparametrically using a Nadaraya-Watson kernel estimator or other nonparametric methods for conditional means. If u_i has unbounded support and a strictly increasing CDF, then there is a one-to-one invertible relationship between the propensity score P_i and $h(z_i)$. Therefore, the selection term $g(h(z_i))$ can be represented as $\lambda(P_i)$, where the function λ is unknown. The selection model can be represented using the partially linear model:

$$y_i = x_i\beta + \lambda(P_i) + \tilde{\varepsilon}_i. \tag{3.67}$$

A sufficient condition for the identification of β (without a parametric assumption on λ) is that $\mathbb{E}(x_i\, x_i' \mid P_i)$ has full rank. Given equation (3.67) and nonparametric estimates of propensity scores, we can estimate β and the function λ using standard estimators for partially linear model such as sieve methods, kernel-based methods like Robinson (1988), or differencing methods like Yatchew (2003).

Olley and Pakes method to control for endogenous exit

Now, we describe the Olley-Pakes procedure for the estimation of the production function taking into account endogenous exit. The first step of the method (that is, the estimation of α_L) is not affected by the selection problem because we are controlling for ω_{it} using a control function approach. However, there is endogenous selection in the second step of the method.

For simplicity consider that the productivity shock follows an AR(1) process: $\omega_{it} = \rho\ \omega_{i,t-1} - \xi_{it}$. Then, the "outcome" equation is:

$$\phi_{it} = \begin{cases} \alpha_K\ k_{it} + \rho\ \phi_{i,t-1} + (-\rho\,\alpha_K)\ k_{i,t-1} + \xi_{it} & \text{if} \quad d_{it} = 1 \\ \text{unobserved} & \text{if} \quad d_{it} = 0 \end{cases} \tag{3.68}$$

The exit/stay decision is: $\{d_{it} = 1\}$ iff $\{\omega_{it} \geq v(\ell_{it-1}, k_{it})\}$. Taking into account that $\omega_{it} = \rho\,\omega_{i,t-1} + \xi_{it}$, and that $\omega_{i,t-1} = \phi_{i,t-1} - \alpha_K\ k_{it-1}$, we have that the condition $\{\omega_{it} \geq v(\ell_{it-1}, k_{it})\}$ is equivalent to:

$$d_{it} = \begin{cases} 1 & \text{if} \quad \xi_{it} \leq v(\ell_{it-1}, k_{it}) - \rho\,(\phi_{i,t-1} - \alpha_K k_{it-1}) \\ \\ 0 & \text{if} \quad \xi_{it} > v(\ell_{it-1}, k_{it}) - \rho\,(\phi_{i,t-1} - \alpha_K k_{it-1}) \end{cases} \tag{3.69}$$

The propensity score is $P_{it} \equiv \mathbb{E}\,(d_{it} \mid \ell_{it-1}, k_{it}, \phi_{i,t-1}, k_{it-1})$ such that P_{it} is a function of $(\ell_{it-1}, k_{it}, \phi_{i,t-1}, k_{it-1})$. The equation controlling for selection is:

$$\phi_{it} = \beta_1\ k_{it} + \beta_2\ \phi_{i,t-1} + \beta_3\ k_{i,t-1} + \lambda\,(P_{it}) + \tilde{\xi}_{it} \tag{3.70}$$

where $\beta_1 = \alpha_K$, $\beta_2 = \rho$, and $\beta_2 = -\rho\,\alpha_K$. By construction, $\tilde{\xi}_{it}$ is mean independent of k_{it}, k_{it-1}, $\phi_{i,t-1}$, and P_{it}. We can estimate parameters β_1, β_2, and β_3 and function $\lambda\,(.)$ in the regression equation (3.70) by using standard methods for semiparametric partially linear models.

In reality, the method to control for selection in Olley and Pakes (1996) is a bit more involved because the stochastic process for the productivity shock is nonparametrically specified: $\omega_{it} = h(\omega_{i,t-1}) - \xi_{it}$. Therefore, the regression model is:

$$\phi_{it} = \alpha_K\ k_{it} + h\,(\phi_{i,t-1} - \alpha_K\ k_{i,t-1}) + \lambda\,(P_{it}) + \tilde{\xi}_{it} \tag{3.71}$$

such that we have two nonparametric functions, $h(.)$ and $\lambda\,(.)$. However, the identification and estimation of the model proceeds in a very similar way. For instance, we can consider a polynomial approximation to these nonparametric functions and estimate the parameters by least squares.

3.5 Determinants of productivity

3.5.1 What determines productivity?

There are large and persistent differences in TFP across firms. This evidence is ubiquitous even within narrowly defined industries and products.

Large TFP differences. A commonly used measure of the heterogeneity in TFP across firms is the ratio between the 90th to 10th percentile in the (cross-sectional) distribution. Using data from U.S. manufacturing industries – 4-digit SIC industries – Syverson (2004) reports that the ratio between the 90th to 10th percentile is on average equal 1.92. For industries in Denmark, Fox and Smeets (2011) report an average ratio of 3.75. This ratio is even larger in developing countries. For instance, Hsieh and Klenow (2009) report average ratios above 5 for China and India.

Persistent TFP differences. A statistic that is commonly used to measure this persistence is the slope parameter in the simple regression of the log-TFP of a firm on its log-TFP in the previous year. Most studies report estimates of this autoregressive coefficient between 0.6 to 0.8.

Relevant TFP differences. Studies show that these differences in productivity have an important impact on different decisions such as market exit, exporting, or investing in R&D.

Why do firms differ in their productivity levels? What mechanism can support such large differences in productivity in market equilibrium? Can producers control the factors that influence productivity or are they purely external effects of the environment? If firms can partly control their TFP, what type of choices increase it?

3.5.2 TFP dispersion in equilibrium

Following Syverson (2004), we present here a very stylized model to illustrate how dispersion in TFP within the same industry is perfectly possible in equilibrium, and that it can be driven by very common forces that exist in most markets. Consider a homogeneous product industry and let the profit of a firm be $\pi_i = R(q_i,d) - C(q_i,A_i,w) - F$, where: $R(q_i,d)$ is the revenue function; $C(q_i,A_i,w)$ is the variable cost function; q_i is output; A_i is TFP; d is the state of the demand; w represents input prices; and F is the fixed cost. Firms with different TFPs coexist in the same market if it is not optimal for the firm with the largest TFP to produce all the quantity demanded in the market. The key necessary and sufficient condition for this to occur is that the profit function of a firm must be strictly concave in output q_i. That is, either the revenue function $R(.)$ is strictly concave in q_i (because market power and oligopoly competition), or the cost function $C(.)$ is strictly convex in q_i (because diseconomies of scale or fixed inputs).

For instance, consider a perfectly competitive industry such that the revenue function is $R(q_i,d) = P(d)\ q_i$, that is, it is linear in output q_i. Suppose that there are decreasing returns to scale such that the cost function $C(q_i,A_i,w)$ is strictly convex in q_i. Then, even in this perfectly competitive industry we have that the firm with the highest TFP does not produce all the output demanded in the market.

Consider the – somehow – opposite case. The industry has a constant returns to scale technology such that the cost function is $C(q_i,A_i,w) = c(A_i,w)\ q_i$, that is, it is linear in output. This industry is characterized by Cournot competition. This implies that the revenue function is $R(q_i,d) = P(q_i + Q_{-i},d)\ q_i$, where $P(.)$ is the inverse demand function. This revenue function is strictly concave in q_i – provided the demand curve is downward sloping.

More formally, the equilibrium in the industry can be described by two types of conditions. At the intensive margin, optimal $q_i^* = q^*[A_i,d,w]$ is such that:

$$MR_i \equiv \frac{\partial R(q_i,A_i,d)}{\partial q_i} = \frac{\partial C(q_i,A_i,w)}{\partial q_i} \equiv MC_i \tag{3.72}$$

At the extensive margin, a firm is active in the market if:

$$R(q^*[A_i,d,w],A_i,d) - C(q^*[A_i,d,w],A_i,w) - F \geq 0 \tag{3.73}$$

If variable profit is strictly concave, this equilibrium can support firms with different TFPs. It is not optimal for the firm with the highest TFP to provide all the output in the industry. Firms with different TFPs – above a certain threshold value – coexist and compete in the same market.

3.5.3 How can firms improve their TFP?

There are multiple ways in which firms can affect their TFP. The following is a list of practices – non exhaustive – that firms can follow to increase their TFP, as well as empirical papers that have found evidence for these effects.

(i) Human resources and managerial practices: Bloom and VanReenen (2007); Ichniowski and Shaw (2003).
(ii) Learning-by-doing: Benkard (2000).
(iii) Organizational structure such as outsourcing or (the opposite) vertical integration.
(iv) Adoption of new technologies: Bresnahan, Brynjolfsson, and Hitt (2002).
(v) Investment in R&D and product and process innovation: Griliches (1979); Doraszelski and Jaumandreu (2013).

There is a long literature linking R&D investment and innovation to productivity. Multiple studies show evidence that R&D and innovation are important factors to explain firm heterogeneity in the level and growth of TFP. As usual, the main difficulty in these studies comes from separating causation from correlation. In section 3.6, we review models, methods, and datasets in different empirical applications dealing with the causal effect of R&D and/or innovation on TFP.

3.6 R&D and productivity

Investment in R&D and innovation is expensive. Investors – firms and governments – are interested in measuring its private and social returns. Process R&D is directed towards invention of new methods of production. Product R&D tries to create new and improved goods. Both process and product R&D can increase a firm's TFP. It can have also spillover effects in other firms: competition spillovers, and/or knowledge spillovers.

3.6.1 Knowledge capital model

In an influential paper, Griliches (1979) proposes a model and method to measure knowledge capital, that is, the capital generated by investments in R&D that is intangible and different from physical capital. This model is often referred to as the *knowledge capital model*, and many studies have used it to measure the returns to R&D.

The model is based on the estimation of a production function. Consider a Cobb-Douglas PF in logs:

$$y_{it} = \beta_L \, \ell_{it} + \beta_K \, k_{it} + \beta_M \, m_{it} + \beta_R \, k^R_{it} + \omega_{it} + e_{it}$$

where k_{it} is the logarithm of the stock of physical capital, and k^R_{it} is logarithm of the of stock of knowledge capital. A major difficulty is the measurement of the stock of knowledge capital. Let R_{it} be the investment in R&D of firm i at period t, and let K^R_{it} be the firm's stock of knowledge capital: that is, $K^R_{it} = \exp\{k^R_{it}\}$. Suppose that the researcher observes R_{it} for $t = 1, 2, ...T_i$. However, the researcher does not observe the stock of knowledge capital. Griliches (1979) proposes the following *perpetual inventory method* to obtain the sequence of stocks K^R_{it} for $t = 1, 2, ...T_i$. Suppose that the stock follows the transition rule:

$$K^R_{it} = (1 - \delta_R) \; K^R_{i,t-1} + R_{it}$$

where δ_R is the depreciation rate of knowledge capita. Given values for δ_R and for the the initial condition K^R_{i0}, we can use the data on R&D investments to construct the sequence $\{K^R_{it} : t = 1, 2, ...T_i\}$.

How to choose δ_R and K^R_{i0}? It is difficult to know the true value of the rate of technological obsolescence, δ_R: it can be endogenous, and vary across industries and

firms. Researchers have considered different approaches to estimate this depreciation rate: using patent renewal data (Pakes and Schankerman ,1984; Pakes ,1986); or using Tobin's Q model (Hall ,2007). The estimates of this depreciation rate in the literature range between 10% and 35%. Different authors have performed sensitivity analysis on the estimates of β_R for different value of δ_R. They report small differences, if any, in the estimate of β_R when δ_R varies between 8% and 25%.

3.6.2 An application

Doraszelski and Jaumandreu (2013) propose and estimate a model that extends the knowledge capital model in important ways. In their model, TFP and Knowledge capital (KC) are unobservables to the researcher. They follow stochastic processes that are endogenous and depend on (observable) R&D investments. The model accounts for uncertainty and heterogeneity across firms in the relationship between R&D and TFP. The model takes into account that the outcome of R&D investments is subject to a high degree of uncertainty.

For the estimation of the structural parameters in the PF the and stochastic process of KC, the authors exploit first order conditions for variable inputs.

Model

Consider the production function in logs:

$$y_{it} = \beta_L \, \ell_{it} + \beta_K \, k_{it} + \beta_M \, m_{it} + \omega_{it} + e_{it} \tag{3.74}$$

log-TFP ω_{it} follows a stochastic process with transition probability $p\left(\omega_{it+1} \mid \omega_{it}, \, r_{it}\right)$, where r_{it} is log-R&D expenditure. Every period t a firm chooses static inputs (ℓ_{it}, m_{it}) and investment in physical capital and R&D (i_{it}, r_{it}) to maximize its value.

$$V(s_{it}) = \max_{i_{it}, r_{it}} \left\{ \pi(s_{it}) - c^{(1)}(i_{it}) - c^{(2)}(r_{it}) + \rho \, \mathbb{E}\left[V(s_{it+1}) | s_{it}, i_{it}, r_{it}\right] \right\} \tag{3.75}$$

with $s_{it} = (k_{it}, \omega_{it}$, input prices $[w_{it}]$, demand shifters $[d_{it}])$.

The Markov structure of log-TFP implies:

$$\omega_{it} = \mathbb{E}\left[\omega_{it} \mid \omega_{it-1}, \, r_{it-1}\right] + \xi_{it} = g\left(\omega_{it-1}, \, r_{it-1}\right) + \xi_{it}$$

where $\mathbb{E}\left[\xi_{it} \mid \omega_{it-1}, \, r_{it-1}\right] = 0$. The *productivity innovation* ξ_{it} captures two sources of uncertainty for the firm: the uncertainty linked to the evolution of TFP; and the uncertainty inherent to R&D – for instance, chances of making a new discovery, its degree of applicability, successful implementation, etc.

The authors' identification approach exploits static marginal conditions of optimality. Obtaining these conditions requires an assumption about competition. The authors assume monopolistic competition. More precisely, they assume the following form for the marginal revenue:

$$MR_{it} = P_{it} \left(1 - \frac{1}{\eta(p_{it}, d_{it})} \right) \tag{3.76}$$

where $\eta(p_{it}, d_{it})$ is the price elasticity of demand for firm i, that is, monopolisitc competition.

The marginal condition of optimality for labor provides a closed-form expression for labor demand. Solving for log-TFP in the labor demand equation, we get:

$$\begin{aligned} \omega_{it} &= \lambda - \beta_K\, k_{it} + (1-\beta_L-\beta_M)\, \ell_{it} + (1-\beta_M)\, (w_{it}-p_{it}) \\ &+ \beta_M\, (p_{Mit}-p_{it}) - \ln\left(1-\frac{1}{\eta(p_{it},d_{it})}\right) \end{aligned} \tag{3.77}$$

We represent the RHS as $h(x_{it},\beta)$, such that $\omega_{it} = h(x_{it},\beta)$, with $x_{it} = (k_{it},\, \ell_{it},\, w_{it},\, p_{Mit},\, p_{it},\, d_{it})$.

Identification and estimation

Combining the PF equation with the stochastic process for TFP, and the marginal condition for optimal labor, we have the equation:

$$y_{it} = \beta_L\, \ell_{it} + \beta_K\, k_{it} + \beta_M\, m_{it} + g\left[h(x_{it-1},\beta),\; r_{it-1}\right] + \xi_{it} + e_{it} \tag{3.78}$$

From the marginal condition for labor we have:

$$h(x_{it},\beta) = g\left[h(x_{it-1},\beta),\; r_{it-1}\right] + \xi_{it} \tag{3.79}$$

The "parameters" in this system of equations are: β_L, β_K, β_M, g, and η. The unobservables ξ_{it} and e_{it} are mean independent of any observable variable at period $t-1$ or before. Therefore, x_{it-1} and r_{it-1} are exogenous w.r.t. $\xi_{it}+e_{it}$. Capital stock k_{it} is also uncorrelated to the error term because of time-to-build. However, we need to instrument the regressors ℓ_{it} and m_{it}.

To see that the parameters of the model are identified, it is convenient to consider a simplified version with: $\beta_K = \beta_M = 1/\eta = 0$ and $g\left[\omega_{t-1},\; r_{t-1}\right] = \rho_\omega\, \omega_{t-1} + \rho_r\, r_{t-1}$. Then, we have:

$$y_{it} = \beta_L\, \ell_{it} + \rho_\omega\left[(1-\beta_L)\,\ell_{it-1} + w_{it-1} - p_{it-1}\right] + \rho_r\, r_{it-1} + \xi_{it} + e_{it} \tag{3.80}$$

By using the vector of instruments $Z_{it} = (y_{it-1},\, \ell_{it-1},\, w_{it-1}-p_{it-1}\,,\, r_{it-1})$, we have that the moment conditions $\mathbb{E}\left[Z_{it}\;(\xi_{it}+e_{it})\right] = 0$ identify β_L, ρ_ω, ρ_r. Given the identification of these parameters, we know $\omega_{it} = h(x_{it},\beta) = (1-\beta_L)\ell_{it} + (w_{it}-p_{it})$. The model implies, that:

$$\xi_{it} = h(x_{it},\beta) - \rho_\omega\, h(x_{it},\beta) - \rho_r\, r_{it-1} \tag{3.81}$$

such that ξ_{it} is identified, and so is its variance $Var(\xi_{it})$ that represents uncertainty in the link between R&D and TFP.

The instrument $w_{it-1} - p_{it-1}$ plays a very important role in the identification of the model. Without variation in lagged (real) input prices the model is not identified. But note that the model does not use contemporaneous input prices as instruments because they can be correlated with the innovation ξ_{it}.

Data

The papers uses panel data of Spanish manufacturing firms ($N = 1,870$ firms) from ten industries (SIC 2-digits). The dataset has annual frequency and it covers the period $1990-1999$ (max $T_i = 10$). This was a period of rapid growth in output and physical

capital, coupled with stagnant employment. Table 3.7 presents some descriptive statistics. R&D intensity = R&D expenditure / Sales: the average among all firms is 0.6% (smaller than in France, Germany, or UK, $> 2\%$). R&D intensity among performers (column 13) is between 1% and 3.5%.

TABLE 1
Descriptive statistics

					Rates of growth[a]					With R&D[b]			
Industry	Obs.[a]	Firms[a]	Entry[a] (%)	Exit[a] (%)	Output (s. d.)	Capital (s. d.)	Labour (s. d.)	Materials (s. d.)	Price (s. d.)	Obs. (%)	Stable (%)	Occas. (%)	R&D inten. (s. d.)
	(1)	(2)	(3)	(4)	(5)	(7)	(6)	(8)	(9)	(10)	(11)	(12)	(13)
1. Metals and metal products	1235	289	88 (30.4)	17 (5.9)	0.050 (0.238)	0.086 (0.278)	0.010 (0.183)	0.038 (0.346)	0.012 (0.055)	420 (34.0)	63 (21.8)	72 (24.9)	0.0126 (0.0144)
2. Non-metallic minerals	621	131	20 (15.3)	15 (11.5)	0.037 (0.208)	0.062 (0.238)	−0.001 (0.141)	0.039 (0.308)	0.010 (0.059)	186 (30.0)	16 (12.2)	41 (31.3)	0.0100 (0.0211)
3. Chemical products	1218	275	64 (23.3)	15 (5.5)	0.068 (0.196)	0.093 (0.238)	0.007 (0.146)	0.054 (0.254)	0.007 (0.061)	672 (55.2)	124 (45.1)	55 (20.0)	0.0268 (0.0353)
4. Agric. and ind. machinery	576	132	36 (27.3)	6 (4.5)	0.059 (0.275)	0.078 (0.247)	0.010 (0.170)	0.046 (0.371)	0.013 (0.032)	322 (55.9)	52 (39.4)	35 (26.5)	0.0219 (0.0275)
6. Transport equipment	637	148	39 (26.4)	10 (6.8)	0.087 (0.354)	0.114 (0.255)	0.011 (0.207)	0.087 (0.431)	0.007 (0.037)	361 (56.7)	62 (41.9)	35 (23.6)	0.0224 (0.0345)
7. Food, drink, and tobacco	1408	304	47 (15.5)	22 (7.2)	0.025 (0.224)	0.094 (0.271)	−0.003 (0.186)	0.019 (0.305)	0.022 (0.065)	386 (27.4)	56 (18.4)	64 (21.1)	0.0071 (0.0281)
8. Textile, leather, and shoes	1278	293	77 (26.3)	49 (16.7)	0.020 (0.233)	0.059 (0.235)	−0.007 (0.192)	0.012 (0.356)	0.016 (0.040)	378 (29.6)	39 (13.3)	66 (22.5)	0.0152 (0.0219)
9. Timber and furniture	569	138	52 (37.7)	18 (13.0)	0.038 (0.278)	0.077 (0.257)	0.014 (0.210)	0.029 (0.379)	0.020 (0.035)	66 (12.6)	7 (5.1)	18 (13.8)	0.0138 (0.0326)
10. Paper and printing products	665	160	42 (26.3)	10 (6.3)	0.035 (0.183)	0.099 (0.303)	−0.000 (0.140)	0.026 (0.265)	0.019 (0.089)	113 (17.0)	21 (13.1)	25 (13.8)	0.0143 (0.0250)

Figure 3.7: Doraszelski and Jaumandreu (2013): Descriptive statistics

Estimates

Figure 3.8 presents parameter estimates. Comparing GMM and OLS estimates, we can see that correcting for endogeneity has the expected implications. For instance, β_L and β_M decline, and β_K increases. There are not big differences in the β estimates across industries. The test of of over-identifying restrictions (OIR) cannot reject the validity of the instruments with a 5% confidence level. The test of parameter restrictions (in the two equations) can reject these restrictions at 5% level only in 2 out of 10 industries.

As for the stochastic process for TFP, the model where TFP doesn't depend on R&D is clearly rejected. Models with linear effects or without complementarity between ω_{t-1} and r_{t-1} are rejected. $Var(e)$ is approx. equal to $Var(\omega)$ in most industries. $Var(\xi)/Var(\omega)$ is between 30% and 75%. The authors find significant evidence supporting that the effect of R&D on TFP is stochastic and uncertain to forms. There are significant differences across industries in the magnitude of this uncertainty.

The authors test three different versions of the knowledge capital (KC) model. For the basic KC model (where $\omega_{it} + e_{it} = \beta_R \, k_{it}^R + e_{it}$), the authors can reject this model for all industries. The second model is Hall and Hayashi (1989) and Klette (1996) KC model, where $\omega_{it} = \sigma \, \omega_{it-1} + (1-\sigma) \, r_{it-1} + \xi_{it}$. This model is rejected at 5% level in 8 industries, and at 7% level in all the industries. The third KC model is characterized by the equation $k_{it}^R + \omega_{it} + e_{it}$, and ω_{it} follows an exogenous Markov process. This model is ejected at 5% levle in 2 industries, and at 10% in 6 industries.

TABLE 2
Production function estimates and specification tests

Industry	OLS[a]			GMM[a]			Overidentifying restrictions test		Parameter restrictions test	
	Capital (std. err.)	Labour (std. err.)	Materials (std. err.)	Capital (std. err.)	Labour (std. err.)	Materials (std. err.)	$\chi^2(df)$	p val.	$\chi^2(3)$	p val.
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1. Metals and metal products	0.109 (0.013)	0.252 (0.022)	0.642 (0.020)	0.106 (0.014)	0.111 (0.031)	0.684 (0.011)	62.553 (51)	0.129	11.666	0.009
2. Non-metallic minerals	0.096 (0.021)	0.275 (0.034)	0.655 (0.028)	0.227 (0.014)	0.137 (0.016)	0.633 (0.014)	50.730 (47)	0.329	6.047	0.109
3. Chemical products	0.060 (0.010)	0.239 (0.021)	0.730 (0.020)	0.132 (0.015)	0.122 (0.026)	0.713 (0.011)	48.754 (47)	0.402	0.105	0.991
4. Agric. and ind. machinery	0.051 (0.017)	0.284 (0.038)	0.671 (0.027)	0.079 (0.015)	0.281 (0.029)	0.642 (0.013)	45.833 (44)	0.396	1.798	0.615
6. Transport equipment	0.080 (0.023)	0.289 (0.033)	0.636 (0.046)	0.117 (0.015)	0.158 (0.023)	0.675 (0.016)	40.296 (47)	0.745	0.414	0.937
7. Food, drink, and tobacco	0.094 (0.014)	0.177 (0.016)	0.739 (0.016)	0.068 (0.014)	0.129 (0.024)	0.766 (0.008)	61.070 (46)	0.068	8.866	0.031
8. Textile, leather, and shoes	0.059 (0.010)	0.335 (0.024)	0.605 (0.019)	0.057 (0.011)	0.313 (0.016)	0.593 (0.013)	66.143 (51)	0.075	4.749	0.191
9. Timber and furniture	0.079 (0.019)	0.283 (0.029)	0.670 (0.029)	0.131 (0.009)	0.176 (0.017)	0.697 (0.011)	44.951 (43)	0.390	0.618	0.892
10. Paper and printing products	0.092 (0.016)	0.321 (0.029)	0.621 (0.025)	0.121 (0.013)	0.249 (0.025)	0.617 (0.014)	51.371 (42)	0.152	5.920	0.118

Figure 3.8: Doraszelski and Jaumandreu (2013): PF estimates

Counterfactuals on R&D and TFP.

The distribution of TFP with R&D stochastically dominates distribution without R&D. Differences in means are between 3% and 5% for all industries and firm sizes, except for small firms in industries with low observed R&D intensity.

The magnitude of the elasticity of TFP with respect to R&D has considerable variation between and within industries. Its average is 0.015. The elasticity of TFP with respect to lagged TFP shows substantial persistence, but there is also considerable heterogeneity between and within industries. Non-performers have a higher degree of persistence than performers. The degreee of persistence is negatively related to the degree of uncertainty.

In summary, the authors model TFP growth as the consequence of R&D expenditures with uncertain outcomes. Results show that this model can explain better the relationship between TFP and R&D than standard Knowledge Capital models without uncertainty and non-linearity. R&D is a major determinant of the differences in TFP across firms and of their evolution. They also find that firm-level uncertainty in the outcome of R&D is considerable. Their estimates suggest that engaging in R&D roughly doubles the degree of uncertainty in the evolution of a producer's TFP.

3.7 Exercises

3.7.1 Exercise 1

Consider an industry for an homogeneous product. Firms use capital and labor to produce output according to a Cobb-Douglas technology with parameters α_L and α_K and Total Factor Productivity (TFP) A.

Industry	Exogeneity test		Separability test		$\frac{Var(e_{jt})}{Var(\omega_{jt})}$	$\frac{Var(\xi_{jt})}{Var(\omega_{jt})}$
	$\chi^2(10)$	p val.	$\chi^2(3)$	p val.		
	(1)	(2)	(3)	(4)	(5)	(6)
1. Metals and metal products	65.55	0.000	16.360	0.001	0.735	0.407
2. Non-metallic minerals	92.65	0.000	13.027	0.005	0.842	0.410
3. Chemical products	40.79	0.000	8.647	0.034	0.749	0.244
4. Agric. and ind. machinery	51.88	0.000	11.605	0.009	1.410	0.505
6. Transport equipment	56.85	0.000	18.940	0.000	1.626	0.524
7. Food, drink, and tobacco	38.29	0.000	7.186	0.066	1.526	0.300
8. Textile, leather, and shoes	29.91	0.001	18.417	0.000	1.121	0.750
9. Timber and furniture	118.17	0.000	32.260	0.000	1.417	0.515
10. Paper and printing products	59.73	0.000	23.249	0.000	0.713	0.433

Figure 3.9: Doraszelski and Jaumandreu (2013): Stochastic process TFP

Question 1.1. Write the expression for this Cobb-Douglas production function (PF).

Suppose that firms are price takers in the input markets for labor and capital. Let W_L and W_K be the price of labor and capital, respectively. Capital is a fixed input such that the fixed cost for a firm, say i, is $FC_i = W_K K_i$. The *variable cost function*, $VC(Y)$, is defined as the minimum cost of labor to produce an amount of output Y.

Question 1.2. Derive the expression for the variable cost function of a firm in this industry. Explain your derivation. [Hint: Given that capital is fixed and there is only one variable input, the minimization problem is trivial. The PF implies that there is only one possible amount of labor that give us a certain amount of output].

Question 1.3. Using the expressions for the fixed cost and for the variable cost function in Q1.2:

(a) Explain how an increase in the amount of capital affects the fixed cost and the variable cost of a firm.

(b) Explain how an increase in TFP affects the fixed cost and the variable cost.

Suppose that the output market in this industry is competitive: firms are price takers. The demand function is linear with the following form: $P = 100 - Q$, where P and Q are the industry price and total output, respectively. Suppose that $\alpha_L = \alpha_K = 1/2$, and the value of input prices are $W_L = 1/2$ and $W_K = 2$. Remember that firms' capital stocks are fixed (exogenous), and for simplicity suppose that all the firms have the same capital stock $K = 1$.

Question 1.4. Using these primitives, write the expression for the profit function of a firm (revenue, minus variable cost, minus fixed cost) as a function of the market price, P, the firm's output, Y_i, and its TFP, A_i.

Knowledge capital model tests

Basic		Generalization 1		Generalization 2	
$N(0,1)$	p val.	$N(0,1)$	p val.	$N(0,1)$	p val.
(7)	(8)	(9)	(10)	(11)	(12)
−2.815	0.002	−2.431	0.008	−1.987	0.023
−2.041	0.021	−1.541	0.062	−0.784	0.216
−3.239	0.001	−2.090	0.018	−1.400	0.081
−2.693	0.004	−1.588	0.056	−1.493	0.068
−2.317	0.010	−2.042	0.021	−1.821	0.034
−3.263	0.001	−2.499	0.006	−0.901	0.184
−2.770	0.003	−1.788	0.037	−1.488	0.068
−2.510	0.006	−2.097	0.018	−1.028	0.152
−3.076	0.001	−2.210	0.014	−1.595	0.055

Figure 3.10: Doraszelski and Jaumandreu (2013): Testing Knowledge capital

Question 1.5. Using the condition "price equal to marginal cost", obtain the optimal amount of output of a firm as a function of the market price, P, and the firm's TFP, A_i. Explain your derivation.

Question 1.6. A firm is active in the market (that is, it finds optimal to produce a positive amount of output) only if its profit is greater or equal than zero. Using this condition show that a firm is active in this industry only if its TFP satisfies the condition $A_i \geq 2/P$. Explain your derivation.

Let $(P^*, Q^*, Y_1^*, Y_2^*, ..., Y_N^*)$ the equilibrium price, total output, and individual firms' outputs. Based on the previous results, the market equilibrium can be characterized by the following conditions: (i) the demand equation holds; (ii) total output is equal to the sum of firms' individual outputs; (iii) firm i is active ($Y_i^* > 0$) if and only if its total profit is greater than zero; and (iv) for firms with $Y_i^* > 0$, the optimal amount of output is given by the condition price is equal to marginal cost.

Question 1.7. Write conditions (i) to (iv) for this particular industry.

Question 1.8. Combine conditions (i) to (iv) to show that the equilibrium price can be written as the solution to this equation:

$$P^* = 100 - P^* \left[\sum_{i=1}^{N} A_i^2 \, 1\{A_i \geq 2/P^*\} \right]$$

where $1\{x\}$ is the indicator function that is defined as $1\{x\} = 1$ if condition x is true, and $1\{x\} = 0$ if condition x is false. Explain your derivation.

Suppose that the subindex i sorts firms by their TFP such that firm 1 is the most efficient, then firm 2, etc. That is, $A_1 > A_2 > A_3 > .....$

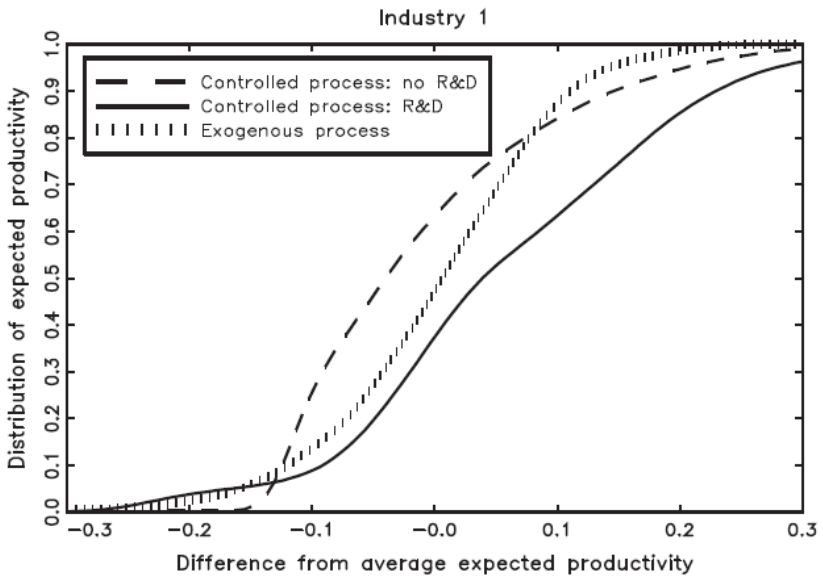


Figure 3.11: Doraszelski and Jaumandreu (2013): R&D and productivity

Question 1.9. Suppose that $A_1 = 7$, $A_2 = 5$, and $A_3 = 1$. Obtain the equilibrium price, total output, and output of each individual firm in this industry. [Hint: Start with the conjecture that only firms 1 and 2 produce in equilibrium. Then, confirm this conjecture. Note that we do not need to know the values of A_4, A_5, etc].

Question 1.10. Explain why the most efficient firm, with the largest TFP, does not produce all the output of the industry.

3.7.2 Exercise 2

The Stata datafile `blundell_bond_2000_production_function.dta` contains annual information on sales, labor, and capital for 509 firms for the period 1982-1989 (8 years). Consider a Cobb-Douglas production function in terms of labor and capital. Use this dataset to implement the following estimators.

Question 2.1. OLS with time dummies. Test the null hypothesis $\alpha_L + \alpha_K = 1$. Provide the code in Stata and the table of estimation results. Comment the results.

Question 2.2. Fixed Effects estimator with time dummies. Test the null hypothesis of no time-invariant unobserved heterogeneity: $\eta_i = \eta_i$ for every firm i. Provide the code in Stata and the table of estimation results. Comment the results.

Question 2.3. Fixed Effects - Cochrane Orcutt estimator with time dummies. Test the two over-identifying restrictions of the model. Provide the code in Stata and the table of estimation results. Comment the results.

Industry	Elasticity wrt. R_{jt-1}[a]			
	Q_1	Q_2	Q_3	Mean
	(1)	(2)	(3)	(4)
1. Metals and metal products	−0.013	0.007	0.021	0.022
2. Non-metallic minerals	−0.018	−0.012	0.000	−0.006
3. Chemical products	0.009	0.011	0.014	0.013
4. Agric. and ind. machinery	−0.017	−0.009	0.021	0.005
6. Transport equipment	−0.034	−0.008	0.010	0.020
7. Food, drink, and tobacco	−0.008	0.010	0.026	0.020
8. Textile, leather, and shoes	−0.003	0.014	0.051	0.046
9. Timber and furniture	−0.031	0.005	0.048	0.004
10. Paper and printing products	−0.036	0.022	0.049	0.013

Figure 3.12: Doraszelski and Jaumandreu (2013): Elasticity TFP lagged R&D

Question 2.4. Arellano-Bond estimator with time dummies and non-serially correlated transitory shock. Provide the code in Stata and the table of estimation results. Comment the results.

Question 2.5. Arellano-Bond estimator with time dummies and AR(1) transitory shock. Provide the code in Stata and the table of estimation results. Comment the results.

Question 2.6. Blundell-Bond system estimator with time dummies and non-serially correlated transitory shock. Provide the code in Stata and the table of estimation results. Comment the results.

Question 2.7. Blundell-Bond system estimator with time dummies and AR(1) transitory shock. Provide the code in Stata and the table of estimation results. Comment the results.

Question 2.8. Based on the previous results, select your preferred estimates of the production function. Explain your choice.

3.7.3 Exercise 3

The Stata datafile `data_mines_eco2901_2017.dta` contains annual information on output and inputs from 330 copper mines for the period 1992-2010 (19 years). The following is a description of the variables.

Elasticity wrt. ω_{jt-1}[b]					
Performers			Non-performers		
Q_1	Q_2	Q_3	Q_1	Q_2	Q_3
(5)	(6)	(7)	(8)	(9)	(10)
0.504	0.619	0.755	0.441	0.759	0.901
0.433	0.477	0.575	0.377	0.646	0.878
0.459	0.523	0.634	0.547	0.815	0.947
0.434	0.721	0.791	0.729	0.894	0.979
0.404	0.615	0.727	0.423	0.513	0.646
0.445	0.705	0.867	0.822	0.930	0.965
0.090	0.325	0.626	0.491	0.605	0.689
0.458	0.585	0.814	0.303	0.430	0.641
0.405	0.676	0.812	0.569	0.644	0.670

Figure 3.13: Doraszelski and Jaumandreu (2013): Elasticity TFP lagged TFP

Variable name		**Description**
`id`	:	Mine identification number
`year`	:	Year [from 1992 to 2010]
`active`	:	Binary indicator of the event "mine is active during the year"
`prod_tot`	:	Annual production of pure copper of the mine [in thousands of tonnes]
`reserves`	:	Estimated mine reserves [in thousands of ore]
`grade`	:	Average ore grade (in %) of mined ore during the year (% copper / ore)
`labor_n_tot`	:	Total number of workers per year (annual equivalent)
`cap_tot`	:	Measure of capital [maximum production capacity of the mine]
`fuel_cons_tot`	:	Consumption of fuel (in physical units)
`elec_cons_tot`	:	Consumption of electricity (in physical units)
`materials_tot`	:	Consumption of intermediate inputs / materials (in $ value)

Note that some variables have a few missing values even at years when the mine is actively producing.

Question 3.1. Consider a Cobb-Douglas production function in terms of labor, capital, fuel, electricity, and ore grade. Use this dataset to implement the following estimators:

- OLS
- Fixed-Effects
- Arellano-Bond estimator with non-serially correlated transitory shock
- Arellano-Bond estimator with AR(1) transitory shock
- Blundell-Bond estimator with non-serially correlated transitory shock

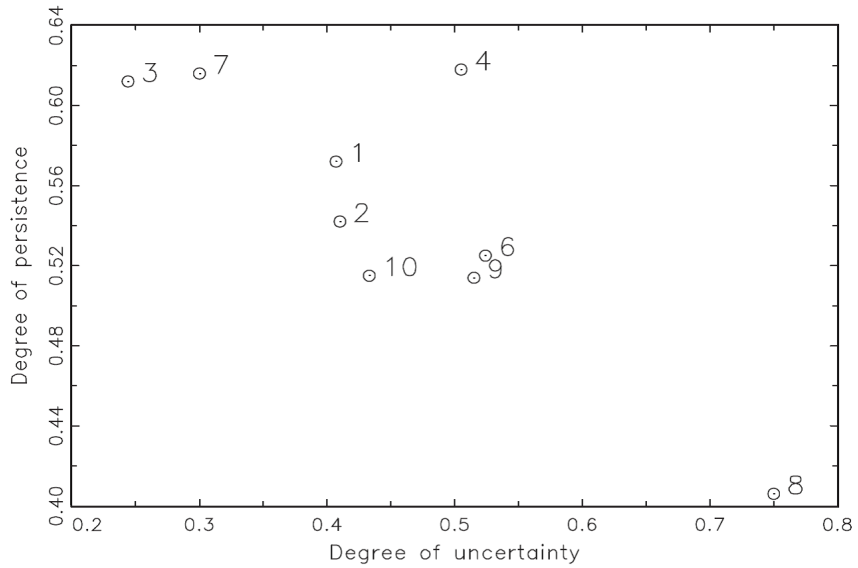


Figure 3.14: Doraszelski and Jaumandreu (2013): Uncertainty and persistence TFP

- Blundell-Bond estimator with AR(1) transitory shock
- Olley-Pakes (Using the first difference in cap_tot as investment)
- Levinshon-Petrin

Question 3.2. Suppose that these mines are price takers in the input markets. Consider that the variable inputs are labor, fuel, and electricity.

(a) Derive the expression for the Variable Cost function for a mine (that is, the minimum cost to produce an amount of output given input prices).
(b) Let $\ln MC_{it}$ be the logarithm of the realized Marginal Cost of mine i at year t. I have not included data on input prices in this dataset, so we will assume that mines face the same prices for variable inputs, and normalize to zero the contribution of these input prices to $\ln MC_{it}$. Calculate the quantiles 5%, 25%, 50%, 75%, and 95% in the cross-sectional distributions of $\ln MC_{it}$ at each year in the sample. Present a figure with the time-series of these five quantiles over the sample period. Comment the results.
(c) For a particular sample year, say 2005, calculate the contribution of each component of $\ln MC_{it}$ (that is, total factor productivity, capital, ore grade, and output) to the cross-sectional variance of $\ln MC_{it}$. Present it in a table. Comment your results.
[Note: To measure the contribution of each component, use the following approach. Consider $y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_K x_K$. A measure of the contribution of x_j to $var(y)$ is $\rho_j \equiv \dfrac{var(y) - var(y \mid x_j = constant)}{var(y)}$. Note that $\rho_j \in (0,1)$ for any variable x_j. However, in general, $\sum_{j=1}^{K} \rho_j$ can be either

smaller or greater than one, depending the sign of the covariances between the components.]

(d) Consider the balance panel of mines that are active in the industry every year during the sample period. Repeat exercises (b) and (c) for this balanced panel. Compared your results with those in (c) and (d). Comment the results.

4. Competition in prices/quantities

4.1 Introduction

The decisions of how much to produce and what price to charge are fundamental determinants of firms' profits. These decisions are also main sources of strategic interactions: a firm's profit not only depends on its own decisions but also on other firms' actions. In the market for a homogeneous good, the price declines with total output such that a firm's profit also declines with the amount of output produced by its competitors. In a differentiated product industry, demand for a firm's product increases with the prices of products sold by other firms. These strategic interactions have first order importance to understand competition and outcomes in most industries. For this reason, models of competition where firms choose prices or quantities are at the core of Industrial Organization.

The answers to many economy questions in IO require not only the estimation of demand and cost functions but also the explicit specification of an equilibrium model of competition. For instance, evaluating the effects on prices, profits, and welfare of an increase in the minimum wage (or in the sales tax rate) requires to understand firms' incentives to change their prices or outputs in response to a change in costs. This incentive depends on their beliefs about what other firms will do: that is, it depends on how firms compete in the market.

The estimation of competition models can provide information on firms' marginal costs, on the form of competition, and on the demand function. In many empirical applications, the researcher has information on firms' prices and quantities sold, but information on firms' costs is not always available. The researcher may not observe the amounts of firms' inputs, such that it is not even possible to obtain costs by estimating the production function as described in chapter 3. In this context, empirical models of competition in prices or quantities may provide an approach to obtain estimates of firms' marginal costs and the structure of the marginal cost function, such as the magnitude of economies of scale or scope. Given an assumption about the form of competition (for instance, perfect competition, Cournot, Bertrand, Stackelberg, or collusion), the model predicts that a firm's marginal cost should be equal to the marginal revenue

implied by that form of competition. This is the key condition that is used to estimate firms' marginal costs in this class of models. Typically, the first step in the econometric analysis of these models consists in the estimation of the demand function or demand system. Given the estimated demand, we can construct an estimate of the realized marginal revenue for every observation in the sample. This measure of marginal revenue provides, directly, an estimate of the realized marginal cost at each sample observation. Finally, we use this sample of realized marginal costs to estimate the marginal cost function, and how the marginal cost depends on the firm's output of different products (that is, economies of scale and scope), and possibly on other firm characteristics such as historical cumulative output, installed capacity, or geographic distance between the firm's production plants (that is, economies of density).

The value of a firm's marginal revenue depends on the form of competition in the industry, or *the nature of competition*. Given the same demand function, the marginal revenue is different under perfect competition, Cournot, Bertrand, or collusion. The researcher's selection of a model of competition should answer the following questions: (a) is the product homogeneous or differentiated; (b) do firms compete in prices or in quantities?; (c) is there collusion between some or all the firms in the industry?; and (d) what does a firm believe about the behavior of other firms in the market? For instance, if the researcher assumes that the product is homogenous, that firms compete in quantities, that there is no collusion in the industry, and that firms choose their levels of output under the belief that the other firms will not change their respective output levels (that is, Nash assumption), then the form of competition is the one specified in the Cournot model. In principle, some of these assumptions may be supported by the researcher's knowledge of the industry. However, in general, some of these assumptions are difficult to justify. Ideally, we would like to learn from our data about the nature of competition. Suppose that the researcher has data on firms' marginal costs (or estimates of these costs based on a production function) and has estimated the demand system. Then, given an assumption about the form of competition in this industry (for instance, perfect competition, Cournot, collusion), the researcher can use the demand to obtain firms' marginal revenues and check whether they are equal to the observed marginal costs. That is, the researcher can test if a particular form of competition is consistent with the data. In this way, it is possible to find the form of competition that is consistent with the data, for instance, identify if there is evidence of collusive behavior. We will see in this chapter that, even if the researcher does not have data on firms' costs, it is still possible to combine the demand system and the equilibrium conditions to jointly identify marginal costs and the *nature of competition* in the industry. This is the main purpose of the so called *conjectural variation approach*.

Section 4.2 presents empirical models of competition in a homogenous product industry. Section 4.3 deals with competition in a differentiated product industry. We present the conjectural variation approach both in homogenous and differentiated product industries. Section 4.4 describes models of price and quantity competition when firms have asymmetric or incomplete information.

4.2 Homogenous product industry

4.2.1 Estimating marginal costs

First, we consider the situation where the researcher does not have direct measures of marginal costs and uses the equilibrium conditions to estimate these costs.

Perfect competition

We first illustrate this approach in the context of a perfectly competitive industry for a homogeneous product. Suppose that the researcher knows, or is willing to assume, that the industry under study is perfectly competitive, and she has data on the market price and on firms' output for T periods of time (or T geographic markets) that we index by t. The dataset consists of $\{p_t, q_{it}\}$ for $i = 1,2,...,N_t$ and $t = 1,2,...,T$, where N_t is the number of firms active at period t. The variable profit of firm i is $p_t\ q_{it} - C_i(q_{it})$. Under perfect competition, the marginal revenue of any firm i is the market price, p_t. The marginal condition of profit maximization for firm i is $p_t = MC_i(q_{it})$ where $MC_i(q_{it})$ is the marginal cost, $MC_i(q_{it}) \equiv C_i'(q_{it})$. Since all the firms face the same market price, a first important implication of the first order condition of optimality under perfect competition is that all the firms should have the same realized marginal costs. This is a testable restriction of the assumption of perfect competition with a homogeneous product.

Consider a particular specification of the cost function. With a Cobb-Douglas production function, we have that (see section 3.2.1 above):

$$MC_i(q_{it}) = q_{it}^{\theta}\ w_{1it}^{\alpha_1}\ ...\ w_{Jit}^{\alpha_J}\ \exp\{\varepsilon_{it}^{MC}\} \tag{4.1}$$

w_{jit} is the price of variable input j for firm i, and α's are the technological parameters in the Cobb-Douglas production function. Variable ε_{it}^{MC} is unobservable to the researcher and it captures the cost (in)efficiency of a firm that depends on the firm's total factor productivity, and input prices that are not observable. The technological parameter θ is equal to $(1-\alpha_V)/\alpha_V$, where α_V is the sum of the Cobb-Douglas coefficients of all the variable inputs: $\alpha_V \equiv \alpha_1 + ... + \alpha_J$. Therefore, the equilibrium condition $p_t = MC_i(q_{it})$ implies the following regression model in logarithms:

$$\ln(p_t) = \theta\ \ln(q_{it}) + \alpha_1\ \ln(w_{1it}) + ... + \alpha_J\ \ln(w_{Jit}) + \varepsilon_{it}^{MC} \tag{4.2}$$

We can distinguish three cases for parameter θ. Constant Returns to Scale (CRS), with $\alpha_V = 1$ such that $\theta = 0$ and the marginal cost does not depend on the level of output; and Decreasing (Increasing) Returns to Scale, with $\alpha_V < 1$ ($\alpha_V > 1$) such that $\theta > 0$ ($\theta < 0$) and the log-marginal cost function is an increasing (decreasing) linear function of log-output.

Using data on market price, firms' quantities and firms' inputs, we can estimate the slope parameter θ in this regression equation. Even if the researcher does not have data on any of the firms' inputs, we can estimate parameter θ in this regression equation, as all the inputs become part of the error term ε_{it}^{MC}. As we explain below, we should be careful with endogeneity problems due to the correlation between this error term an a firm's output. Given an estimate of parameter θ, we can estimate ε_{it}^{MC} as a residual from this regression. Therefore, we can estimate the marginal cost function of each firm. Since the dependent variable of the regression, $\ln(p_t)$, is constant over firms, then, by

construction, if $theta > 0$, then firms that produce more output should have a smaller value for the term $\alpha_1 \ln(w_{1it}) + ... + \alpha_J \ln(w_{Jit}) + \varepsilon_{it}^{MC}$: that is, they should be more cost-efficient.

Estimation of equation (4.2) by OLS suffers from an endogeneity problem. The equilibrium condition implies that firms with a large value of ε_{it}^{MC} are less cost-efficient and, all else equal, should have a lower level of output. Therefore, the regressor $\ln(q_{it})$ is negatively correlated with the error term ε_{it}^{MC}. This negative correlation between the regressor and the error term implies that the OLS estimator provides a downward biased estimate of the true θ. For instance, the OLS estimate could show increasing returns to scale, $\theta < 0$, when in fact the true technology has decreasing returns to scale, $\theta > 0$. This endogeneity problem does not disappear if we consider the model in market means.

We can deal with this endogeneity problem by using instrumental variables. Suppose that $\mathbf{x}_t^D$ is an observable variable (or vector of variables) that affects the demand of the product but not the marginal costs of the firms. The equilibrium of the model implies that these demand variables should be correlated with firms' outputs, $\ln(q_{it})$: exogenous variables that shift the demand curve should have an impact on the amount output of each firm in the market. The condition that $\mathbf{x}_t^D$ is correlated with firms' output is testable. Under the assumption that these observable demand variables $\mathbf{x}_t^D$ are not correlated with the unobserved term in the marginal cost, we can use these variables as instruments for log-output in the regression equation (4.2) to obtain a consistent estimator of θ.

Cournot competition

Now, suppose that the researcher assumes that the market is not perfectly competitive and that firms compete à la Nash-Cournot. Demand can be represented using the inverse demand function $p_t = P\left(Q_t, \mathbf{x}_t^D\right)$, where $Q_t \equiv \sum_{i=1}^N q_{it}$ is the market total output, and $\mathbf{x}_t^D$ is a vector of exogenous market characteristics affecting demand. Each firm chooses its own output q_{it} to maximize profit. Profit maximization implies the condition that marginal revenue equals marginal cost, where the marginal revenue function is:

$$MR_{it} \;=\; p_t + P'_Q\left(Q_t, \mathbf{x}_t^D\right) \left[1 + \frac{dQ_{(-i)t}}{dq_{it}}\right] q_{it} \tag{4.3}$$

where $P'_Q\left(Q_t, \mathbf{x}_t^D\right)$ is the derivative of the inverse demand function with respect to total output. Variable $Q_{(-i)t}$ is the aggregate output of firms other than i. The derivative $dQ_{(-i)t}/dq_{it}$ represents the *belief* or *conjecture* that firm i has about how other firms will respond by changing their output when firm i changes marginally its own output. Under the assumption of Nash-Cournot competition, this *belief* or *conjecture* is zero:

$$Nash - Cournot \;\Leftrightarrow\; \frac{dQ_{(-i)t}}{dq_{it}} = 0 \tag{4.4}$$

Firm i takes as fixed the quantity produced by the rest of the firms, $Q_{(-i)t}$, and chooses her own output q_{it} to maximize her profit. Therefore, the first order condition of optimality under Nash-Cournot competition is:

$$MR_{it} \;=\; p_t + P'_Q\left(Q_t, \mathbf{x}_t^D\right) q_{it} \;=\; MC_i(q_{it}) \tag{4.5}$$

We assume that the profit function is globally concave in q_{it} for any positive value of $Q_{(-i)t}$, such that there is a unique value of q_{it} that maximizes the firm's profit, and it is

fully characterized by the marginal condition of optimality that establishes that marginal revenue equals marginal cost.

Suppose that the demand function has been estimated in a first step such that there is a consistent estimate of demand. Therefore, the researcher can construct consistent estimates of marginal revenues $MR_{it} \equiv p_t + P'_Q\left(Q_t, \mathbf{x}_t^D\right) q_{it}$ for every firm i. Consider the same Cobb-Douglas specification of the cost function as in equation (4.1). Then, the econometric model can be described in terms of the following linear regression model in logarithms:[1]

$$\ln\left(MR_{it}\right) = \theta \ \ln(q_{it}) + \alpha_1 \ \ln(w_{1it}) + ... + \alpha_J \ \ln(w_{Jit}) + \varepsilon_{it}^{MC} \tag{4.6}$$

We are interested in the estimation of the parameters θ and α's, and in the firms' cost inefficiency, ε_{it}^{MC}.

OLS estimation of this regression function suffers from the same endogeneity problem as in the perfect competition case described above. The model implies a negative correlation between a firm's output and its unobserved inefficiency. To deal with this endogeneity problem, we can use instrumental variables. As in the case of perfect competition, we can use observable variables that affect demand but not costs as instruments. With Cournot competition, we may have additional types of instruments, as we explain next.

Suppose that the researcher observes some exogenous input prices $\mathbf{w}_{it} = (w_{1it}, ..., w_{Jit})$ and that at least one of these prices has cross-sectional variation over firms. For instance, suppose that there is information at the firm level on the firm's wage rate, or its capital stock, or its installed capacity. Note that, in equilibrium, the input prices of competitors have an effect on the level of output of a firm. That is, given its own input prices $\mathbf{w}_{it}$, log-output $\ln(q_{it})$ still depends on the input prices of other firms competing in the market, $\mathbf{w}_{jt}$ for $j \neq i$. A firm's output increases if, all else equal, the wage rates of a competitor increase. Note that the partial correlation between $\mathbf{w}_{jt}$ and $\ln(q_{it})$ is a testable condition. Under the assumption that the vector $\mathbf{w}_{jt}$ is exogenous, that is, $\mathbb{E}(\mathbf{w}_{jt}$ $\varepsilon_{it}^{MC}) = 0$, a standard approach to estimate this model is using IV or GMM based on moment conditions that use the characteristics of other firms as an instrument for output. For instance, the moment conditions can be:

$$\mathbb{E}\left(\begin{bmatrix} \ln(\mathbf{w}_{it}) \\ \sum_{j \neq i} \ln(\mathbf{w}_{jt}) \end{bmatrix} \left[\ln\left(MR_{it}\right) - \theta \ \ln(q_{it}) - \ln \mathbf{w}'_{it} \ \alpha\right]\right) = \mathbf{0} \tag{4.7}$$

4.2.2 The nature of competition

Model

Consider an industry where the inverse demand curve is $p_t = P\left(Q_t, \mathbf{x}_t^D\right)$, and firms, indexed by i, have cost functions $C_i(q_{it})$. Every firm i chooses its amount of output, q_{it}, to maximize its profit, $p_t \ q_{it} - C_i(q_{it})$. The marginal condition for the profit maximization implies marginal revenue equals marginal cost. The marginal revenue of firm i has the

[1] For notational simplicity, here I omit the estimation error from the estimation of the demand function in the first step. Note that, in this case, this estimation error only implies measurement error in the dependent variable and it does not affect the consistency of the instrumental variables estimator described below or the estimation of robust standard errors.

expression in equation (4.3). As mentioned above, the term $dQ_{(-i)t}/dq_{it}$ represents the *belief* that firm i has about how the other firms in the market will respond if it changes its own output marginally. We denote this belief as the *conjectural variation* of firm i at period t, and denote it as CV_{it}.

As researchers, we can choose between different assumptions about firms' beliefs or conjectural variations. An assumption about CVs implies a model of competition with its corresponding equilibrium outcomes. Nash (1951) proposed the following conjecture: when a player constructs her best response, she believes that the other players will not respond to a change in her decision. In the Cournot model, the Nash conjecture implies that $CV_{it} = 0$. For every firm i, the "perceived" marginal revenue is $MR_{it} = p_t + P'_Q\left(Q_t\ \mathbf{x}_t^D\right)\ q_{it}$, and the condition $p_t + P'_Q\left(Q_t\ \mathbf{x}_t^D\right)\ q_{it} = MC_i(q_{it})$ implies the Cournot equilibrium.

Similarly, there are assumptions about CVs that generate the perfect competition equilibrium and the collusive or cartel equilibrium.

Perfect competition. For every firm i, $CV_{it} = -1$. A firm believes that if it increases (reduces) its own output in, say, q units, the other firms will respond by reducing (increasing) their output by the same amount such that total market output does not change. That is, a firm believes that it cannot have any influence on total market output. This conjecture implies that: $MR_{it} = p_t$, and the equilibrium conditions $p_t = MC_i(q_{it})$ under perfect competition.

Perfect collusion. For every firm i, $CV_{it} = N_t - 1$. A firm believes that if it increases (reduces) its own output in, say, q units, each of the other firms in the market will imitate this decision, increasing (reducing) its output by the same amount, such that total market output increases (declines) in $N_t q$ units. This conjecture implies that $MR_{it} = p_t + P'_Q\left(Q_t, \mathbf{x}_t^D\right)\ N_t\ q_{it}$, which generates the equilibrium conditions $p_t + P'_Q\left(Q_t, \mathbf{x}_t^D\right)\ N_t\ q_{it} = MC_i(q_{it})$. When firms have constant and homogeneous MCs, this condition implies $p_t + P'_Q\left(Q_t, \mathbf{x}_t^D\right)\ Q_t = MC_t$, as $Q_t = N_t q_t$, which is the equilibrium condition under monopoly.

The value of the beliefs / CV parameters are related to the nature of competition:

$$\begin{cases} \textit{Perfect competition:} & CV_{it} = -1; & MR_{it} = p_t \\ \textit{Nash-Cournot:} & CV_{it} = 0; & MR_{it} = p_t + P'_Q(Q_t)\ q_{it} \\ \textit{Collusion of n firms:} & CV_{it} = n-1; & MR_{it} = p_t + P'_Q(Q_t)\ n\ q_{it} \\ \textit{Perfect collusion:} & CV_{it} = N_t - 1; & MR_{it} = p_t + P'_Q(Q_t)\ Q_t \end{cases} \tag{4.8}$$

The expressions in (**??**) show that firms' beliefs about competitors' behavior, as represented by *CV*s, are closely related to the nature of competition. Importantly, these results are not making any assumption about how firms' conjectures *CV* are determined. These beliefs can endogenously determined, together with the other outcomes of the model, price and outputs. However, the model is silent about how *CV*s are achieved.[2] For the

[2] A possible approach for endogenizing *CV*s is to consider a dynamic game with multiple periods $t = 1, 2, ...$ where every period firms choose their CVs and their amounts of output. Firms can learn

moment, we do not specify the determinants of CVs, but it is important to keep in mind that they are endogenous objects. Interpreting CV_{it} as an exogenous parameter is not correct. Conjectural variations represent firms' beliefs, and as such they are endogenous outcomes from the model.

Some applications or views of the *conjectural variations model* go beyond the results above for perfect competition, Cournot, and collusion, and consider that CVs can take any continuous value between -1 and $N_t - 1$. Under this interpretation, if CV is negative, the degree of competition is stronger than Cournot, and the closer to -1, the more competitive. If CV is positive, the degree of competition is weaker than Cournot, and the closer to $N_t - 1$, the less competitive. It seems reasonable to expect that CV should not be smaller than -1 or greater than $N_t - 1$. Values smaller than -1 imply a competitors' respond that generates negative profits. Values greater than $N_t - 1$ imply that the cartel is not maximizing the joint profit.[3] However, this view of the conjectural variations approach has been criticized as these "intermediate values" of CVs cannot be obtained as equilibrium values of a dynamic game. See Corts (1999) for an analysis of this issue that has been influential in empirical IO.

Estimation with information on marginal costs

Consider a homogeneous product industry and a researcher with data on firms' quantities and marginal costs, and on market prices over T periods of time: $\{p_t, MC_{it}, q_{it}\}$ for $i = 1, 2, ..., N_t$ and $t = 1, 2, ..., T$. Under the assumption that every firm chooses the amount of output that maximizes its profit given its belief CV_{it}, we have that the following condition holds:

$$p_t + P'_Q\left(Q_t\ \mathbf{x}_t^D\right)\ [1 + CV_{it}]\ q_{it} = MC_{it} \tag{4.9}$$

And solving for the conjectural variation, we have:

$$CV_{it} = \frac{p_t - MC_{it}}{-P'_Q\left(Q_t.\mathbf{x}_t^D\right)\ q_{it}} - 1 = \left[\frac{p_t - MC_{it}}{p_t}\right]\ \left[\frac{1}{q_{it}/Q_t}\right]\ \eta_t - 1 \tag{4.10}$$

where η_t is the price elasticity of demand (in absolute value): that is, $\eta_t = -(p_t/Q_t)$ $(1/P'_Q(Q_t, \mathbf{x}_t^D))$. Note that $(p_t - MC_{it})/p_t$ is the Lerner index, and q_{it}/Q_t is the market share of firm i. This equation shows that, given data on output, price, demand, and marginal cost, we can identify a firm's belief that is consistent with these data and profit maximization.

Let us denote $\left[\frac{p_t - MC_{it}}{p_t}\right]\ \left[\frac{1}{q_{it}/Q_t}\right]$ as the Lerner-index-to-market-share ratio of a firm. If this ratio is close to zero, then the estimated value of CV is close to -1 unless the absolute demand elasticity is large. In contrast, if the Lerner-index-to-market-share ratio is large (that is, larger than the inverse demand elasticity), then the estimate of CV is greater than zero, and the researcher can reject the hypothesis of Cournot competition in favor of some collusion.

over time and update their beliefs CV. In the equilibrium of this dynamic game, CVs are determined endogenously. We present this type of dynamic game in chapter 8.

[3] Nevertheless, in the context of dynamic games, we could have values of CV that can be smaller than -1 due to competitive wars that try to induce other firms' exit from the market. For instance, see the dynamic game in Beviá, Corchón, and Yasuda (2020).

Under the restriction that all the firms have both the same marginal costs and conjectural variations, equation (4.10) becomes:

$$\frac{p_t - MC_t}{p_t} = \left[\frac{1 + CV_t}{N_t}\right] \frac{1}{\eta_t} \tag{4.11}$$

where N_t is the number of firms in the market. This is the equation that we use in the empirical application that we describe at the end of this section. According to this expression, market power, as measured by the Lerner Index, is related to the elasticity of demand (negatively), the number of firms in the market (negatively), and the conjectural variation (positively). Importantly, one *should not* interpret equation (4.11) as a causal relationship where the Lerner index (in the left hand side) depends on exogenous variables in the right hand side. In this equation, all the variables – Lerner index, conjectural variation, and number of firms – are endogenous, and are jointly determined in the equilibrium of this industry as functions of exogenous variables affecting demand and costs. Nevertheless, equation 4.11 is still a very useful equation for empirical analysis and estimation, as it determines the value of one of the endogenous variables once we have measures for the others.

Estimation without information on marginal costs

So far, we have considered the estimation of CV parameters when the researcher knows both demand and firms' marginal costs. We now consider the case where the researcher knows the demand, but it does not know firms' marginal costs. Identification of CVs requires also the identification of marginal costs. We show here that, under some conditions, we can jointly identify CVs and MCs using the marginal condition of optimality and demand.

The researcher observes data $\left\{p_t,\ q_{it},\ \mathbf{x}_t^D,\ \mathbf{w}_t :\ i = 1,...N_t; t = 1,...,T\right\}$, where $\mathbf{x}_t^D$ are exogenous variables affecting consumer demand, for instance, average income or population, and $\mathbf{w}_t$ are variables affecting marginal costs, for instance, some input prices. Consider the linear (inverse) demand equation:

$$p_t \ = \ \alpha_0 + \mathbf{x}_t^D \alpha_1 \ - \alpha_2 \ Q_t + \varepsilon_t^D \tag{4.12}$$

with $\alpha_2 \geq 0$, and ε_t^D is unobservable to the researcher. Consider the marginal cost function:

$$MC_{it} \ = \ \beta_0 + \mathbf{w}_t \beta_1 + \beta_2 \ q_{it} + \varepsilon_{it}^{MC} \tag{4.13}$$

with $\beta_2 \geq 0$, and ε_{it}^{MC} is unobservable to the researcher. Profit maximization implies the marginal condition $p_t + dP_t/dQ_t \left[1 + CV_{it}\right] q_{it} = MC_{it}$. Since the demand function is linear and $dP_t/dQ_t = -\alpha_2$, we can write the marginal condition as follows:

$$p_t \ = \ \beta_0 + \mathbf{w}_t \beta_1 + \left[\beta_2 + \alpha_2 (1 + CV_{it})\right] \ q_{it} + \varepsilon_{it}^{MC} \tag{4.14}$$

This model is typically completed with the assumption that conjectural variations are constant over time: $CV_{it} = CV_i$. The assumption of CV constant over time is plausible when the industry is mature and has not experienced structural changes during the sample period. Nevertheless, some empirical studies analyze specific events, such as important regulatory changes or mergers, and allow CV to be different before and after this change. Some empirical applications impose also the restriction that CV is the same

for all the firms in the market. This assumption of homogeneous CVs across firms is not always plausible. For instance, there may be leaders and followers in the industry, or cartels that include only some firms. Furthermore, this restriction is not necessary if the researcher has data on output at the firm level, q_{it}, and not only on total market output, Q_t. Accordingly, in this section we assume that firms' beliefs/conjectures are constant over time but can vary across firms.

The structural equations of the model are the demand equation in (4.12) and the equilibrium condition in (4.14). Using this model and data, can we identify (that is, estimate consistently) the CV parameter? Without further restrictions, the answer to this question is negative. However, we show below that a simple and plausible condition in this model implies the identification of both CV and MC parameters. We first describe the identification problem.

Identification of demand parameters. The estimation of the regression equation for the demand function needs to deal with the well-known simultaneity problem. In equilibrium, output Q_t is correlated with the error term ε_t^D. The model implies a valid instrument to estimate demand. In equilibrium, Q_t depends on the exogenous cost variables $\mathbf{w}_t$. This variable does not enter in the demand equation. If $\mathbf{w}_t$ is not correlated with ε_t^D, then this variable(s) satisfies all the conditions of a valid instrument. Parameters α_0, α_1, and α_2 are therefore identified using this IV estimator.

Identification of CV and MCs. In the regression equation (4.14), we also need to deal with an endogeneity problem. In equilibrium, output q_{it} is correlated with the error term ε_{it}^{MC}. The model implies a valid instrument to estimate this equation. In equilibrium, q_{it} depends on the exogenous demand shifters $\mathbf{x}_t^D$. Note that $\mathbf{x}_t^D$ does not enter in the marginal cost and in the right hand side of the regression equation (4.14). If $\mathbf{x}_t^D$ is not correlated with ε_{it}^{MC}, then this variable satisfies all the conditions for being a valid instrument such that the parameters β_0, β_1, and $\gamma_i \equiv \beta_2 + \alpha_2(1 + CV_i)$ are identified using this IV estimator.

Now, the identification of parameter $\gamma_i \equiv \beta_2 + \alpha_2(1 + CV_i)$ and of the slope of the inverse demand function, α_2, is not sufficient to identify separately CV_i and the slope of the marginal cost function, β_2. That is, given known values for γ_i and α_2, equation

$$\gamma_i \;=\; \beta_2 \;+\; \alpha_2 \,(1 + CV_i) \tag{4.15}$$

implies a linear relationship between CV_i and β_2 and there are infinite values of these parameters that satisfy this restriction. Even if we restrict CV_i to belonging to the values consistent with an equilibrium concept, such that $CV_i \in \{-1, 0, N-1\}$ and β_2 to being greater or equal than zero, we do not have point identification of these parameters. For instance, suppose that $N = 2$, $\gamma_i = 2$, and $\alpha_2 = 1$ such that equation (4.15) becomes $2 = \beta_2 + (1 + CV_i)$, or equivalently, $\beta_2 + CV_i = 1$. This equation is satisfied by any of the following forms of competition and values of $\beta_2 \geq 0$: perfect competition, with $CV_i = -1$ and $\beta_2 = 2$; Cournot competition, with $CV_i = 0$ and $\beta_2 = 1$; and perfect collusion, with $CV = 1$ and $\beta_2 = 0$.

This identification problem has an intuitive interpretation. The identified parameter γ_i captures the true causal effect of firm i's output on market price. There are two different channels for this causal effect: through the change in marginal cost; and through the change in marginal revenue, that depends on the firm's conjectural variation.

Identification of the causal effect parameter γ_t is not sufficient to disentangle the relative contribution of the two channels.

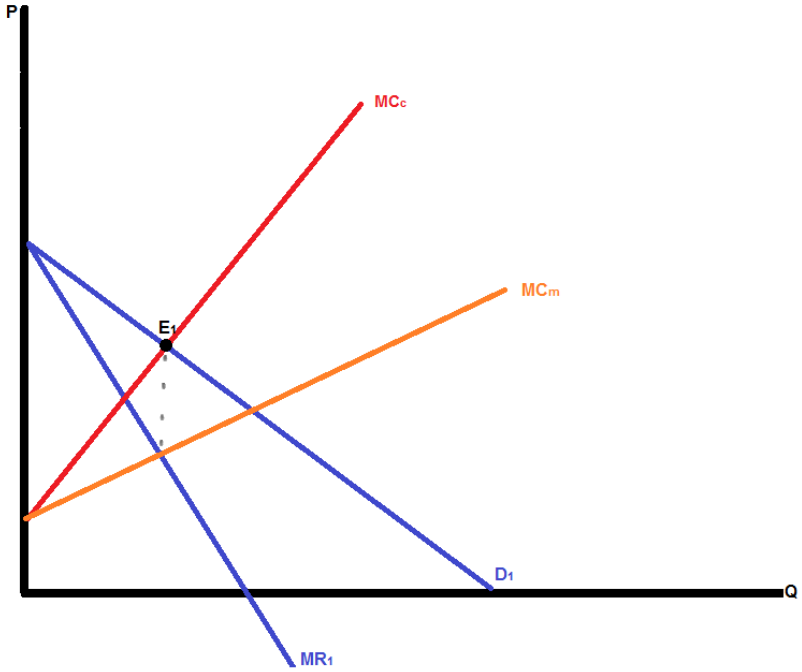


Figure 4.1: One data point: No identification of competition (c) vs. collusion (m)

Following Bresnahan (1981), we can provide a graphical representation of this identification problem. Suppose that we have followed the approach described above to estimate consistently the demand parameters – that imply the demand curve D_1 and the monopoly marginal revenue curve MR_1 in figure 4.1 – the marginal cost parameters β_0 and β_1, and the parameter γ_i. We can define two hypothetical marginal cost functions: the marginal cost under the hypothesis of perfect competition ($CV_i = -1$ such that $\beta_2 = \gamma_i$), $MC_c = \beta_0 + \mathbf{w}\beta_1 + \gamma_i q$; and the marginal cost under the hypothesis of monopoly or perfect collusion ($CV = N - 1$ such that $\beta_2 = \gamma_i - \alpha_2 N$), $MC_m = \beta_0 + \mathbf{w}\beta_1(\gamma_i - \alpha_2 N)$ q. That is, MC_c and MC_m are the marginal cost functions that rationalize the observed price and output under the hypotheses of perfect competition and monopoly, respectively. Figure 4.1 shows that the observed price and quantity – represented by the point $E_1 = (q_1, p_1)$ – can be rationalized either as the point where the demand function D_1 crosses the competitive marginal cost MC_c, or as the monopoly outcome defined by the marginal revenue MR_1 and the monopoly marginal cost MC_m.

Data on prices and quantities at multiple time periods do not help to solve this identification problem. This is illustrated in Figure 4.2. Consider the demand curves D_1 and D_2 at periods $t = 1$ and $t = 2$, respectively. Importantly, under the demand function in equation (4.12), every change in the demand curve (that is, a change in $\mathbf{x}_t^D$ or in ε_t^D) implies a parallel vertical shift, keeping the slope constant. Therefore, demand curves D_1 and D_2 are parallel, and so they are the corresponding marginal revenue curves MR_1 and MR_2, as shown in Figure 4.2. Again, the observed points $E_1 = (q_1, p_1)$ and $E_2 = (q_2, p_2)$ can be rationalized either as perfectly competitive equilibria that come from the intersection of demand curve D_t and marginal cost MC_c, or as monopoly

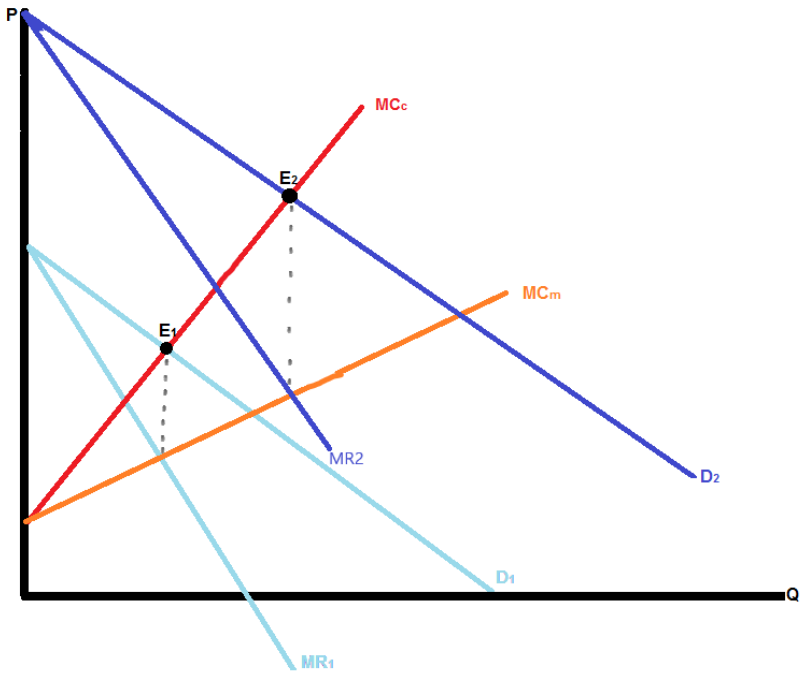


Figure 4.2: Multiple data points: No identification of competition (c) vs. collusion (m)

outcomes that are determined by the intersection of the marginal revenue curve MR_t and the marginal cost MC_m.

This graphical analysis provides also an intuitive interpretation of a solution to this identification problem. This solution involves generalizing the demand function so that changes in exogenous variables do more than just a parallel shift in the demand curve and the marginal revenue. We introduce additional exogenous variables that are capable of **rotating** the demand curve. Consider Figure 4.3. Now, the demand curve D_2 represents a rotation of demand curve D_1 around point E_1. Under perfect competition, this rotation in the demand curve should not have any effect in equilibrium prices and quantities. Therefore, under perfect competition, E_1 is the equilibrium point under the two demand curves. This is not the case under monopoly (collusion). When firms have market power, a change in the slope of the demand has an effect on prices and quantities. Therefore, given demand curves D_1 and D_2, if the data shows different values of the (quantity, price) points E_1 and E_2, as in Figure 4.3, we can reject the hypothesis of perfect competition in favor of firms having market power.

We now present more formally the identification of the model illustrated in Figure 4.3. Consider now the following demand equation:

$$p_t = \alpha_0 + \mathbf{x}_t^D \alpha_1 - \alpha_2 Q_t - \alpha_3 [z_t Q_t] + \varepsilon_t^D \tag{4.16}$$

Variable z_t is observable to the researcher and affects the slope of the demand. Some possible candidates for these variables are the price of a substitute or complement product, seasonal dummies capturing changes in the composition of the population of consumers during the year, or consumer demographics. The key condition is that the parameter α_3 is different from zero. That is, when z_t varies, there is a rotation in the demand curve. Note that this condition is testable. Given this demand model, we have

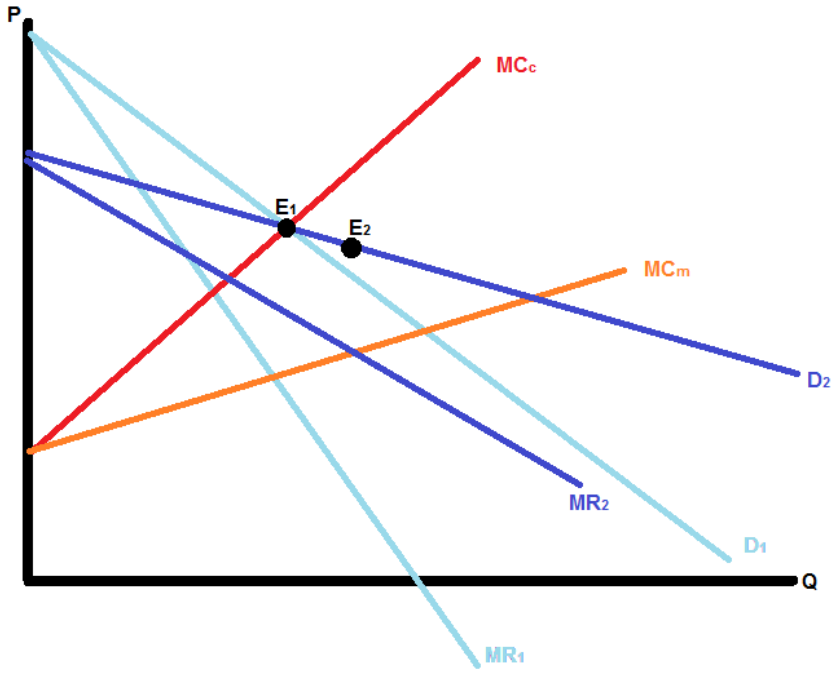


Figure 4.3: Rotating demand curve: Rejecting perfect competition

that the slope of the demand curve is $dP_t/dQ_t = -\alpha_2 - \alpha_3\ z_t$, and the marginal condition for profit maximization implies the following regression model:

$$p_t \ = \ \beta_0 + \mathbf{w}_t\beta_1 + \gamma_{1,i}\ q_{it} + \gamma_{2,i}\ (z_t\ q_{it}) + \varepsilon_{it}^{MC} \tag{4.17}$$

with $\gamma_{1,i} \equiv \beta_2 + \alpha_2\,[1 + CV_i]$ and $\gamma_{2,i} \equiv \alpha_3\,[1 + CV_i]$.

Equations (4.16) and (4.17) describe the structural model. Using this model and data, we now show that we can separately identify CV_i and the slope of the marginal cost, $\beta 2$. Demand parameters can be identified similarly as before, using $\mathbf{w}_t$ as instruments for output. Parameters α_0, α_1, α_2, and α_3 are identified using this IV estimator. The model also implies valid instruments to estimate the parameters in the equilibrium equation in (4.17). We can instrument q_{it} using $\mathbf{x}_t^D$ and $z_t\ q_{it}$ using $z_t\ \mathbf{x}_t^D$. Parameters β_0, β_1, $\gamma_{1,i}$, and $\gamma_{2,i}$ are identified. Note that:

$$\begin{cases} \gamma_{1,i} \ = \ \beta_2 + \alpha_2\,[1 + CV_i] \\ \gamma_{2,i} \ = \ \alpha_3\,[1 + CV_i] \end{cases} \tag{4.18}$$

Given estimates of α_2, α_3, $\gamma_{1,i}$, and $\gamma_{2,i}$, we have that (4.18) is a system of two equations with two unknowns (β_2 and CV_i) that has a unique solution if and only if α_3 is different to zero. The solution of this system implies that $1 + CV_i = \gamma_{2,i}/\alpha_3$. The conjectural variation is identified by the ratio between the sensitivity of price with respect to $(z_t\ q_{it})$ in the equilibrium equation and the sensitivity of price with respect to $(z_t\ Q_t)$ in the demand equation.

The sample variation in the slope of the inverse demand plays a key role in the identification of the CV parameter. An increase in the slope means that the demand

becomes less price sensitive, more inelastic. For a monopolist, when the demand becomes more inelastic, the optimal price should increase. In general, for a firm with a high level of market power (high CV), we should observe an important increase in prices associated with an increase in the slope. On the contrary, if the industry is characterized by very low market power (low CV), the increase in price should be practically zero. Therefore, the response of price to an exogenous change in the slope of the demand contains key information for the estimation of CV.

This identification result still holds if the exogenous variable z_t that generates the change in the slope of the demand curve also also an effect in the marginal cost. That is, z_t can be included in the vector of cost shifters $\mathbf{w}_t$. However, a key identifying restriction that cannot relaxed in this approach is that z_t cannot affect the slope of the marginal cost function. That is, variation in z_t does not affect the degree of diseconomies of scale in the production of the good.

An application: The sugar industry

Genesove and Mullin (1998) (GM) study competition in the US sugar industry during the period 1890-1914. One of the purposes of this study is to test the validity of the conjectural variation approach by focusing on an industry where firms' marginal costs can be very accurately measured. This motivation plays an important role in the authors' selection of this industry and historical period. During this period, the production technology of refined sugar was very simple, and the marginal cost function was characterized in terms of a simple linear function of the cost of raw sugar, the main intermediate input in the production of refined sugar. Furthermore, during this period there was an investigation of the industry by the US antitrust authority. As a result of that investigation, there were reports from multiple expert witnesses who provided a very coherent description of the structure and magnitude of production costs in this industry. GM use this information on marginal costs to test the validity of the standard conjectural variation approach for the estimation of price cost margins and marginal costs.

Let $p_t = P(Q_t, S_t)$ be the inverse demand function at year t in the industry, where S_t represents exogenous variables affecting demand, and that we specify below. Under the assumption that all the firms are identical in their marginal costs and in their conjectural variations, the marginal revenue at period t is:

$$MR_t \;=\; p_t - P'(Q_t, S_t)\;[1 + CV_t]\;\frac{Q_t}{N_t} \tag{4.19}$$

where $P'(Q_t, S_t)$ is the slope of the demand curve. The condition for profit maximization (marginal revenue equals marginal cost) implies the following relationship between the Lerner Index and the conjectural variation:

$$\frac{p_t - MC_t}{p_t} = \left[\frac{1 + CV_t}{N_t}\right]\;\frac{1}{\eta_t} \tag{4.20}$$

where η_t is the price demand elasticity, in absolute value.

Given equation 4.20, if we observe price and marginal cost and we can estimate the demand elasticity, then there is a simple and direct estimate of the conjectural variation. Without knowledge of the marginal cost, the estimation of the CV should depend on two conditions: (a) the existence of an observable exogenous variable that rotates the

demand curve; and (b) the exclusion restrictions that observable demand shifters do not affect marginal costs. If assumptions (a) or (b) are not correct, our estimation of the CV (and of the Lerner Index) will be biased. GM evaluate these assumptions by comparing the estimate of CV obtained under conditions (a) and (b) (and without using data on marginal costs) with the direct estimate of this parameter using data on marginal costs.

The rest of this section describes the following aspects of this empirical application: (i) the industry; (ii) the data; (iii) estimation of demand parameters; (iv) predicted markups under different conduct parameters; and (v) estimation of CV.

(i) The industry

During 1890-1914, refined sugar was a homogeneous, and the industry in the US was highly concentrated. The industry leader, the *American Sugar Refining Company (ASR)*, had more than 65% of the market share during most of these years.[4]

Production technology. Refined sugar companies bought raw sugar from suppliers in national and international markets, transform it into refined sugar, and sell it to grocers. They sent sugar to grocers in barrels, without any product differentiation. Raw sugar is 96% sucrose and 4% water. Refined sugar is 100% sucrose. The process of transforming raw sugar into refined sugar was called "melting", and it consisted of eliminating the 4% of water in raw sugar. Industry experts reported that firms in the industry used a fixed coefficient (or Leontieff) production technology that can be described by the following production function:

$$Q_t \;=\; \min\{\lambda\; Q_t^{raw}\;;\; f(L_t, K_t)\} \tag{4.21}$$

where Q_t is refined sugar output, Q_t^{raw} is the input of raw sugar, $\lambda \in (0,1)$ is a technological parameter, and $f(L_t, K_t)$ is a function of labor and capital inputs. Production efficiency and cost minimization imply that $Q_t = \lambda\; Q_t^{raw} = f(L_t, K_t)$. That is, 1 ton of raw sugar generates λ tons units of refined sugar. Since raw sugar is only 96% sucrose, the largest possible value of λ is 0.96. Industry experts at that time unanimously reported that there was some loss of sugar in the refining process such that the value of the parameter λ was close to 0.93.

Marginal cost function. For this production technology, the marginal cost function is:

$$MC_t \;=\; c_0 + c_1\; p_t^{raw} + c_2\; q_t \tag{4.22}$$

where c_0, c_1, and c_2 are parameters, p_t^{raw} is the price of raw sugar in dollars per pound, and q_t is output per firm. The Leontieff production function in equation (4.21) implies that $c_2 = 0$, $c_1 = 1/\lambda$, and c_0 is a component of the marginal cost that depends on labor. According to industry experts, during the sample period the values of the parameters in the marginal cost were $c_0 = \$0.26$ per pound, $c_1 = 1/\lambda = 1/0.93 = 1.075$, and $c_2 = 0$. Therefore, the marginal cost at period (quarter) t, in dollars per pound of sugar, was:

$$MC_t \;=\; 0.26 + 1.075\; p_t^{raw} \tag{4.23}$$

[4] ASR operated one of the world's largest sugar refineries at that time, the Domino Sugar Refinery in Brooklyn, New York. The ASR company became known as Domino Sugar in 1900.

(ii) The data

The dataset contains 97 quarterly observations on industry output, price, price of raw sugar, imports of raw sugar, and a seasonal dummy.

$$\text{Data} = \{\ Q_t,\ p_t,\ p_t^{raw},\ IMP_t,\ S_t : t = 1,2,...,97\} \tag{4.24}$$

IMP_t represents imports of raw sugar from Cuba, and S_t is a dummy variable for the Summer season: $S_t = 1$ if observation t is a Summer quarter, and $S_t = 0$ otherwise. The summer was a high demand season for sugar because most the production of canned fruits was concentrated during that season, and the canned fruit industry accounted for an important fraction of the demand of sugar.

(iii) Estimation of demand parameters

GM estimate four different models of demand: linear, quadratic, log-linear, and exponential. The main results are consistent for the four models. Here we concentrate on results using the linear (inverse) demand function:

$$p_t \ = \ \alpha_0 + \alpha_1\ S_t - \alpha_2\ Q_t - \alpha_3\ S_t Q_t + \varepsilon_t^D \tag{4.25}$$

Parameters α_0 and α_2 represent the intercept and slope of the demand curve during the "Low season" (when $S_t = 0$). Similarly, parameters $\alpha_0 + \alpha_1$ and $\alpha_2 + \alpha_3$ are the intercept and slope of the demand curve in the "High season" (when $S_t = 1$).

As we have discussed before, Q_t is an endogenous regressor in this regression equation. We need to use IV to deal with this endogeneity problem. In principle, it seems that we could use p_t^{raw} as an instrument. However, GM have a reasonable concern about the validity of this instrument. The demand of raw sugar from the US accounts for a significant fraction of the world demand of raw sugar. Therefore, shocks in the US domestic demand of refined sugar, as represented by ε_t^D, can generate an increase in the world demand of raw sugar and in p_t^{raw} such that p_t^{raw} and ε_t^D can be positively correlated. Instead, GM use imports of raw sugar from Cuba as an instrument. Almost 100% of the production of raw sugar in Cuba was exported to the US, and the authors claim that variations in Cuban production of raw sugar was driven by supply/weather conditions and not by the demand from the US.

Table 4.1: Genesove and Mullin: Demand estimates
Based on Table 3 (column 2) in Genesove and Mullin (1998)

Parameter	Estimate	Standard Error
Intercept Low, α_0	5.813	(0.826)
Intercept High, $\alpha_0 + \alpha_1$	7.897	(1.154)
Slope Low, α_2	0.434	(0.194)
Slope High, $\alpha_2 + \alpha_3$	0.735	(0.321)
Average elasticity Low, η_L	2.24	
Average elasticity High, η_H	1.04	

Table 4.1 presents parameter estimates of demand parameters. In the high season, the demand shifts upwards by \$2.09 per ton ($\alpha_1 = \$2.09 > 0$) and becomes steeper ($\alpha_3 = 0.301 > 0$). The estimated price elasticities of demand in the low and the high season are $\eta_L = 2.24$ and $\eta_H = 1.04$, respectively. According to this, any model of oligopoly competition where firms have some market power predicts that the price cost margin should increase during the high season due to the lower price sensitivity of demand.

(iv) Predicted markups under different conduct parameters

Before we discuss the estimates of the conjectural variation parameter, it is interesting to illustrate the errors that researchers can make when – in the absence of information about marginal costs – they estimate price cost margins by making an incorrect assumption about the value of CV in the industry.

As mentioned above, the industry was highly concentrated during this period. Though there were approximately 6 firms active during most of the sample period, one of the firms accounted for more than two-thirds of total output. Consider three different researchers investigating this industry, that we label as researchers M, C, and S. These researchers do not know the true marginal cost and they have different views about the nature of competition in this industry.Researcher M considers that the industry was basically a Monopoly/Cartel during this period.[5] Therefore, she assumes that $[1+CV]/N = 1$. Researcher C considers that the industry can be characterized by Cournot competition between the 6 firms, such that $[1+CV]/N = 1/6$. Finally, researcher S thinks that this industry can be better described by a Stackelberg model with 1 leader and 5 Cournot followers, and therefore $[1+CV]/N = 1/(2*6-1) = 1/11$.

Table 4.2: Genesove and Mullin: Markups under different conduct parameters

Assumption	Predicted Lerner Low season $\frac{1+CV}{N\,\eta_L}$	Actual Lerner Low season $\frac{p_L-MC}{p_L}$	Predicted Lerner High season $\frac{1+CV}{N\,\eta_H}$	Actual Lerner High season $\frac{p_H-MC}{p_H}$
Monopoly: $\frac{1+CV}{N}=1$	44.6%	3.8%	96.1%	6.5%
Cournot: $\frac{1+CV}{N}=\frac{1}{6}$	7.4%	3.8%	16.0%	6.5%
Stackelberg: $\frac{1+CV}{N}=\frac{1}{11}$	4.0%	3.8%	8.7%	6.5%

Table 4.2 presents the predictions of the Lerner index – in the low and high season – from these three researchers and also the actual value of the Lerner index based on our information on marginal costs. Researcher M makes a very seriously biased prediction of market power. Since the elasticity of demand is quite low in this industry, especially during the high season, the assumption of Cartel implies a very high Lerner index, much

[5] In fact, there was an anti-trust investigation, such that there were some suspicions of collusive behavior.

higher than the actual one. Researcher *C* also over-estimates the actual Lerner index. The estimates of researcher *S* are only slightly upward biased.

Consider the judge of an anti-trust case in which there is not reliable information on the actual value of MCs. The picture of industry competition that this judge gets from the three researchers is very different. This judge would be interested in measures of market power in this industry that are based on a scientific estimate of the conjectural parameter.

(iv) Estimation of conjectural variation

Now, GM consider the hypothetical scenario where the researcher does not observe the marginal cost and applies the method described above to jointly estimate CV and marginal cost parameters, c_0, c_1, and c_2. The marginal condition for profit maximization implies the following equation:

$$p_t \;=\; c_0 + c_1\, p_t^{raw} + \gamma_1\, Q_t + \gamma_2\, S_t Q_t + \varepsilon_t^{MC} \tag{4.26}$$

with $\gamma_1 \equiv [c_2 + \alpha_2(1+CV)]/N$, and $\gamma_2 \equiv \alpha_3(1+CV)]/N$. We treat c_0 and c_1 as parameters to estimate because we consider the estimation of CV under the hypothetical situation where the researcher does not know that $c_0 = 0.26$, $c_1 = 1.075$, and $c_2 = 0$.

Since Q_t is endogeneously determined, it should be correlated with ε_t^{MC}. To deal with this endogeneity problem, GM use instrumental variables. Again, they use imports from Cuba as an instrument for Q_t. Table 4.3 presents the IV estimates of c_0, c_1 and $(1+CV)/N$ and their standard errors (in parentheses). For comparison, we also include the "true" values of these parameters based on the information on marginal costs.

Table 4.3: Genesove and Mullin:
Estimates of conduct and marginal cost parameters

Parameter	Estimate (s.e.)		"True" value
$(1+CV)/N$	0.038	(0.024)	0.10
c_0	0.466	(0.285)	0.26
c_1	1.052	(0.085)	1.075

The estimates of $(1+CV)/N$, c_0, and c_1, are not too far from their "true" values. This seems to validate the CV approach for this particular industry and historical period. Based on this estimate of $(1+CV)/N$, the predicted values for the Lerner index is $0.038/2.24 = 1.7\%$ in the low season, and $0.038/1.04 = 3.6\%$ in the high season. Remember that the true values of the Lerner index using information on marginal costs were 3.8% in the low season and 6.5% in the high season. Therefore, the estimates using the CV method only slightly under-estimate the actual market power in the industry. Furthermore, using either information on marginal costs or the CV method, we can clearly reject the null hypothesis of a perfect cartel, that is, $(1+CV)/N = 1$

4.3 Differentiated product industry

4.3.1 Model

Consider an industry with J differentiated products, for instance, automobiles, indexed by $j \in \mathscr{J} = \{1,2,...,J\}$. Consumer demand for each of these products can be represented using the demand system:

$$q_j = D_j(\mathbf{p},\mathbf{x}) \qquad \text{for} j \in \mathscr{J} \tag{4.27}$$

where $\mathbf{p} = (p_1, p_2, ..., p_J)$ is the vector of prices, and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_J)$ is a vector of other product attributes. There are F firms in the industry, indexed by $f \in \{1,2,...,F\}$. Each firm f owns a subset $\mathscr{J}_f \subset \mathscr{J}$ of the brands. The profit of firm f is the sum of the profits from each product it owns. That is:

$$\Pi_f = \sum_{j \in \mathscr{J}_f} p_j \, q_j - C_j(q_j) \tag{4.28}$$

where $C_j(q_j)$ is the cost of producing a quantity q_j of product j. Firms compete in prices.

(i) Nash-Bertrand competition

We start with the case where firms compete in prices a la Nash-Bertrand. Each firm chooses its own prices to maximize profits and takes the prices of other firms as given. The first order conditions of optimality for profit maximization of firm f are: for any $j \in \mathscr{J}_f$

$$q_j + \sum_{k \in \mathscr{J}_f} [p_k - MC_k] \, \frac{\partial D_k}{\partial p_j} = 0 \tag{4.29}$$

where MC_j is the marginal cost $C_j'(q_j)$. We can write this system in vector form. Let $\mathbf{q}^f$, $\mathbf{p}^f$, and $\mathbf{MC}^f$ be the column vectors with the quantities, prices, and marginal costs, respectively, for every product $j \in \mathscr{J}_f$. And let $\Delta\mathbf{D}^f$ be the square Jacobian matrix with the demand-price derivatives $\partial D_k / \partial p_j$ for every $j,k \in \mathscr{J}_f$. Then, the system of optimality conditions for firm f has the following vector form:

$$\mathbf{q}^f + \Delta\mathbf{D}^f \left[\mathbf{p}^f - \mathbf{MC}^f\right] = 0 \tag{4.30}$$

Under the condition that the Jacobian matrix is non-singular, we can solve for price-cost margins in this system:

$$\mathbf{p}^f - \mathbf{MC}^f = -\left[\Delta\mathbf{D}^f\right]^{-1} \mathbf{q}^f \tag{4.31}$$

The right-hand-side of this equation depends only on demand parameters, and not on costs. Given an estimated demand system and an ownership structure of brands, the vector of Price-Cost Margins under Nash-Bertrand competition is known to the researcher.

(ii) Example: Single product firms with Logit demand

For single product firms, the marginal condition of optimality is:

$$p_j - MC_j = -\left[\frac{\partial D_j}{\partial p_j}\right]^{-1} q_j \tag{4.32}$$

In the logit demand system, we have that:

$$D_j(\mathbf{p},\mathbf{x}) = H \frac{\exp\left\{\mathbf{x}_j'\beta - \alpha\, p_j\right\}}{1+\sum_{k=1}^{J}\exp\left\{\mathbf{x}_k'\beta - \alpha\, p_k\right\}} \tag{4.33}$$

where H represents market size, and β and α are parameters. This logit demand system implies that $\partial D_j/\partial p_j = -\alpha\ H\ s_j(1-s_j)$ where s_j is the market share $s_j \equiv q_j/H$. Therefore, in this model:

$$PCM_j \equiv p_j - MC_j = \frac{1}{\alpha(1-s_j)} \tag{4.34}$$

We see that in this model the price-cost margin of a firm declines with the price sensitivity of demand, α, and increases with the own market share, s_j.

(iii) Example: Multi-product firms with Logit demand

In the logit demand system, we have that $\partial D_j/\partial p_j = -\alpha\, H\, s_j(1-s_j)$, and for $k \neq j$, $\partial D_j/\partial p_k = \alpha\ H\ s_j\ s_k$. Plugging these expressions into the first order conditions of optimality in equation (4.29), we get:

$$PCM_j \quad = \quad \frac{1}{\alpha} + \sum_{k\in\mathscr{J}_f} PCM_k\ s_k \tag{4.35}$$

The right-hand-side is firm-specific but it does not vary across products within the same firm. This condition implies that all the products owned by a firm have the same price-cost margin. According to this condition, the price-cost margin is:

$$PCM_j = \overline{PCM}_f = \frac{1}{\alpha\left(1-\sum_{k\in\mathscr{J}_f} s_k\right)} \tag{4.36}$$

For the Logit demand model, a multi-product firm charges the same price-cost margin for all of its products. This prediction does not extend to more general demand systems.

(iv) Owning multiple products implies higher price-cost margins

In the logit model, the difference between the price-cost margins of a multi-product and a single-product firm is:

$$\frac{1}{\alpha\left(1-\sum_{k\in\mathscr{J}_f} s_k\right)} - \frac{1}{\alpha\left(1-s_j\right)} > 0 \tag{4.37}$$

which is always positive. This prediction extends to a general demand system as long as products are substitutes. For a general demand system, the marginal condition for multi-product firm f and product j can be written as:

$$\begin{aligned} PCM_j &= \left[\frac{-\partial D_j}{\partial p_j}\right]^{-1} q_j \\ &+ \left[\frac{-\partial D_j}{\partial p_j}\right]^{-1} \left[\sum_{k\in J_f;\, k\neq j} PCM_k \frac{\partial D_k}{\partial p_j}\right] \end{aligned} \tag{4.38}$$

In the right-hand-side, the first term is the price-cost margin of a single-product firm. When products are substitutes, the second term is positive because $\partial D_k/\partial p_j > 0$ and $PCM_k > 0$ for every $k \neq j$. Selling multiple products contributes to increasing the price-cost margin of each of the products. This has an intuitive interpretation in terms of a multi-product firm's concern for the cannibalization of its own products. A reduction in the price of product j implies stealing market share from other competing firms, but also cannibalizing market share of the firm's own products other than j.

(v) Collusion and nature of competition

In the homogeneous product case, we have represented the *nature of competition* using firms' conjectural variations or beliefs CV. In section 4.3.5 below, we present the conjectural variation approach in the context of this model with differentiated products. For the moment, we consider here a different representation of the *nature of competition*. We can represent a collusive setting – or the nature of competition – as a $F \times F$ matrix Θ of zeroes and ones. Element (f,g) in this matrix, that we represent as $\theta_{f,g}$, is a dummy variable that equals one if firm f believes that it is colluding with firm g, and it zero otherwise. Of course, all the elements in the diagonal of Θ are ones. But other than this, there are not other restrictions in this matrix. For instance, the matrix can be asymmetric if some firms have not been able to coordinate their collusion beliefs. If there is no collusion at all in the industry, Θ is the identity matrix. The other extreme case is when all the firms in the industry form a cartel: in this case, Θ is a matrix of ones. This representation of the nature of competition can be extended to allow the elements $\theta_{f,g}$ to be real numbers in the interval $[0,1]$ such that they can be interpreted as probabilistic beliefs or as the degree collusion.

A firm f chooses the prices of its own products to maximize the profit of its collusion rink, that has the following expression:

$$\Pi_f^{\Theta} = \sum_{g=1}^{F} \theta_{f,g} \sum_{j\in\mathscr{J}_g} \left[p_j\, q_j - C_j(q_j)\right] \tag{4.39}$$

The marginal condition of optimality for firm f and product $j \in \mathscr{J}_f$ is:

$$q_j + \sum_{g=1}^{F} \theta_{f,g} \sum_{j\in\mathscr{J}_g} [p_k - MC_k] \frac{\partial D_k}{\partial p_j} = 0 \tag{4.40}$$

In vector form, using all the J products, we have:

$$\mathbf{q} + \Theta^* \, \Delta\mathbf{D} \, \mathbf{PCM} = 0. \tag{4.41}$$

$\mathbf{q}$ and $\mathbf{PCM}$ are $J \times 1$ vectors of quantities and price-cost margins for all the products; Δ is the $J \times J$ Jacobian matrix of demand-price derivatives $\partial D_k / \partial p_j$; and Θ^* is a $J \times J$ matrix with elements $\theta_{f(j),f(k)}$, where $f(j)$ represents the index of the firm that owns product j. If this matrix $\Theta^* \ \Delta\mathbf{D}$ is non-singular, we can obtain price cost margins as:

$$\mathbf{PCM} \ = \ -\left[\Theta^* \ \Delta\mathbf{D}\right]^{-1} \mathbf{q} \tag{4.42}$$

4.3.2 Estimating marginal costs

We first consider the estimation of marginal costs given that the researcher knows the nature of competition, as represented by matrix Θ. For instance, a standard set of assumptions in this context is that there is no collusion (that is, Θ is the identity matrix) and firms compete a la Nash-Bertrand to maximize their own profit.

The researcher has data on J products over T markets, and knows the ownership structure: $\left\{p_{jt}, q_{jt}, \mathbf{x}_{jt} : \ j = 1, ..., J; t = 1, 2, ..., T\right\}$. Suppose that the researcher has estimated in a first step the parameters in the demand system, such that there is a consistent estimator of the Jacobian matrix $\Delta\mathbf{D}$. Therefore, using the marginal conditions of optimality in equation (4.42), we can solve for the vector of marginal costs to obtain:

$$\mathbf{MC}_t \ = \ \mathbf{p}_t + \left[\Theta^* \ \Delta\mathbf{D}_t\right]^{-1} \mathbf{q}_t \tag{4.43}$$

Given the same demand system, different hypotheses about collusion or ownership structures of products (for instance, mergers), imply different estimates of price-cost margins and of marginal costs.

Under the assumption of constant marginal costs – that is, these costs do not depend on the level of output – the realized marginal costs that we recover from equation (4.43) provide the whole marginal cost function. However, in some industries, the assumption of constant marginal costs may not be plausible, and the researcher needs to estimate how these costs depend on the level of output. The estimation of this function is necessary for predictions and counterfactual experiments involving substantial changes in output relative to those observed in the data. In this case, identification of realized marginal costs is not enough and we need to estimate the marginal cost function.

Consider the following cost function,

$$C(q_{jt}) \ = \ \frac{1}{\gamma_q + 1} q_{jt}^{\gamma_q + 1} \ \exp\{\mathbf{x}'_{jt}\gamma_x + \varepsilon_{jt}^{MC}\}, \tag{4.44}$$

with the corresponding marginal cost function,

$$MC_{jt} \ = \ q_{jt}^{\gamma_q} \ \exp\{\mathbf{x}'_{jt}\gamma_x + \varepsilon_{jt}^{MC}\}, \tag{4.45}$$

where γ_q and γ_x are parameters, and ε_{jt}^{MC} is unobservable to the researcher. Taking logarithms, we have the following linear-in-parameters regression model:

$$\ln\left(MC_{jt}\right) \ = \ \gamma_q \ \ln\left(q_{jt}\right) + \mathbf{x}'_{jt}\gamma_x + \varepsilon_{jt}^{MC} \tag{4.46}$$

Note that the realized log- marginal cost, $\left(MC_{jt}\right)$, is known to the researcher as it has been identified using equation (4.43). We are interested in the estimation of the parameters γ_q and γ_x.

The equilibrium model implies that the amount of output q_{jt} is negatively correlated with the unobservable cost inefficiency ε_{jt}^{MC}. Firms/products with larger ε_{jt}^{MC} are less cost-efficient, and this, all else equal, implies a smaller amount of output. Therefore, regressor $\ln(q_{jt})$ is endogenous in the regression equation that represents the log- marginal cost function. Fortunately, the model implies a exclusion restriction that can be used to obtain valid instruments. Note that, given the own characteristics of product j, $\mathbf{x}_{jt}$, and the own amount of output q_{jt}, the marginal cost of this product does not depend on the characteristics of other products in the market, $\{\mathbf{x}_{kt} : k \neq j\}$. However, the model of competition implies that the equilibrium amount of output for a product depends not only on the own characteristics but also on the attributes of competing products. Suppose that $\mathbf{x}_{jt}$ is not correlated with the unobservable cost inefficiency ε_{jt}^{MC}. Then, the model implies that we can use $\{\mathbf{x}_{kt} : k \neq j\}$ as instruments for the endogenous regressor $\ln(q_{jt})$ in the regression equation (4.46).

4.3.3 Testing hypotheses on nature of competition

Researchers can be interested in using price data to learn about the nature of competition, instead of imposing an assumption about matrix Θ. Can we identify collusive behavior? Can we identify matrix Θ? As in the homogeneous product case, we can distinguish two cases for this identification problem: with and without data on marginal costs.

Suppose that the researcher observes the true MC_{jt}. Perhaps more realistically, suppose that the researcher observes some measures of marginal costs that we represent using $q \times 1$ vector $\mathbf{c}^{MC}$. For instance, $\mathbf{c}^{MC}$ may include the mean value of marginal costs for all the products and firms in the industry and for one year during the sample period; or the mean value of realized marginal costs for a particular firm. In the best case scenario, $\mathbf{c}^{MC}$ includes the marginal cost of every product at every sample period.

Given an estimated demand system and a hypothesis about the nature of competition, as represented by a matrix Θ, we can use equation (4.43) to obtain the corresponding vector of marginal costs, and then we can use these values to construct the predicted value of $\mathbf{c}^{MC}$ implied by this value of Θ. We denote this predicted value as $\mathbf{c}^{MC}(\Theta)$. Then, we can compare the actual value $\mathbf{c}^{MC}$ and the predicted value $\mathbf{c}^{MC}(\Theta)$. More formally, we can use the difference between actual and predicted value to construct a test for the null hypothesis that our conjecture about Θ is correct. For instance, if $\mathbf{c}^{MC}$ is a vector of sample means (or more generally, a vector of moments) we can construct a Chi-square goodness-of-fit test. Under the null hypothesis:

$$\left[\mathbf{c}^{MC} - \mathbf{c}^{MC}(\Theta)\right]' \left[Var(\mathbf{c}^{MC}(\Theta))\right]^{-1} \left[\mathbf{c}^{MC} - \mathbf{c}^{MC}(\Theta)\right] \sim \chi^2_q \tag{4.47}$$

This approach has been used in a good number of papers to test collusion and other hypotheses about the nature of competition. Bresnahan (1987) on the US automobile industry was the pioneering study using this approach. He finds evidence of collusive behavior. Other very influential paper using this method is Nevo (2001) on the US Ready-to-Eat cereal industry. Nevo rejects collusive behavior, and finds that the multi-product feature of firms accounts for a very substantial fraction of market power. Some authors, such as Gasmi, Laffont, and Vuong (1992), have used non-nested testing procedures (e.g., Vuong-Test) to select between alternative hypothesis about the nature of competition. Gasmi, Laffont, and Vuong (1992) study competition in prices and advertising between

Coca-Cola and Pepsi-Cola during 1968-1986. They cannot reject the null hypothesis of collusion between these firms.

4.3.4 Estimating the nature of competition

Suppose that the elements of matrix Θ are real numbers within the interval $[0,1]$. That is, $\theta_{f,g}$ represents the degree to which firm f internalizes the profits of firm g when setting prices of its own products. Can we identify these parameters without data on marginal costs? We show here conditions under which these parameters are identified, and describe an estimation method.

It is helpful to illustrate identification and estimation using a simple version of the model with only two single-product firms, firms 1 and 2. This model has two conjectural parameters, θ_{12} and θ_{21}. The marginal condition of optimality in equation (4.40) has the following form, for firm 1:

$$q_{1t} + (p_{1t} - MC_{1t}) \frac{\partial D_{1t}}{\partial p_{1t}} + \theta_{12} (p_{2t} - MC_{2t}) \frac{\partial D_{2t}}{\partial p_{1t}} = 0 \tag{4.48}$$

We can re-write the equation as follows

$$p_{1t} - \left(\frac{\partial D_{1t}}{\partial p_{1t}}\right)^{-1} q_{1t} = MC_{1t} + \theta_{12} (p_{2t} - MC_{2t}) \left(\frac{\partial D_{1t}}{\partial p_{1t}}\right)^{-1} \frac{\partial D_{2t}}{\partial p_{1t}} \tag{4.49}$$

The econometric model is completed with a specification of the marginal cost function. For instance:

$$MC_{jt} = \mathbf{x}'_{jt}\gamma + \varepsilon^{MC}_{jt} \tag{4.50}$$

Plugging this marginal cost function into the marginal condition of optimality, we get the following regression equation for product 1:

$$y_{1t} = \theta_{12}\, \widetilde{p}_{2t} + \mathbf{x}'_{1t}\gamma + \widetilde{\mathbf{x}}'_{2t}\pi_1 + u_{1t} \tag{4.51}$$

with: $y_{1t} \equiv p_{1t} - (\partial D_{1t}/\partial p_{1t})^{-1} q_{1t}$; $\widetilde{p}_{2t} \equiv p_{2t}(\partial D_{1t}/\partial p_{1t})^{-1}(\partial D_{2t}/\partial p_{1t})$; $\widetilde{\mathbf{x}}_{2t} \equiv \mathbf{x}_{2t}$ $(\partial D_{1t}/\partial p_{1t})^{-1}(\partial D_{2t}/\partial p_{1t})$; $u_{1t} \equiv \varepsilon^{MC}_{1t} - \theta_{12}\varepsilon^{MC}_{2t}(\partial D_{1t}/\partial p_{1t})^{-1}(\partial D_{2t}/\partial p_{1t})$; and $\pi_1 \equiv -\theta_{12}\gamma$. We have a similar regression equation for product 2.

The estimation of the parameters in this regression equation needs to deal with the endogeneity of prices. Regressors $\widetilde{p}_{2t}$ and $\widetilde{\mathbf{x}}_{2t}$ are endogenous because they depend on prices, and prices are correlated with the unobserved cost inefficiencies ε^{MC}_{jt} which enter into the error term u_{1t}. In this context, using the so-called *BLP instruments* (that is, the observable characteristics $\mathbf{x}$ of other products is tricky because these variables already enter in the regressor $\widetilde{\mathbf{x}}_{2t}$. We need additional instruments for the endogenous regressor $\widetilde{p}_{2t}$. Possible identification strategies are: *Hausman-Nevo instruments*, when the dataset includes multiple geographic markets and demand unobservables are not correlated across markets after controlling for product fixed effects; or *Arellano-Bond instruments*, when the dataset includes multiple time periods and demand unobservables are not serially correlated after controlling for product fixed effects. We discuss below an empirical application that uses a different identification strategy.[6]

[6] In principle, we could use BLP instruments if we impose the restrictions between the parameters θ_{12}, γ, and π_1: that is, for any product attribute, say k, in the vector $\mathbf{x}$, we have that $\pi_{1k}/\gamma_k = \theta 12$.

Example: Collusion in the Ready-to-Eat (RTE) cereal industry.
Michel and Weiergraeber (2018) study competition in the US RTE cereal industry during the period 1991-1996. There were two important events in this industry during this period: the merger of two leading firms, *Post* and *Nabisco* in 1993; and a massive wholesale price reduction in 1996. The paper emphasizes the importance of allowing *conduct parameters* θ to vary over time and across firms when an industry is subject to important shocks. This view is consistent with interpretation of conduct parameters as endogenous objects in a broader dynamic game of the industry, as we have discussed above in this chapter. The authors are also concerned with finding powerful instruments to separately identify conduct and marginal costs parameters. They propose novel instruments that exploit information on firms' promotional activities.

The main data consists of consumer scanner data from the Dominick's Finer Food (DFF) between February 1991 and October 1996. It includes 58 supermarket stores located in the Chicago metropolitan area. The authors aggregate the data at the monthly level (69 months) and focus on 26 brands of cereal from the 6 nationwide manufacturers: Kellogg's, General Mills, Post, Nabisco, Quaker Oats, and Ralston Purina. Brands are classified into three groups: adult, family, and kids. Importantly for the purpose of this paper, the dataset contains information on wholesale prices and in-store promotional activities.

It is well-known that this is a highly concentrated industry. During this period and market, the leader (Kellogg's) had a market share of 45%, and the top-2 firms accounted for 75%. Firms market shares were relatively stable over the sample period, though there some changes after the 1993 merger between Post and Nabisco.

In the antitrust authority's evaluation of the proposed merger between Post and Nabisco, the main concern was the strong substitutability in the adult cereal segment between Post's and Nabisco's products. The merger did not lead to any product entry or exit or any changes in existing products. Following the merger, Post+Nabisco increased significantly its prices, and this price increase was followed by the rest of the firms. In principle, this response could be explained under Nash-Bertrand competition (before and after the merger), without the need of any change in conduct parameters.

On April 1996, Post decreased its wholesale prices by 20%. This was followed, a few weeks later, by significant price cuts by the other firms. The average decrease in the wholesale price between April and October 1996 was 9.66% (and 7.5% in retail price). The main purpose of this paper is explaining the role that different factors played in this price reductions – including potential changes in firms' conduct.

The authors estimate a random coefficients nested logit model for the demand system. This demand system is similar to the one in Nevo (2001), but it has an important distinguishing feature: it includes as a product characteristic the variable PRO_{jt} that represents the total (aggregated over stores and type of promotion) in-store promotions of product j during month t. The estimate this demand system using BLP-instruments (characteristics of other products) as instrumental variables. In particular, the authors exploit the substantial time variation in the promotion variables.

Given the estimated demand system, the authors then estimated the conduct parameters θ in a regression model very similar to the one in equation (4.51) but for six multi-product firms, instead of two single-product firms. That is, the authors estimate the whole matrix Θ of conduct parameters, together with marginal cost parameters, and

allow this matrix to vary across three different subperiods: before Post+Nabisco merger in 1993; between 1993 and April 1996; after April 1996. To deal with the endogeneity of variables $\widetilde{p}$ and $\widetilde{\mathbf{x}}$ in the regression equation (4.51), the authors use promotional variables of other products as instruments. Demand elasticities are significantly affected by these variables. They have substantial variation across products, over time, and markets. Still, there is the concern that promotional variables are endogenous: they can be correlated with the unobservable component of the marginal cost. Promotions are chosen by firms: it is more profitable to make promotions when marginal costs are low. To deal with this endogeneity, the authors assume that the error term follows and AR(1) ($u_{jt} = \rho\, u_{j,t-1} + v_{jt}$, where v_{jt} is i.i.d.), and they argue that promotions are negotiated between manufacturers and retailers at least one month in advance. Then, they take a quasi-first-difference of the regression equation (that is, a Cochrane-Orcutt transformation, $y_{jt} - \rho\, y_{j,t-1}$). In this transformed equation, $PROMO_{kt}$ is not correlated with the i.i.d. shock v_{jt} because promotions are determined at least one month in advance.

The estimation results show strong evidence for coordination between 1991-1992. On average the conduct parameter is 0.277: that is, a firm values \$1 of its rivals' profits as much as \$0.277 of its own profits. Because of this coordination, pre-merger price-cost margins are 25.6% higher than under multi-product Bertrand-Nash pricing. After the Post + Nabisco merger in 1993, the degree of coordination increased significantly, on average to 0.454. Towards year 1996, the degree of coordination becomes close to 0, consistent with multi-product Bertrand-Nash pricing. Counterfactual experiments show that if firms had competed à la Bertrand-Nash before 1996, consumer welfare would have increased by between $\$1.6 - \$2.0$ million per year, and the median wholesale price would have been 9.5% and 16.3% lower during the pre-merger and post-merger periods, respectively.

4.3.5 Conjectural variations with differentiated product

So far, in the model with differentiated product, we have incorporated the *nature of competition* by including parameters $\theta_{f,g}$ that represent to what extent firm f values the profit of firm g relative to its own profit. This is a reasonable way of modelling collusion. However, it seems quite different to the *conjectural variation* model that we studied for the homogeneous product model. In this section, we present the conjectural variation model in differentiated product industry with price competition. We show that the marginal conditions of optimality from this model have a similar form as those from the model with profit-weights $\theta_{f,g}$.

For simplicity, consider a differentiated product industry with two single-product firms: firm 1 and firm 2. The profit function of firm j is $\Pi_j = p_j\, q_j - C_j(q_j)$. Define the conjecture parameter CV_1 as firm 1's belief about how firm 2 will change its price when firm 1 changes marginally its own price. That is, CV_1 represents firm 1's belief about $\partial p_2/\partial p_1$. Similarly, CV_2 represents firm 2 's belief about $\partial p_2/\partial p_1$. Nash-Bertand competition implies $CV_j = 0$ for every firm j. Perfect collusion, implies $CV_j = 1$ for every firm j. Taking the conjecture CV as given, the marginal condition for profit maximization for firm 1 is:

$$q_1 + (p_1 - MC_1)\,\frac{\partial D_1}{\partial p_1} + CV_1\,(p_1 - MC_1)\,\frac{\partial D_1}{\partial p_2} \;=\; 0 \tag{4.52}$$

There are both similarities and differences between this equation and the marginal

condition with profit-weights in equation (4.48). The two equations are equivalent when the firms have the same marginal costs and there is symmetric product differentiation.[7] However, there are quantitative differences between the predictions of the two models when firms are heterogeneous in marginal costs or product quality.

Example: Logit demand model with conjectural variations.
Suppose that the demand system has a logit structure where the average utility of product j is $\beta_j - \alpha\, p_j$, where β_j represents the qaulity of product j. This model implies the following equation for the marginal condition of product 1:

$$p_1 - MC_1 \;=\; \frac{1}{\alpha\,(1 - s_1 - s_2\,CV_1)} \tag{4.53}$$

where s_j is the market share of product j.

Suppose that the researcher does not know the magnitude of the marginal costs MC_1 and MC_2, but she knows that the two firms use the same production technology, the same type of variable inputs, and purchase these inputs in the same markets where they are price takers. Therefore, the researcher knows that $MC_1 = MC_2 = MC$, though she does not know the magnitude of MC. This information, together with the marginal conditions of optimality, imply the following equation for the difference between prices:

$$p_1 - p_2 \;=\; \frac{1}{\alpha\,(1 - s_1 - s_2\,CV_1)} - \frac{1}{\alpha\,(1 - s_2 - s_1\,CV_2)} \tag{4.54}$$

The researcher observes prices $p_1 = \$200$ and $p_2 = \$195$ and market shares $s_1 = 0.5$ and $s_2 = 0.2$. Firm 1 has both a larger price and a larger market share because its product has better quality.[8] The researcher has estimated the demand system and knows that $\alpha = 0.01$. Solving these data into the previous equation, we have:

$$\$200 - \$195 = \frac{100}{1 - 0.5 - 0.2\,CV_1} - \frac{100}{1 - 0.2 - 0.5\,CV_2} \tag{4.55}$$

This is a condition that the parameters CV_1 and CV_2 should satisfy. Using this equation we can show that the hypothesis of Nash-Bertrand competition (that requires $CV_1 = CV_2 = 0$) implies a prediction about the price difference $p_1 - p_2$ that is substantially larger than the price difference that we observe in the data. The hypothesis of Nash-Bertrand competition, $CV_1 = CV_2 = 0$, implies that the right hand side of equation (4.55) is:

$$\frac{100}{0.5} - \frac{100}{0.8} = 200 - 125 = \$75 \tag{4.56}$$

That is, Nash-Bertrand implies a price difference of \$75 but the price difference in the data is only \$5. The hypothesis of Collusion, $CV_1 = CV_2 = 1$, implies that the right hand side of the equation in (4.55) is:

$$\frac{100}{0.5 - 0.2} - \frac{100}{0.8 - 0.5} = \$0 \tag{4.57}$$

[7] We have symmetric product differentiation if $\partial D_j/\partial p_j$ is the same for every product j, and $\partial D_j/\partial p_k = \partial D_k/\partial p_j$ for every pair of products j,k. For instance, this is the case in a logit demand model where all the products have the same quality, or in Hotelling (1929) linear-city and Salop (1979) circle-city models when firms are equidistant from each other.

[8] In this industry, higher product quality requires a larger fixed cost but it does not affect marginal cost.

That is, Collusion implies a price difference of \$0, which is closer to the price difference of \$5 that we observe in the data. Under the restriction $CV_1 = CV_2$, we can use equation (4.55) to obtain the value of the conjecture parameter. It implies a quadratic equation in CV, and the positive root is $CV = 0.984$.

4.4 Incomplete information

In this chapter, we have considered different factors that can affect price and quantity competition and market power in an industry. Heterogeneity in marginal costs, product differentiation, multi-product firms, or conduct/nature of competition are among the most important features that we have considered so far. All the models that we have considered assume that firms have perfect knowledge about demand, their own costs, and the costs of their competitors. In game theory, this type of model is a *game of complete information*. This assumption can be quite unrealistic in some industries. Firms have uncertainty about current and future realizations of demand, costs, market regulations, or the behavior of competitors. This uncertainty can have substantial implications for their decisions and profits, and for the efficiency of the market. For example, some firms may be more efficient in gathering and processing information, and they can use this information in their pricing or production strategies to improve their profits.

The assumption of firms' complete information has been the status quo in empirical models of Cournot or Bertrand competition. In reality, firms often face significant uncertainty about demand and about their rivals costs and strategies. Firms are different in their ability and their costs for collecting and processing information, for similar reasons as they are heterogeneous in their costs of production or investment. In this section, we study models of price and quantity competition that allow for firms' incomplete and asymmetric information. Our main purpose is to study how limited information affects competition and market outcomes.

4.4.1 Cournot competition with private information

Vives (2002) studies theoretically the importance of firms' private information as a determinant of prices, market power, and consumer welfare. He considers a market in which firms compete à la Cournot and have private information. Then, he studies the relative contribution of private information and market power in accounting for the welfare losses. He shows that in large enough markets, abstracting from market power provides a much better approximation than abstracting from private information. If M represents market size, then the effect of market power is of the order of $1/M$ for prices and $1/M^2$ for per-capita deadweight loss, while the effect of private information is of the order of $1/\sqrt{M}$ for prices and $1/M$ for per-capita deadweight loss. Numerical simulations of the model show that there is a critical value for market size M^* (that depends on the values of structural parameters) such that the effect of private information dominates the effect of market power when market size is greater than this threshold value.

(i) Demand, costs, and information structure

Consider the market for a homogeneous product where firms compete à la Cournot and there is free market entry. A firm's marginal cost is subject to idiosyncratic shocks that

are private information to the firm. The demand function and the marginal cost functions are linear such that the model is linear-quadratic. This feature facilitates substantially the characterization of a Bayesian Nash equilibrium in this model with incomplete information.

There are M consumers in the market and each consumer has an indirect utility function $U(x) = \alpha\ x - \beta\ x^2/2 - p\ x$, where x is the consumption of the good, p is the market price, and $\alpha > 0$ and $\beta > 0$ are parameters. This utility function implies the market level inverse demand function, $p = P(Q) = \alpha - \beta_M\ Q$, where $\beta_M \equiv \beta/M$. Firms are indexed by i. If firm i is actively producing in the market, its cost function is $C(q_i, \theta_i) = \theta_i\ q_i + (\gamma/2)\ q_i^2$ such that its marginal cost is $MC_i = \theta_i + \gamma\ q_i$. Variable θ_i is private information to firm i. In games of incomplete information, θ_i is denoted as player i's (in this case, firm i's) *type*. Firms' types are random variables which are i.i.d. with mean μ_θ and variance σ_θ^2. This distribution is *common knowledge* to all firms.[9] Every active firm producing in the market should pay a fixed cost $F > 0$.

(ii) Bayesian Nash equilibrium

The model is a two-stage game. In the first stage, firms decide whether to enter the market or not. If a firm decides to enter, it pays a fixed cost $F > 0$. When a firm makes its entry decision it does not know yet the realization of its type θ_i. Therefore, the entry decision is based on the maximization of expected profits. At the second stage, each active firm i that has decided to enter observes its own θ_i but not the θ's of the other active firms, and competes according to a Bayesian Nash-Cournot equilibrium. This equilibrium concept is a version of Nash equilibrium for games of incomplete information, and we describe it below.

We now recursively solve the equilibrium of the model starting at the second stage. For the moment, suppose that there are n firms active in the market: we later obtain the equilibrium value of n. The expected profit of firm i is:

$$\begin{aligned} \pi_i(\theta_i) &= \mathbb{E}\left[P(Q) \mid \theta_i\right]\ q_i - \theta_i\ q_i - \frac{\gamma}{2} q_i^2 \\ &= \left(\alpha - \beta_M\ \left(q_i + \mathbb{E}\left[\sum_{j \neq i} q_j\right]\right)\right)\ q_i - \theta_i\ q_i - \frac{\gamma}{2} q_i^2, \end{aligned} \tag{4.58}$$

where the expectation $\mathbb{E}[.]$ is over the distribution of the variables θ_j for $j \neq i$, which are not known to firm i. A *Bayesian Nash Equilibrium (BNE)* is an n-tuple of strategy functions, $[\sigma_1(\theta_1), \sigma_2(\theta_n), ..., \sigma_n(\theta_n)]$, such that for every firm's strategy maximizes its own expected profit taking as given other firms' strategies. That is, for every firm i:

$$\sigma_i(\theta_i) = \arg\max_{q_i}\ \mathbb{E}\left[P(Q) \mid \theta_i, \sigma_j \text{ for } j \neq i\right]\ q_i - \theta_i\ q_i - \frac{\gamma}{2} q_i^2 \tag{4.59}$$

The first order condition of optimality for the best response of firm i implies:

$$q_i = \sigma_i(\theta_i) = \left[\gamma + 2\beta_M\ \right]^{-1} \left[\alpha - \theta_i - \beta_M\ \sum_{j \neq i} \mathbb{E}\left(\sigma_j(\theta_j)\right)\right] \tag{4.60}$$

[9] In game theory, an object or event is *common knowledge* if everybody knows that everybody knows that ... knows it.

Since firms are identical up to the private information θ_i, it seems reasonable to focus on a symmetric BNE such that $\sigma_i(\theta_i) = \sigma(\theta_i)$ for every firm i. Imposing this restriction in the best response condition (4.60), taking expectations over the distribution of θ_i, and solving for $\sigma^e \equiv \mathbb{E}(\sigma(\theta_i))$, we obtain that:

$$\sigma^e \equiv \mathbb{E}(\sigma(\theta_i)) = \frac{\alpha - \mu_\theta}{\gamma + \beta_M (n+1)} \tag{4.61}$$

Solving this expression in (4.60), we obtain the following closed-form expression for the equilibrium strategy function under BNE, in the second stage of the game:

$$q_i = \sigma(\theta_i) = \frac{\alpha - \mu_\theta}{\gamma + \beta_M (n+1)} - \frac{\theta_i - \mu_\theta}{\gamma + 2\beta_M} \tag{4.62}$$

Now, we proceed to the first stage of the game to obtain the equilibrium number of active firms in the market. Under the BNE in the second stage, the expected profit of an active firm, before knowing the realization of its own θ_i is:

$$\mathbb{E}[\pi(\theta_i)] = [\beta_M + \gamma/2] \; \mathbb{E}\left[\sigma(\theta_i)^2\right] = \frac{\left[\alpha - \mu_\theta^2\right]}{\left[\gamma + \beta_M (n+1)\right]^2} + \frac{\sigma_\theta^2}{\left[\gamma + 2\beta_M\right]^2} \tag{4.63}$$

Given this expected profit, we can obtain the the equilibrium number of entrants in the first stage of the game. Given a market of size M, the free-entry number of firms $n^*(M)$ is approximated by the solution to $\mathbb{E}[\pi(\theta_i)] - F = 0$. Given the expression for the equilibrium profit, it is simple to verify that $n^*(M)$ is of the same order as market size M. That is, the ratio $n^*(M)/M$ of the firms per consumer is bounded away from zero and infinity.

(iii) Welfare analysis

From the point of view of a social planner, the optimal allocation in this industry can be achieved if firms share all their information and behave as price takers. Let us label this equilibrium as $CI-PT$: *complete information* with *price taking* behavior. If p and W are the price and the total welfare, respectively, under the "true" model (with both Cournot conduct and private information), then the differences $p - p_{CI-PT}$ and $W - W_{CI-PT}$ represent the combined effect of incomplete information and Cournot behavior on prices and on welfare.

To measure the separate effects of incomplete information and Cournot behavior, it is convenient to define other two models: a model of Cournot competition with complete information, that we label as CI; and a model of incomplete information that assumes that firms are price takers, that we label as PT.[10] Using these models, we can make the following decomposition:

$$\begin{aligned} p - p_{CI-PT} &= [p - p_{PT}] + [p_{PT} - p_{CI-PT}] \\ W_{CI-PT} - W &= [W_{CI-PT} - W_{PT}] + [W_{PT} - W] \end{aligned} \tag{4.64}$$

[10] In the complete information Cournot model, equilibrium output is: $q_i^{CI} = \frac{\alpha - \widetilde{\theta}_n}{\gamma + \beta_M (n+1)} - \frac{\theta_i - \widetilde{\theta}_n}{\gamma + 2\beta_M}$, where $\widetilde{\theta}_n \equiv (n-1)^{-1} \sum_{j \neq i} \theta_j$.

The term $p - p_{PT}$ captures the effect of Cournot behavior (market power) on prices, and the term $p_{PT} - p_{CI-PT}$ captures the effect of incomplete information. Similarly, $W_{CI-PT} - W$ is the total deadweight loss, $[W_{CI-PT} - W_{PT}]$ is the contribution of incomplete information, and $[W_{PT} - W]$ is the contribution of Cournot competition.[11]

Vives (2002) shows that as market size M (and therefore n) goes to infinity, market price and welfare per capita converge to the optimal allocation: that is, $[p - p_{CI-PT}] \longrightarrow 0$ and $[W_{CI-PT} - W]/M \longrightarrow 0$. Private information and Cournot behavior have an effect only when the market is not too large. Vives shows also that there is a critical value for market size, M^* (that depends on the values of structural parameters), such that if market size is greater this threshold value, then the effect of private information on prices and consumer welfare dominates the effect of market power.

This result has interesting policy implications. Antitrust authorities look with suspicion at the information exchanges between firms because they can help collusive agreements. The collusion concern is most important in the presence of a few players because collusion is easier to be sustained in this case (repeated game). Vives (2002)'s results show that with few firms, market power (Cournot) has the most important contribution to the welfare loss, so it seems reasonable to control these information exchanges. When market size and the number of firms increase, information asymmetry becomes a more important factor in welfare loss and it is optimal to allow for some information sharing between firms.

(iv) An empirical application

Armantier and Richard (2003) study empirically how asymmetric information on marginal costs affects competition and outcomes in the US airline industry. They investigate how marketing alliances between American Airlines and United Airlines facilitate information sharing and how this affects market outcomes. The authors find that such information exchanges would benefit airlines with a very moderate cost in terms of consumer welfare.

[11] Note that this is one of different ways we can decompose these effects. For instance, we could also consider the decomposition, $p - p_{CI-PT} = [p - p_{CI}] + [p_{CI} - p_{CI-PT}]$ and $W_{CI-PT} - W = [W_{CI-PT} - W_{CI}] + [W_{CI} - W]$. The main results are the same regardless of the decomposition chosen.

4.5 Exercises

4.5.1 Exercise 1

Consider an industry with a differentiated product. There are two firms in this industry, firms 1 and 2. Each firm produces and sells only one brand of the differentiated product: brand 1 is produced by firm 1, and brand 2 by firm 2. The demand system is a logit demand model, where consumers choose between three different alternatives: $j = 0$, represents the consumer decision of no purchasing any product; and $j = 1$ and $j = 2$ represent the consumer purchase of product 1 and 2, respectively. The utility of no purchase ($j = 0$) is zero. The utility of purchasing product $j \in \{1,2\}$ is $\beta\, x_j - \alpha\, p_j + \varepsilon_j$, where the variables and parameters have the interpretation that we have seen in class. Variable x_j is a measure of the quality of product j, for instance, the number of stars of the product according to consumer ratings. Therefore, we have that $\beta > 0$. The random variables ε_1 and ε_2 are independently and identically distributed over consumers with a type I extreme value distribution, that is, Logit model of demand. Let H be the number of consumers in the market. Let s_0, s_1, and s_2 be the market shares of the three choice alternatives, such that s_j represents the proportion of consumers choosing alternative j and $s_0 + s_1 + s_2 = 1$.

Question 1.1. Based on this model, write the equation for the market share s_1 as a function of the prices and the qualities x's of all the products.

Question 1.2. Obtain the expression for the derivatives: (a) $\dfrac{\partial s_1}{\partial p_1}$; (b) $\dfrac{\partial s_1}{\partial p_2}$; (c) $\dfrac{\partial s_1}{\partial x_1}$; and (d) $\dfrac{\partial s_1}{\partial x_2}$. Write the expression for these derivatives in terms only of the market shares s_1 and s_2 and the parameters of the model.

The profit function of firm $j \in \{0,1\}$ is $\pi_j = p_j\, q_j - c_j\, q_j - FC(x_j)$, where: q_j is the quantity sold by firm j (that is, $q_j = H\, s_j$); c_j is firm $j's$ marginal cost, that is assumed constant, that is, linear cost function; and $FC(x_j)$ is a fixed cost that depends on the level of quality of the firm.

Question 1.3. Suppose that firms take their qualities x_1 and x_2 as given and compete in prices ala Bertrand.
(a) Obtain the equation that describes the marginal condition of profit maximization of firm 1 in this Bertrand game. Write this equation taking into account the specific form of $\dfrac{\partial s_1}{\partial p_1}$ in the Logit model.
(b) Given this equation, write the expression for the equilibrium price-cost margin $p_1 - c_1$ as a function of s_1 and the demand parameter α.

Now, suppose that the researcher is not willing to impose the assumption of Bertrand competition and considers a conjectural variations model. Define the conjecture parameter CV_1 as the belief or conjecture that firm 1 has about how firm 2 will change its price when firm 1 changes marginally its price. That is, CV_1 represents the belief or conjecture of firm 1 about $\dfrac{\partial p_2}{\partial p_1}$. Similarly, CV_2 represents the belief or conjecture of firm 2 about $\dfrac{\partial p_2}{\partial p_1}$.

Question 1.4. Suppose that firm 1 has a conjectural variation CV_1.
(a) Obtain the equation that describes the marginal condition of profit maximization of firm 1 under this conjectural variation. Write this equation taking into account the specific form of $\frac{\partial s_1}{\partial p_1}$ in the Logit model. [Hint: Now, we have that: $\frac{dq_1}{dp_1} = \frac{\partial q_1}{\partial p_1} + \frac{\partial q_1}{\partial p_2}\frac{\partial p_2}{\partial p_1}$, where $\frac{\partial q_1}{\partial p_1}$ and $\frac{\partial q_1}{\partial p_2}$ are the expressions you have derived in Q1.2].
(b) Given this equation, write the expression for the equilibrium price-cost margin $p_1 - c_1$ as a function of the market shares s_1 and s_2, and the parameters α and CV_1.

Question 1.5. Suppose that the researcher does not know the magnitude of the marginal costs c_1 and c_2, but she knows that the two firms use the same production technology, they use the same type of variable inputs, and they purchase these inputs in the same markets where they are price takers. Under these conditions, the researcher knows that $c_1 = c_2 = c$, though she does not know the magnitude of the marginal cost c.
(a) The marginal conditions for profit maximization in Q1.4(b), for the two firms, together with the condition $c_1 = c_2 = c$, imply that price difference between these two firms, $p_1 - p_2$, is a particular function of their markets shares and their conjectural variations. Derive the equation that represents this condition.
(b) The researcher observes prices $p_1 = \$200$ and $p_2 = \$195$ and market shares $s_1 = 0.5$ and $s_2 = 0.2$. Firm 1 has both a larger price and a larger market share because its product has better quality, that is, $x_1 > x_2$. The researcher has estimated the demand system and knows that $\alpha = 0.01$. Plug in these data into the equation in Q1.5(a) to obtain a condition that the parameters CV_1 and CV_2 should satisfy in this market.
(c) Using the equation in Q1.5(b), show that the hypothesis of Nash-Bertrand competition (that requires $CV_1 = CV_2 = 0$) implies a prediction about the price difference $p_1 - p_2$ that is substantially larger than the price difference that we observe in the data.
(d) Using the equation in Q1.5(b), show that the hypothesis of Collusion (that requires $CV_1 = CV_2 = 1$) implies a prediction about the price difference $p_1 - p_2$ that is much closer to the price difference that we observe in the data.

4.5.2 Exercise 2

To answer the questions in this part of the problem set you need to use the dataset `verboven_cars.dta` Use this dataset to implement the estimations describe below. Please, provide the STATA code that you use to obtain the results. For all the models that you estimate below, impose the following conditions:

- For market size (number of consumers), use Population/4, that is, `pop/4`
- Use prices measured in euros (`eurpr`).
- For the product characteristics in the demand system, include the characteristics: `hp, li, wi, cy, le,` and `he.`
- Include also as explanatory variables the market characteristics: ln(pop) and log(gdp).
- In all the OLS estimations include fixed effects for market (`ma`), year (`ye`), and brand (`brd`).
- Include the price in logarithms, that is, ln(`eurpr`).

- Allow the coefficient for log-price to be different for different markets (countries). That is, include as explanatory variables the log price, but also the log price interacting (multiplying) each of the market (country) dummies except one country dummy (say the dummy for Germany) that you use as a benchmark.

Question 2.1.
(a) Obtain the OLS-Fixed effects estimator of the Standard logit model. Interpret the results.
(b) Test the null hypothesis that all countries have the same price coefficient.
(c) Based on the estimated model, obtain the average price elasticity of demand for each country evaluated at the mean values of prices and market shares for that country.

Question 2.2. Consider the equilibrium condition (first order conditions of profit maximization) under the assumption that each product is produced by only one firm.
(a) Write the equation for this equilibrium condition. Write this equilibrium condition as an equation for the Lerner Index, $\frac{p_j - MC_j}{p_j}$.
(b) Using the previous equation in Q2.2(a) and the estimated demand in Q2.1, calculate the Lerner index for every car-market-year observation in the data.
(c) Report the mean values of the Lerner Index for each of the counties/markets. Comment the results.
(d) Report the mean values of the Lerner Index for each of the top five car manufacturers (that is, the five car manufacturers with largest total aggregate sales over these markets and sample period). Comment the results.

Question 2.3.
(a) Using the equilibrium condition and the estimated demand, obtain an estimate of the marginal cost for every car-market-year observation in the data.
(b) Run an OLS-Fixed effects regression where the dependent variable is the estimated value of the marginal cost, and the explanatory variables (regressors) are the product characteristics `hp, li, wi, cy, le,` and `he.` Interpret the results.

5. Market Entry

5.1 Introduction

In a model of market entry the endogenous variables are firms' decisions to be active in the market and, in some cases, the characteristics of the products that firms provide. In the previous chapters, we have taken the number of firms and products in a market as exogenously given or, more precisely, as predetermined in the first stage of a two-stage game of competition. In this chapter, we study the first stage of the competition game.

Empirical games of market entry in retail markets share as common features that the payoff of being active in the market depends on market size, entry cost, and the number and characteristics of other active firms. The set of structural parameters of the model varies considerably across models and applications, but it typically includes parameters that represent the entry cost and the strategic interactions between firms (competition effects). These parameters play a key role in the determination of the number of firms in the market, their characteristics, and their spatial configuration. These costs cannot be identified from the estimation of demand equations, production functions, or marginal conditions of optimality for prices or quantities. Instead, in a structural entry model, entry costs are identified using the principle of revealed preference: if we observe a firm operating in a market it is because its value in that market is greater than the value of shutting down and putting its assets to alternative uses. Under this principle, firms' entry decisions reveal information about the underlying or latent profit function. Empirical games of market entry can be also useful to identify strategic interactions between firms that occur through variable profits. In empirical applications where sample variation in prices is very small but there is substantial variation in entry decisions, an entry model can provide more information about demand substitution between stores and products than the standard approach of using prices and quantities to estimate demand. Furthermore, data on prices and quantities at the store level are sometimes difficult to obtain, while data on firms entry/exit decisions are more commonly available.

In empirical applications of games of market entry, structural parameters are estimated using data on firms' entry decisions in a sample of markets. The estimated model is used to answer empirical questions on the nature of competition and the structure of

costs in an industry, and to make predictions about the effects of changes in structural parameters or of counterfactual public policies affecting firms' profits, for example, subsidies, taxes, or zoning laws.

An important application of models of entry is the study of firms' decision about the spatial location of their products, their production plants, or their stores. Competition in differentiated product markets is often characterized by the importance of product location in the space of product characteristics, and therefore, the geographic location of stores is important in retail markets. As shown in previous chapters, the characteristics of firms' products relative to those of competing products can have substantial effects on demand and costs, and consequently on prices, quantities, profits, and consumer welfare. Firms need to choose product location carefully so that they are accessible to many potential customers. For instance, opening a store in attractive locations is typically more expensive (for example, higher land prices) and it can be associated with stronger competition. Firms should consider this trade-off when choosing the best store location. The study of the determinants of spatial location of products is necessary to inform public policy and business debates such as the value of a merger between multiproduct firms, spatial pre-emption, cannibalization between products of the same firm, or the magnitude of economies of scope. Therefore, it is not surprising that models of market entry, store location, and spatial competition have played a fundamental role in the theory of industrial organization at least since the work of Hotelling (1929). However, empirical work on structural estimation of these models has been much more recent and it has followed the seminal work by Bresnahan and Reiss (1990, 1991).

5.2 General ideas

5.2.1 What is a model of market entry?

Models of market entry in IO can be characterized in terms of three main features. First, the key endogenous variable is a firm decision to operate or not in a market. Entry in a market should be understood in a broad sense. The standard example is the decision of a firm to enter in an industry for the first time. However, applications of entry models include also decisions of opening a new store, introducing a new product, adopting a new technology, the release of a new movie, or the decision to bid in an auction, among others. A second important feature is that there is an entry cost associated with being active in the market. Finally, the payoff of being active in the market depends on the number (and the characteristics) of other firms active in the market, that is, the model is a game.

Consider a market with N firms that decide whether to be active. We index firms with $i \in \{1,2,...,N\}$. Let $a_i \in \{0,1\}$ be a binary variable that represents the decision of firm i of being active in a market ($a_i = 1$) or not ($a_i = 0$). The profit of not being active is zero. The profit of an active firm is $V_i(n) - F_i$ where V_i is the variable profit of firm i when there are n firms active in the market, and F_i is the entry cost for firm i. The number of active firms, n, is endogenous and is equal to $n = \sum_{i=1}^{N} a_i$. Under the Nash assumption, every firm takes as given the actions of the other firms and makes a decision that maximizes its own profit. Therefore, the best response of firm i under the

Nash equilibrium is:

$$a_i = \begin{cases} 1 & \text{if} \quad V_i\left(1+\sum_{j\neq i} a_j\right) - F_i \geq 0 \\ 0 & \text{if} \quad V_i\left(1+\sum_{j\neq i} a_j\right) - F_i < 0 \end{cases} \tag{5.1}$$

For instance, consider a market with two potential entrants with $V_1(n) = V_2(n) = 100 - 20\,n$ and $F_1 = F_2 = F$, such that $V_i\left(1+a_j\right) - F_i = 80 - F - 20\,a_j$. The best responses are:

	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$(0\ ,\ 0)$	$(0\ ,\ 80-F)$
$a_1 = 1$	$(80-F\ ,\ 0)$	$(60-F\ ,\ 60-F)$

(5.2)

We can see that the model has different predictions about market structure depending on the value of the fixed cost. If $F \leq 60$, duopoly, $(a_1, a_2) = (1,1)$, is the unique Nash equilibrium. If $60 < F \leq 80$, then either the monopoly of firm 1 $(a_1, a_2) = (1,0)$ or the monopoly of firm 2 $(a_1, a_2) = (0,1)$ are Nash equilibria. If $F > 80$, then no firm in the market $(a_1, a_2) = (0,0)$ is the unique Nash equilibrium. The observed actions of the potential entrants reveal information about profits, and about fixed costs.

Principle of Revealed Preference. The estimation of structural models of market entry is based on the principle of Revealed Preference. In the context of these models, this principle establishes that if we observe a firm operating in a market it is because its value in that market is greater than the value of shutting down and putting its assets in alternative uses. Under this principle, firms' entry decisions reveal information about the underlying latent firm's profit (or value).

Static models. In this chapter, we study static games of market entry. We study dynamic models of market entry in chapters 7 and 8. There are several differences between static and dynamic models of market entry. But there is a simple difference that should be already pointed out. For static models of entry, we should understand entry as "being active in the market" and not as a transition from being "out" of the market to being "in" the market. That is, in these static models we ignore the fact that, when choosing whether to be active or not in the market, some firms are already active (incumbents) and other firms not (potential entrants). In other words, we ignore that the choice of not being active in the market means "exit" for some firms and "stay out" for others.

5.2.2 Why estimating entry models?

The specification and estimation of models of market entry is motivated by the need to endogenize the number of firms in the market, as well as some characteristics that operate at the extensive margin. Endogenizing the number of firms in the market is a key aspect in any model of IO where market structure is treated as endogenous. Once we endogenize the number of firms in the market, we need to identify entry cost parameters, and these parameters cannot be identified from demand equations, production functions, and marginal conditions of optimality for prices and quantities. We identify entry costs from the own entry model. More generally, we can distinguish the following motives for the estimation of models of market entry.

(a) Identification of entry cost parameters. Parameters such as fixed production costs, entry costs, or investment costs do not appear in demand or production equations, or in

the marginal conditions of optimality in firms' decisions of prices or quantities. However, fixed costs contribute to the market entry decision. These parameters can be important in the determination of market structure and market power in an industry.

(b) Data on prices and quantities may not be available at the level of individual firm, product, and market. Many countries have excellent surveys of manufacturers or retailers with information at the level of the specific industry (5 or 6 digits NAICS, SIC) and local markets (census tracts) on the number of establishments and some measure of firm size such as aggregate revenue. Though we observe aggregate revenue at the industry-market level, we do not observe P and Q at that level. Under some assumptions, it is possible to identify structural parameters using these data and the structure of an entry model.

(c) Econometric efficiency. The equilibrium entry conditions contain useful information for the identification of structural parameters. Using this information can increase significantly the precision of our estimates. In fact, when the sample variability in prices and quantities is small, the equilibrium entry conditions may have a more significant contribution to the identification of demand and cost parameters than demand equations or production functions.

(d) Dealing with endogenous selection problem in the estimation of demand or production functions. In some applications, the estimation of a demand system or a production function requires dealing with the endogeneity of firms' and products' entry. For instance, Olley and Pakes (1996) show that ignoring the endogeneity of a firm's decision to exit from the market can generate significant biases in the estimation of production functions. Similarly, in the estimation of demand of differentiated products, not all the products are available in every market and time period. We observe a product only in markets where demand for this product is high enough to make it profitable to introduce that product. Ignoring the endogeneity of the presence of products can introduce important biases in the estimation of demand (Ciliberto, Murry, and Tamer, 2020; Gandhi and Houde, 2019; and Li et al., 2018). Dealing with the endogeneity of product presence may require the specification and estimation of a model of market product entry.

The type of data used, the information structure of the entry game, and the assumptions about unobserved heterogeneity, are important characteristics of an entry game that have implications on the identification, estimation, and predictions of the model.

5.3 Data

The datasets that have been used in empirical applications of structural models of entry in retail markets consist of a sample of geographic markets with information on firms' entry decisions and consumer socio-economic characteristics over one or several periods of time. In these applications, the number of firms and time periods is typically small such that statistical inference (that is, the construction of sample moments and the application of law of large numbers and central limit theorems) is based on a 'large' number of markets. In most applications, the number of geographic markets is between a few hundred and a few thousand. Within these common features, there is substantial heterogeneity in the type of data that have been used in empirical applications.

In this section, we concentrate on four features of the data that are particularly important, as they have substantial implications on the type of model that can be estimated, the empirical questions that we can answer, and the econometric methods to be used. These features are: (1) selection of geographic markets; (2) presence or not of within-market spatial differentiation; (3) information on prices, quantities, or sales at the store level; and (4) information on potential entrants.

5.3.1 Geographic markets

In a seminal paper, Bresnahan and Reiss (1990) use cross-sectional data from 149 small US towns to estimate a model of entry of automobile dealerships. For each town, the dataset contains information on the number of stores in the market, demographic characteristics such as population and income, and input prices such as land prices. The selection of the 149 small towns is based on the following criteria: the town belongs to a county with fewer than 10 000 people; there is no other town with a population of over 1000 people within 25 miles of the central town; and there is no large city within 125 miles. These conditions for the selection of a sample of markets are typically described as the 'isolated small towns' market selection. This approach has been very influential and has been followed in many empirical applications of entry in retail markets.

The main motivation for using this sample selection is in the assumptions of spatial competition in the Bresnahan–Reiss model. The model assumes that the location of a store within a market does not have any implication on its profits or in the degree of competition with other stores. This assumption is plausible only in small towns where the possibilities for spatial differentiation are very limited. If this model were estimated using a sample of large cities, we would spuriously find very small competition effects simply because there is negligible or no competition at all between stores located far away from each other within the city. The model also assumes that there is no competition between stores located in different markets. This assumption is plausible only if the market under study is not geographically close to other markets; otherwise the model would ignore relevant competition from stores outside the market.

Although the 'isolated small towns' approach has generated a good number of important applications, it has some limitations. The extrapolation to urban markets of the empirical findings obtained in these samples of rural markets is in general not plausible. Focusing on rural areas makes the approach impractical for many interesting retail industries that are predominantly urban. Furthermore, when looking at national retail chains, these rural markets account for a very small fraction of these firms' total profits.

5.3.2 Spatial competition

The limitations of the 'isolated small towns' approach have motivated the development of empirical models of entry in retail markets that take into account the spatial locations and differentiation of stores within a city market. The work by Seim (2006) was seminal in this evolution of the literature. In Seim's model, a city is partitioned into many small locations or blocks, for example, census tracts, or a uniform grid of square blocks. A city can be partitioned into dozens, hundreds, or even thousands of these contiguous blocks or locations. In contrast to the 'isolated small towns' approach, these locations are not isolated, and the model allows for competition effects between stores at different

locations.

The datasets in these applications contain information on the number of stores, consumer demographics, and input prices at the block level. This typically means that the information on store locations should be geocoded, that is, should include the exact latitude and longitude of each store location. Information on consumer demographics is usually available at a more aggregate geographic level.

The researcher's choice for the size of a block depends on multiple considerations, including the retail industry under study, data availability, specification of the unobservables, and computational cost. In principle, a finer grid entails a more flexible model in measuring spatial substitution between stores. The computational cost of estimating the model can increase rapidly with the number of locations. The assumption on the distribution of the unobservables across locations is also important.

A common approach is to define a block/location where demographic information is available. For example, the set of locations can be equal to the set of census tracts within the city. While convenient, a drawback of this approach is that some blocks, especially those in the periphery of a city, tend to be very large. These large blocks are often problematic because (1) within-block spatial differentiation seems plausible, and (2) the distance to other blocks becomes highly sensitive to choices of block centroids. In particular, a mere use of geometric centroids in these large blocks can be quite misleading as the spatial distribution of population is often quite skewed.

To avoid this problem, Seim (2006) uses population weighted centroids rather than (unweighted) geometric centroids. An alternative approach to avoid this problem is to draw a square grid on the entire city and use each square as a possible location, as in Datta and Sudhir (2013) and Nishida (2015). The value of consumer demographics in a square block is equal to the weighted average of the demographics at the census tracts that overlap with the square. The advantage of this approach is that each submarket has a uniform shape. In practice, implementation of this approach requires the removal of certain squares where entry cost is prohibitive. These areas include those with some particular natural features (for example, lakes, mountains, and wetlands) or where commercial space is prohibited by zoning. For example, Nishida (2015) excludes areas with zero population, and Datta and Sudhir (2013) remove areas that do not have any 'big box' stores, as these areas are very likely to be zoned for either residential use or small stores.

So far, all the papers that have estimated this type of model have considered a sample of cities (but not locations within a city) that is still in the spirit of the Bresnahan–Reiss isolated small markets approach. For instance, Seim selects US cities with population between 40 000 and 150 000, and without other cities with more than 25 000 people within 20 miles. The main reason for this is to avoid the possibility of outside competition at the boundaries of a city.

It is interesting that in the current generation of these applications, statistical inference is based on the number of cities and not on the number of locations. A relevant question is whether this model can be estimated consistently using data from a single city with many locations, that is, the estimator is consistent when the number of locations goes to infinity. This type of application can be motivated by the fact that city characteristics that are relevant for these models, such as the appropriate measure of geographic distance, transportation costs, or land use regulations and zoning, can

be city specific. Xu (2018) studies an empirical game of market entry for a single city (network) and presents conditions for consistency and asymptotic normality of estimators as the number of locations increases. As far as we know, there are not yet empirical applications following that approach.

5.3.3 Store level data

Most applications of models of entry in retail markets do not use data on prices and quantities due to the lack of such data. The most popular alternative is to estimate the structural (or semi-structural) parameters of the model using market entry data only, for example,Bresnahan and Reiss (1990), Mazzeo (2002), Seim (2006), or Jia (2008), among many others. Typically, these studies either do not try to separately identify variable profits from fixed costs, or they do it by assuming that the variable profit is proportional to an observable measure of market size. Data on prices and quantities at the store level can substantially help the identification of these models. In particular, it is possible to consider a richer specification of the model that distinguishes between demand, variable cost, and fixed cost parameters, and includes unobservable variables into each of these components of the model.

A sequential estimation approach is quite convenient for the estimation of this type of model. In a first step, data on prices and quantities at the store level can be used to estimate a spatial demand system as in Davis (2006) for movie theatres or Houde (2012) for gas stations. Note that, in contrast to standard applications of demand estimation of differentiated products, the estimation of demand models of this class should deal with the endogeneity of store locations. In other words, in these demand models, not only are prices endogenous, but also the set of products or stores available at each location, as they are potentially correlated with unobserved errors in the demand system. In a second step, variable costs can be estimated using firms' best response functions in a Bertrand or Cournot model. Finally, in a third step, we estimate fixed cost parameters using the entry game and information of firms' entry and store location decisions. It is important to emphasize that the estimation of a demand system of spatial differentiation in the first step provides the structure of spatial competition effects between stores at different locations, such that the researcher does not need to consider other types of semi-reduced form specifications of strategic interactions, as in Seim (2006) among others.

In some applications, price and quantity are not available, but there is information on revenue at the store level. This information can be used to estimate a (semi reduced form) variable profit function in a first step, and then in a second step the structure of fixed costs is estimated. This is the case in the applications in Ellickson and Misra 2012, Suzuki 2013), and Aguirregabiria, Clark, and Wang 2016.

5.3.4 Potential entrants

An important modelling decision in empirical entry games is to define the set of potential entrants. In most cases, researchers have limited information on the number of potential entrants, let alone their identity. This problem is particularly severe when entrants are mostly independent small stores (for example, mom-and-pop stores). A practical approach is to estimate the model under different numbers of potential entrants and examine how estimates are sensitive to these choices, for example, Seim (2006) and Jia (2008). The problem is less severe when most entrants belong to national chains (for

example, big box stores) because the names of these chains are often obvious and the number is typically small.

It is important to distinguish three types of data sets. The specification and the identification of the model is different for each of these three types of data.

(1) Only global potential entrants. The same N firms are the potential entrants in every market. We know the identity of these "global" potential entrants. Therefore, we observe the decision of each of these firms in every independent market. We observe market characteristics, and sometimes firm characteristics which may vary or not across markets. The data set is $\{s_m, x_{im}, a_{im} : m = 1,2,...,M;\ i = 1,2,...,N\}$ where m is the market index; i is the firm index; s_m is a vector of characteristics of market m such as market size, average consumer income, or other demographic variables; x_{im} is a vector of characteristics of firm i; and a_{im} is the indicator of the event "firm i is active in market m".

Examples. Berry (1992) considers entry in airline markets. A market is a city pair (for instance, Boston-Chicago). The set of markets consists of all the pairs of US cities with airports. Every airline company operating in the US is a potential entrant in each of these markets. a_{im} is the indicator of the event "airline i operates in city pair m". Toivanen and Waterson (2005) consider entry in local markets by fast-food restaurants in UK. Potential entrants are Burger King, McDonalds, KFC, Wendys, etc.

(2) Only local potential entrants. We do not know the identity of the potential entrants. In fact, most potential entrants may be local, that is, they consider entry in only one local market. For this type of data we only observe market characteristics and the number of active firms in the market. The data set is: $\{s_m, n_m : m = 1,2,...,M\}$ where n_m is the number of firms operating in market m. Notice also that we do not know the number of potential entrants N, and this may vary over markets.

Examples. Bresnahan and Reiss (1990). Car dealers in small towns. Bresnahan and Reiss (1991). Restaurants, dentists and other retailers and professional services in small towns. Seim (2006). Video rental stores.

(3) Both global and local potential entrants. This case combines and encompasses the previous two cases. There are N_G firms which are potential entrants in all the markets, and we now the identity of these firms. But there are also other potential entrants that are just local. We observe $\{s_m, n_m, z_{im}, a_{im} : m = 1,2,...,M;\ i = 1,2,...,N_G\}$.With this data we can nonparametrically identify $\Pr(n_m, a_m | x_m)$. We can allow for heterogeneity between global players in a very general way. Heterogeneity between local players should be much more restrictive.

5.4 Models

Road map.

(a) Bresnahan and Reiss. We start with a simple and pioneer model in this literature: the models in Bresnahan and Reiss (1991). This paper together with Bresnahan and Reiss (1990) were significant contributions to the structural estimation of models of market entry that opened a new literature that has grown significantly during the last 20 years. In their paper, Bresnahan and Reiss show that given a cross-section of "isolated"

local markets where we observe the number of firms active, and some exogenous market characteristics, including market size, it is possible to identify fixed costs and the "degree of competition" or the "nature of competition" in the industry. By "nature of competition", these authors (and after them, this literature) mean a measure of how a firm's variable profit declines with the number of competitors. What is most remarkable about Bresnahan and Reiss's result is how with quite limited information (for instance, no information about prices of quantities) the researcher can identify the degree of competition using an entry model.

(b) Relaxing the assumption of homogeneous firms. Bresnahan and Reiss's model is based on some important assumptions. In particular, firms are homogeneous and they have complete information. The assumption of firm homogeneity (both in demand and costs) is strong and can be clearly rejected in many industries. Perhaps more importantly, ignoring firm heterogeneity when it is in fact present can lead to biased and misleading results about the degree of competition in a industry. Therefore, the first assumption that we relax is the one of homogeneous firms.

As shown originally in the own work of Bresnahan and Reiss (1991), relaxing the assumption of firm homogeneity implies two significant econometric challenges. First, the entry model becomes a system of simultaneous equations with endogenous binary choice variables. Dealing with endogeneity in a binary choice system of equations is not a simple econometric problem. In general, IV estimators are not available. Furthermore, the model now has multiple equilibria. Dealing with both endogeneity and multiple equilibria in this class of nonlinear models is an interesting but challenging problem in econometrics.

(c) Dealing with endogeneity and multiple equilibria in games of complete information. We will go through different approaches that have been used in this literature to deal with the problems of endogeneity and multiple equilibria. It is worthwhile to distinguish two groups of approaches or methods.

The first group of methods is characterized by imposing restrictions that imply equilibrium uniqueness for any value of the exogenous variables. Of course, firm homogeneity is a type of assumption that implies equilibrium uniqueness. But there are other assumptions that imply uniqueness even when firms are heterogeneous. For instance, a triangular structure in the strategic interactions between firms (Heckman, 1978), or sequential entry decisions (Berry, 1992) imply equilibrium uniqueness. Given these assumptions, these papers deal with the endogeneity problem by using a maximum likelihood approach.

The second group of methods do not impose equilibrium uniqueness. The pioneering work by Jovanovic (1989) and Tamer (2003) were important contributions to this approach. These authors showed (Jovanovic for a general but stylized econometric model, and Tamer for a two-player binary choice game) that identification and multiple equilibria are very different issues in econometric models.

Models with multiple equilibria can be (point or set) identified, and we do not need to impose equilibrium uniqueness as a form to get identification. Multiple equilibria can be a computational nuisance in the estimation of these models, but it is not an identification problem. This simple idea has generated a significant and growing literature that deals with computationally simple methods to estimate models with multiple equilibria, and more specifically with the estimation of discrete games.

(d) Games of incomplete information. Our next step will be to relax the assumption of complete information by introducing some variables that are private information to each firm. We will see that the identification and estimation of these models can be significantly simpler than in the case of models of complete information.

5.4.1 Single- and Multi-store firms

Single- and Multi-store firms

We start with the description of a static entry game between single-store firms. Later, we extend this framework to incorporate dynamics and multi-store firms. There are N retail firms that are potential entrants in a market. We index firms by $i \in \{1,2,\ldots,N\}$. From a geographic point of view, the market is a compact set $\mathbb{C}$ in the Euclidean space $\mathbb{R}^2$, and it contains L locations where firms can operate stores. These locations are exogenously given and they are indexed by $\ell \in \{1,2,\ldots,L\}$.

Firms play a two-stage game. In the first stage, firms make their entry and store location decisions. Each firm decides whether to be active or not in the market, and if active, chooses the location of its store. We can represent a firm's decision using an L-dimensional vector of binary variables, $a_i \equiv \{a_{i\ell} : \ell = 1,2,\ldots,L\}$, where $a_{i\ell} \in \{0,1\}$ is the indicator of the event 'firm i has a store in location ℓ'. For single-store firms, there is at most one component in the vector a_i that is equal to one while the rest of the binary variables must be zero. In the second stage they compete in prices (or quantities) taking entry decisions as given. The equilibrium in the second stage determines equilibrium prices and quantities at each active store.

The market is populated by consumers. Each consumer is characterized by her preference for the products that firms sell and by her geographical location or home address h that belongs to the set of consumer home addresses $\{1,2,\ldots,H\}$. The set of consumer home addresses and the set of feasible business locations may be different. Following Smith (2004), Davis (2006), or Houde (2012), aggregate consumer demand comes from a discrete choice model of differentiated products where both product characteristics and transportation costs affect demand. For instance, in a spatial logit model, the demand for firm i with a store in location ℓ is:

$$q_{i\ell} = \sum_{h=1}^{H} M(h) \frac{a_{i\ell} \ \exp\{x_i \, \beta - \alpha \ p_{i\ell} - \tau(d_{h\ell})\}}{1 + \sum_{j=1}^{N} \sum_{\ell'=1}^{L} a_{j\ell'} \ \exp\{x_j \, \beta - \alpha \ p_{j\ell'} - \tau \ d_{h\ell'}\}} \tag{5.3}$$

where $q_{i\ell}$ and $p_{i\ell}$ are the quantity sold and the price, respectively, at store (i,ℓ); $M(h)$ represents the mass of consumers living in address h; the term within the square brackets is the market share of store (i,ℓ) among consumers living in address h; x_i is a vector of observable characteristics (other than price) of the product of firm i; and β is the vector of marginal utilities of these characteristics; α is the marginal utility of income; $d_{h\ell}$ represents the geographic distance between home address h and business location ℓ; and $\tau(d_{h\ell})$ is an increasing real-valued function that represents consumer transportation costs.

Given this demand system, active stores compete in prices à la Nash–Bertrand to maximize their respective variable profits, $(p_{i\ell} - c_{i\ell}) \ q_{i\ell}$, where $c_{i\ell}$ is the marginal cost of store (i,ℓ), that is exogenously given. The solution of the system of best response functions can be described as a vector of equilibrium prices for each active firm/store.

Let $p_i^*(\ell, a_{-i}, x)$ and $q_i^*(\ell, a_{-i}, x)$ represent the equilibrium price and quantity for firm i given that this firm has a store at location ℓ. The rest of the firms' entry/location decisions are represented by the vector $a_{-i} \equiv \{a_j : j \neq i\}$, and the firms' characteristics are denoted by $x \equiv (x_1, x_2, \ldots, x_N)$. Similarly, we can define the equilibrium (indirect) variable profit,

$$VP_i^*(\ell, a_{-i}, x) = [p(\ell, a_{-i}, x) - c_{i\ell}]\, q_i^*(\ell, a_{-i}, x) \tag{5.4}$$

Consider now the entry stage of the game. The profit of firm i if it has a store in location ℓ is:

$$\pi_i(\ell, a_{-i}, x) = VP_i^*(\ell, a_{-i}, x) - EC_{i\ell} \tag{5.5}$$

where $EC_{i\ell}$ represents the entry cost of firm i at location ℓ, that for the moment is exogenously given. The profit of a firm that is not active in the market is normalized to zero, that is, $\pi_i(0, a_{-i}, x) = 0$, where with some abuse of notation, we use $\ell = 0$ to represent the choice alternative of no entry in any of the L locations.

The description of an equilibrium in this model depends on whether firms have complete or incomplete information about other firms' costs. The empirical literature on entry games has considered both cases.

Complete information game. In the complete information model, a Nash equilibrium is an N-tuple $\{a_i^* : i = 1, 2, \ldots, N\}$ such that for every firm i the following best response condition is satisfied:

$$a_{i\ell}^* = 1\{\pi_i(\ell, a_{-i}^*, x) \geq \pi_i(\ell', a_{-i}^*, x) \text{ for any } \ell' \neq \ell\} \tag{5.6}$$

where $1\{.\}$ is the indicator function. In equilibrium, each firm is maximizing its own profit given the entry and location decisions of the other firms.

Incomplete information game. In a game of incomplete information, there is a component of a firm's profit that is private information to the firm. For instance, suppose that the entry cost of firm i is $EC_{i\ell} = ec_{i\ell} + \varepsilon_{i\ell}$, where $ec_{i\ell}$ is public information for all the firms, and $\varepsilon_{i\ell}$ is private information to firm i. These private cost shocks can be correlated across locations for a given firm, but they are independently distributed across firms, that is, $\varepsilon_i \equiv \{\varepsilon_{i\ell} : \ell = 1, 2, \ldots, L\}$ is independently distributed across firms with a distribution function F_i that is continuously differentiable over $\mathbb{R}^L$ and common knowledge to all the firms.

A firm's strategy is an L-dimensional mapping $\alpha_i(\varepsilon_i; x) \equiv \{\alpha_{i\ell}(\varepsilon_i; x) : \ell = 1, 2, \ldots, L\}$, where $\alpha_{i\ell}(\varepsilon_i; x)$ is a binary-valued function from the set of possible private information values $\mathbb{R}^L$ and the support of x into $\{0, 1\}$, such that $\alpha_{i\ell}(\varepsilon_i; x) = 1$ means that firm i enters location ℓ when the value of private information is ε_i. A firm has uncertainty about the actual entry decisions of other firms because it does not know the realization of other firms' private information. Therefore, firms maximize expected profits. Let $\pi_i^e(\ell, \alpha_{-i}, x)$ be the expected profit of firm i if it has a store at location ℓ and the other firms follow their respective strategies in α_{-i}^*. By definition, $\pi_i^e(\ell, \alpha_{-i}, x) \equiv \mathbb{E}_{\varepsilon_{-i}}[\pi_i(\ell, \alpha_{-i}(\varepsilon_{-i}; x), x)]$, where $\mathbb{E}_{\varepsilon_{-i}}$ represents the expectation over the distribution of the private information of firms other than i. A Bayesian Nash equilibrium in this game of incomplete information is an N-tuple of strategy functions $\{\alpha_{-i}^* : i = 1, 2, \ldots, N\}$ such that every firm maximizes its expected profit: for any ε_i,

$$\alpha_{i\ell}^*(\varepsilon_i; x) = 1\{\pi_i^e(\ell, \alpha_{-i}^*, x) \geq \pi_i^e(\ell', \alpha_{-i}^*, x) \text{ for any } \ell' \neq \ell\} \tag{5.7}$$

In an entry game of incomplete information, firms' strategies (and therefore, a Bayesian Nash equilibrium) can also be described using firms' probabilities of market entry, instead of the strategy functions $\alpha_i(\varepsilon_i; x)$. In sections 2.2.1 and 2.2.4, we present examples of this representation in the context of more specific models.

Multi-store firms

Multi-store firms, or retail chains, have become prominent in many retail industries such as supermarkets, department stores, apparel, electronics, fast food restaurants, or coffee shops, among others. Cannibalization and economies of scope between stores of the same chain are two important factors in the entry and location decisions of a multi-store firm. The term cannibalization refers to the business stealing effects between stores of the same chain. Economies of scope may appear if some operating costs are shared between stores of the same retail chain such that these costs are not duplicated when the number of stores in the chain increases. For instance, some advertising, inventory, personnel, or distribution costs can be shared among the stores of the same firm. These economies of scope may become quantitatively more important when store locations are geographically closer to each other. This type of economies of scope is called economies of density.

The recent empirical literature on retail chains has emphasized the importance of these economies of density, that is, Holmes (2011), Jia (2008), Ellickson, Houghton, and Timmins (2013), and Nishida (2015). For instance, the transportation cost associated with the distribution of products from wholesalers to retail stores can be smaller if stores are close to each other. Also, geographic proximity can facilitate sharing inventories and even personnel across stores of the same chain. We now present an extension of the basic framework that accounts for multi-store firms.

A multi-store firm decides its number of stores and their locations. We can represent a firm's entry decision using the L-dimensional vector $a_i \equiv \{a_{i\ell} : \ell = 1, 2, \ldots, L\}$, where $a_{i\ell} \in \{0, 1\}$ is still the indicator of the event 'firm i has a store in location ℓ'. In contrast to the case with single-store firms, now the vector a_i can take any value within the choice set $\{0, 1\}^L$. The demand system still can be described using equation (5.4.1). The variable profit of a firm is the sum of variable profits over every location where the firm has stores, $\sum_{\ell=1}^{L} a_{i\ell} \, (p_{i\ell} - c_{i\ell}) q_{i\ell}$.

Firms compete in prices taking their store locations as given. A retail chain may choose to have a uniform price across all its stores, or to charge a different price at each store. In the Bertrand pricing game with spatial price discrimination (that is, different prices at each store), the best response of firm i can be characterized by the first-order conditions:

$$q_{i\ell} + (p_{i\ell} - c_{i\ell}) \frac{\partial q_{i\ell}}{\partial p_{i\ell}} + \sum_{\ell' \neq \ell} (p_{i\ell'} - c_{i\ell'}) \frac{\partial q_{i\ell'}}{\partial p_{i\ell}} = 0 \tag{5.8}$$

The first two terms represent the standard marginal profit of a single-store firm. The last term represents the effect on the variable profits of all other stores within the firm, and it captures how the pricing decision of the firm internalizes the cannibalization effect among its own stores.

A Nash-Bertrand equilibrium is a solution in prices to the system of best response

equations in (5.4.1). The equilibrium (indirect) variable profit of firm i is:

$$VP_i^*(a_i, a_{-i}; x) = \sum_{\ell=1}^{L} (p_i^*(\ell, a_{-i}; x) - c_{i\ell})\, q_i^*(\ell, a_{-i}; x) \tag{5.9}$$

where $p_{i\ell}^*(\ell, a_{-i}; x)$ and $q_i^*(\ell, a_{-i}; x)$ represent Bertrand equilibrium prices and quantities, respectively.

The total profit of the retail chain is equal to total variable profit minus total entry cost: $\pi_i(a_i, a_{-i}; x) = VP_i^*(a_i, a_{-i}; x) - EC_i(a_i)$. The entry costs of a retail chain may depend on the number of stores (that is, (dis)economies of scale) and on the distance between the stores (for example, economies of density). In section 2.2.5, we provide examples of specifications of entry costs for multi-store retailers.

The description of an equilibrium in this game of entry between retail chains is similar to the game between single-store firms. With complete information, a Nash equilibrium is an N-tuple $\{a_i^* : i = 1, 2, \ldots, N\}$ that satisfies the following best response conditions:

$$\pi_i(a_i^*, a_{-i}^*; x) \geq \pi_i(a_i, a_{-i}^*; x) \text{ for any } a_i \neq a_i^* \tag{5.10}$$

With incomplete information, a Bayesian Nash equilibrium is an N-tuple of strategy functions $\{\alpha_i^*(\varepsilon_i; x) : i = 1, 2, \ldots, N\}$ such that every firm maximizes its expected profit: for any ε_i:

$$\pi_i^e(\alpha_i^*(\varepsilon_i; x), \alpha_{-i}^*, x) \geq \pi_i^e(a_i, \alpha_{-i}^*, x) \quad \text{for any } a_i \neq \alpha_i^*(\varepsilon_i; x) \tag{5.11}$$

Specification assumptions

The games of entry in retail markets that have been estimated in empirical applications have imposed different types of restrictions on the framework that we have presented above. For example, restrictions on firm and market heterogeneity, firms' information, spatial competition, multi-store firms, dynamics, or the form of the structural functions.

The motivations for these restrictions are diverse. Some restrictions are imposed to achieve identification or precise enough estimates of the parameters of interest, given the researcher's limited information on the characteristics of markets and firms. For instance, as we describe in section 5.3.3, prices and quantities at the store level are typically not observable to the researcher, and most sample information comes from firms' entry decisions. These limitations in the available data have motivated researchers to use simple specifications for the indirect variable profit function.

Other restrictions are imposed for computational convenience in the solution and estimation of the model, for example, to obtain closed form solutions, to guarantee equilibrium uniqueness as it facilitates the estimation of the model, or to reduce the dimensionality of the space of firms' actions or states. In this subsection, we review some important models in this literature and discuss their main identification assumptions. We have organized these models in an approximate chronological order.

5.4.2 Homogeneous firms

Work in this field was pioneered by Bresnahan and Reiss. In Bresnahan and Reiss (1991), they study several retail and professional industries in the US, specifically pharmacies, tire dealers, doctors, and dentists. The main purpose of the paper is

to estimate the 'nature' or 'degree' of competition for each of the industries: how fast variable profits decline when the number of firms in the market increases. More specifically, the authors are interested in estimating how many entrants are needed to achieve an oligopoly equilibrium equivalent to the competitive equilibrium, that is, the hypothesis of contestable markets (Baumol 1982; Baumol, Panzar, and Willig 1982; Martin 2000).

For each industry, their dataset consists of a cross-section of M small 'isolated markets'. In section 5.3, we discussed the empirical motivation and implementation of the 'isolated markets' restriction. For the purpose of the model, a key aspect of this restriction is that the M local markets are independent in terms of demand and competition such that the equilibrium in one market is independent of the one in the other markets. The model also assumes that each market consists of a single location, that is, $L = 1$, such that spatial competition is not explicitly incorporated in the model. For each local market, the researcher observes the number of active firms (n), a measure of market size (s), and some exogenous market characteristics that may affect demand and/or costs (x).

Given this limited information, the researcher needs to restrict firm heterogeneity. Bresnahan and Reiss propose a static game between single-store firms where all the potential entrants in a market are identical and have complete information on demand and costs. The profit of a store is:

$$\pi(n) = s\,V(x,n) \;-\; EC(x) \;-\varepsilon, \tag{5.12}$$

where $V(x,n)$ represents variable profit per capita (per consumer) that depends on the number of active firms n, and $EC(x)+\varepsilon$ is the entry cost, where ε is unobservable to the researcher. The form of competition between active firms is not explicitly modelled. Instead, the authors consider a flexible specification of the variable profit per-capita that is strictly decreasing but nonparametric in the number of active stores. Therefore, the specification is consistent with a general model of competition between homogeneous firms, or even between symmetrically differentiated firms.

Given these assumptions, the equilibrium in a local market can be described as a number of firms n^* that satisfies two conditions: (1) every active firm is maximizing profits by being active in the market, that is, $\pi(n^*) \geq 0$; and (2) every inactive firm is maximizing profits by being out of the market, that is, $\pi(n^*+1) < 0$. In other words, every firm is making its best response given the actions of the others. Since the profit function is strictly decreasing in the number of active firms, the equilibrium is unique and it can be represented using the following expression: for any value $n \in \{0,1,2,\ldots\}$,

$$\begin{aligned} \{n^* = n\} \quad &\Leftrightarrow \quad \{\pi(n) \geq 0 \;\text{ and }\; \pi(n+1) < 0\} \\ &\Leftrightarrow \quad \{s\,V(x,n+1) - EC(x) < \varepsilon \leq\; s\,V(x,n) - EC(x)\} \end{aligned} \tag{5.13}$$

Also, this condition implies that the distribution of the equilibrium number of firms given exogenous market characteristics is:

$$\Pr(n^* = n \mid s,x) = F(s\,V(x,n) - EC(x)) - F(s\,V(x,n+1) - EC(x)) \tag{5.14}$$

where $F$is the CDF of ε. This representation of the equilibrium as an ordered discrete choice model is convenient for estimation.

In the absence of price and quantity data, the separate identification of the variable profit function and the entry cost function is based on the exclusion restrictions that variable profit depends on market size and on the number of active firms while the entry cost does not depend on these variables.

Private information. The previous model can be slightly modified to allow for firms' private information. This variant of the original model maintains the property of equilibrium uniqueness and most of the simplicity of the previous model. Suppose that now the entry cost of a firm is $EC(x) + \varepsilon_i$, where ε_i is private information of firm i and it is independently and identically distributed across firms with a CDF F. There are N potential entrants in the local market. The presence of private information implies that, when potential entrants make entry decisions, they do not know ex ante the actual number of firms that will be active in the market. Instead, each firm has beliefs about the probability distribution of the number of other firms that are active. We represent these beliefs, for say firm i, using the function $G_i(n) \equiv \Pr(n^*_{-i} = n|s,x)$, where n^*_{-i} represents the number of firms other than i that are active in the market. Then, the expected profit of a firm if active in the market is:

$$\pi_i^e = \left[\sum_{n=0}^{N-1} G_i(n)\ s\ V(x,n+1)\right] - EC(x) - \varepsilon_i \tag{5.15}$$

The best response of a firm is to be active in the market if and only if its expected profit is positive or zero, that is, $a_i = 1\{\pi_i^e \geq 0\}$. Integrating this best response function over the distribution of the private information ε_i we obtain the best response probability of being active for firm i, that is:

$$P_i \equiv F\Big(\left[\sum_{n=0}^{N-1} G_i(n)\ s\ V(x,n+1)\right] - EC(x)\Big) \tag{5.16}$$

Since all firms are identical, up to their independent private information, it seems reasonable to impose the restriction that in equilibrium they all have the same beliefs and, therefore, the same best response probability of entry. Therefore, in equilibrium, firms' entry decisions $\{a_i\}$ are independent Bernoulli random variables with probability P, and the number of active firms other than i in the market has a Binomial distribution with argument $(N-1,P)$ such that $\Pr(n^*_{-i} = n) = B(n|N-1,P)$.

In equilibrium, the belief function $G(n)$ should be consistent with firms' best response probability P. Therefore, a Bayesian Nash Equilibrium in this model can be described as a probability of market entry P^*, which is the best response probability when firms' beliefs about the distribution of other firms active in the market are $G(n) = B(n|N-1,P^*)$. We can represent this equilibrium condition using the following equation:

$$P^* = F\left(\left[\textstyle\sum_{n=0}^{N-1} B(n|N-1,P^*)\ s\ V(x,n+1)\right] - EC(x)\right) \tag{5.17}$$

When the variable profit $V(x,n)$ is a decreasing function in the number of active stores, the right-hand side in equation (5.17) is also a decreasing function in the probability of entry P, and this implies equilibrium uniqueness. In contrast to the complete information model in Bresnahan and Reiss (1991), this incomplete information model

does not have a closed form solution for the equilibrium distribution of the number of active firms in the market. However, the numerical solution to the fixed point problem in equation (5.17) is computationally very simple, and so are the estimation and comparative statistics using this model.

Given that the only difference between the two models described in this section is in their assumptions about firms' information, it seems reasonable to consider whether these models are observationally different or not. In other words, does the assumption on complete versus incomplete information have implications on the model predictions about competition? Grieco (2014) investigates this question in the context of an empirical application to local grocery markets. In Grieco's model, firms are heterogeneous in terms of (common knowledge) observable variables, and this observable heterogeneity plays a key role in his approach to empirically distinguish between firms' public and private information. Note that the comparison of equilibrium conditions in equations (5.14) and (5.17) shows other testable difference between the two models. In the game of incomplete information, the number of potential entrants N has an effect on the whole probability distribution of the number of active firms: a larger number of potential entrants implies a shift to the right in the whole distribution of the number of active firms. In contrast, in the game of complete information, the value of N affects only the probability $Pr(n^* = N|s,x)$ but not the distribution of the number of active firms at values smaller than N. This empirical prediction has relevant economic implications: with incomplete information, the number of potential entrants has a positive effect on competition even in markets where this number is not binding.

Bresnahan and Reiss (1991)

The authors study several retail and professional industries in the US: Doctors; Dentists; Pharmacies; Plumbers; car dealers; etc. For each industry, the dataset consists of a cross-section of M small, "isolated" markets. We index markets by m. For each market m, we observe the number of active firms (n_m), a measure of market size (s_m), and some exogenous market characteristics that may affect demand and/or costs (x_m).

$$\text{Data} = \{\, n_m,\ s_m, x_m : m = 1,2,...,M\} \tag{5.18}$$

There are several empirical questions that they wish to answer. First, they want to estimate the "nature" or "degree" of competition for each of the industries: that is, how fast variable profits decline when the number of firms in the market increase. Second, but related to the estimation of the degree of competition, BR are also interested in estimating how many entrants are needed to achieve an equilibrium equivalent to the competitive equilibrium, that is, hypothesis of contestable markets.

Model. Consider a market m. There is a number N of potential entrants in the market. Each firm decides whether to be active or not in the market. Let $\Pi_m(n)$ be the profit of an active firm in market m when there are n active firms. The function $\Pi_m(n)$ is strictly decreasing in n. If n_m is the equilibrium number of firms in market m, then it should satisfy the following conditions:

$$\Pi_m(n_m) \geq 0 \qquad \text{and} \qquad \Pi_m(n_m + 1) < 0 \tag{5.19}$$

That is, every firm is making her best response given the actions of the others. For active firms, their best response is to be active, and for inactive firms their best response is to not enter in the market.

To complete the model we have to specify the structure of the profit function $\Pi_m(n)$. Total profit is equal to variable profit, $V_m(n)$, minus fixed costs, $F_m(n)$:

$$\Pi_m(n) = V_m(n) - F_m(n) \tag{5.20}$$

In this model, where we do not observe prices or quantities, the key difference in the specification of variable profit and fixed cost is that variables profits increase with market size (in fact, they are proportional to market size) and fixed costs do not.

The variable profit of a firm in market m when there are n active firms is:

$$V_m(n) = s_m \, v_m(n) = s_m \, \left(x_m^D \beta - \alpha(n) \right) \tag{5.21}$$

where s_m represents market size; $v_m(n)$ is the variable profit per-capita; x_m^D is a vector of market characteristics that may affect the demand of the product, for instance, per capita income, age distribution; β is a vector of parameters; and $\alpha(1)$, $\alpha(2)$, ...$\alpha(N)$ are parameters that capture the degree of competition, such that we expect that $\alpha(1) \leq \alpha(2) \leq \alpha(3) \ldots \leq \alpha(N)$. Given that there is no firm-heterogeneity in the variable profit function, there is an implicit assumption of homogeneous product or symmetrically differentiated product as in, for instance, Salop circle city (Salop, 1979).

The specification for the fixed cost is:

$$F_m(n) = x_m^C \, \gamma + \delta(n) + \varepsilon_m \tag{5.22}$$

where x_m^C is a vector of observable market characteristics that may affect the fixed cost, for instance, rental price; ε_m is a market characteristic that is unobservable to the researchers but observable to the firms; $\delta(1)$, $\delta(2)$, ...$\delta(N)$ are parameters. The dependence of the fixed cost with respect to the number of firms is very unconventional or non-standard in IO. Bresnahan and Reiss allow for this possibility and provide several interpretations. However, the interpretation of the parameters $\delta(n)$ is not completely clear. In some sense, BR allow the fixed cost to depend on the number firms in the market for robustness reasons. There are several possible interpretations for why fixed costs may depend on the number of firms in the market: (a) entry deterrence: incumbents create barriers to entry; (b) a shortcut to allow for firm heterogeneity in fixed costs, in the sense that late entrants are less efficient in fixed costs; and (c) actual endogenous fixed costs, for instance rental prices or other components of the fixed costs, not included in x_m^C, may increase with the number of incumbents (for instance, demand effect on rental prices). For any of these interpretations we expect $\delta(n)$ to be an increasing function of n.

Since both $\alpha(n)$ and $\delta(n)$ increase with n, it is clear that the profit function $\Pi_m(n)$ declines with n. Therefore, as we anticipated above, the equilibrium condition for the number of firms in the market can be represented as follows. For $n \in \{0, 1, ..., N\}$

$$\{n_m = n\} \Leftrightarrow \{ \, \Pi_m(n) \geq 0 \quad \text{AND} \quad \Pi_m(n+1) < 0 \, \} \tag{5.23}$$

It is simple to show that the model has a unique equilibrium for any value of the exogenous variables and structural parameters. This is just a direct implication of the strict monotonicity of the profit function $\Pi_m(n)$.

We have a random sample $\{n_m, s_m, x_m^D, x_m^C : m = 1,2,...,M\}$ and we want to use this sample to estimate the vector of parameters:

$$\theta = \{\beta,\ \gamma,\ \sigma,\ \alpha(1),...,\alpha(N),\ \delta(1),...,\delta(N)\} \tag{5.24}$$

The unobserved component of the entry cost, ε_m, is assumed independent of (s_m, x_m^D, x_m^C) and it is i.i.d. over markets with distribution $N(0,\sigma)$. As usual in discrete choice models, σ is not identified. We normalize $\sigma = 1$, which means that we are really identifying the rest of the parameters up to scale. We should keep this in mind for the interpretation of the estimation results.

Given this model and sample, BR estimate θ by maximum likelihood:

$$\hat{\theta} = \arg\max_{\theta} \sum_{m=1}^{M} \log \Pr(n_m \mid \theta, s_m, x_m^D, x_m^C) \tag{5.25}$$

What is the form of the probabilities $\Pr(n_m|\theta, s_m, x_m)$ in the BR model? This entry model has the structure of an *ordered Probit model*. We can represent the equilibrium condition $\{\Pi_m(n) \geq 0 \ \text{ AND } \ \Pi_m(n+1) < 0\}$ in terms of thresholds for the unobservable variable ε_m:

$$\{n_m = n\} \Leftrightarrow \{T_m(n+1) < \varepsilon_m \leq T_m(n)\}, \tag{5.26}$$

where, for any $n \in \{1,2,...,N\}$,

$$T_m(N) \equiv s_m x_m^D \beta - x_m^C \gamma - \alpha(n) s_m - \delta(n) \tag{5.27}$$

and $T_m(0) = +\infty$, $T_m(N^*+1) = -\infty$. This is the structure of an ordered probit model. Therefore, the distribution of the number of firms conditional on the observed exogenous market characteristics is:

$$\begin{aligned} \Pr(n_m = n|s_m, x_m) &= \Phi\left(T_m(n)\right) - \Phi\left(T_m(n+1)\right) \\ &= \Phi\left(\ s_m x_m^D \beta - x_m^C \gamma - \alpha(n) s_m - \delta(n)\ \right) \\ &- \Phi\left(\ s_m x_m^D \beta - x_m^C \gamma - \alpha(n+1) s_m - \delta(n+1)\ \right) \end{aligned} \tag{5.28}$$

This model is simple to estimate and most econometric software packages include a command for the estimation of the ordered probit.

Data. The dataset consists of a cross-section of 202 "isolated" local markets. Why isolated local markets? It is very important to include in our definition of market all the firms that are actually competing in the market and not more or less. Otherwise, we can introduce significant biases in the estimated parameters. If our definition of market is too narrow, such that we do not include all the firms that are actually in a market, we will conclude that there is little entry either because fixed costs are too large or the degree of competition is strong: that is, we will overestimate the $\alpha's$ or the $\delta's$ or both. If our definition of market is too broad, such that we include firms that are not actually competing in the same market, we will conclude that there is significant entry and to rationalize this we will need fixed costs to be small or to have a low degree of competition between firms. Therefore, we will underestimate the $\alpha's$ or the $\delta's$ or both.

Under a broad definition of a market, the most common mistake is having a large city as a single market. Conversely, under a narrow definition of a market, the most common mistake is having small towns that are close to each other, or close to a large town, as single markets. To avoid these type of errors, BR construct "isolated local markets". The criteria to select isolated markets in the US are: (a) at least 20 miles from the nearest town of 1000 people or more; (b) at least 100 miles from cities with 100,000 people or more.

Empirical results. Let $S(n)$ be the minimum market size to sustain n firms in the market. $S(n)$ are called *market dize entry thresholds* and they can be obtained using the estimated parameters of the model. They do not depend on the normalization $\sigma = 1$. Brenahan and Reiss find that, for most industries, both $\alpha(n)$ and $\delta(n)$ increase with n. There are very significant cross-industry differences in entry thresholds $S(n)$. For most of the industries, entry thresholds $S(n)/N$ become constant for values of n greater than 4 or 5. This result supports the hypothesis of contestable markets (Baumol, 1982).

5.4.3 Endogenous product choice

Mazzeo (2002) studies market entry in the motel industry using local markets along US interstate highways. A local market is defined as a narrow region around a highway exit. Mazzeo's model maintains most of the assumptions in Bresnahan and Reiss (1991), such as no spatial competition (that is, L=1), ex ante homogeneous firms, complete information, no multi-store firms, and no dynamics. However, he extends the Bresnahan–Reiss model in an interesting dimension: he introduces endogenous product differentiation.

More specifically, firms not only decide whether to enter in a market but they also choose the type of product to offer: low-quality product E (that is, economy hotel), or high-quality product H (that is, upscale hotel). Product differentiation makes competition less intense, and it can increase firms' profits. However, firms also have an incentive to offer the type of product for which demand is stronger.

The profit of an active hotel of type $T \in \{E,H\}$ is:

$$\pi_T(n_E,n_H) = s\, V_T(x,n_E,n_H) - EC_T(x) - \varepsilon_T \tag{5.29}$$

where n_E and n_H represent the number of active hotels with low and high quality, respectively, in the local market. Similarly to the Bresnahan–Reiss model, V_T is the variable profit per capita and $EC_T(x) + \varepsilon_T$ is the entry cost for type T hotels, where ε_T is unobservable to the researcher.

Mazzeo solves and estimates his model under two different equilibrium concepts: Stackelberg and what he terms a 'two-stage game'. A computational advantage of the two-stage game is that under the assumptions of the model the equilibrium is unique. In the first stage, the total number of active hotels, $n \equiv n_E + n_H$, is determined in a similar way as in the Bresnahan–Reiss model. Hotels enter the market as long as there is some configuration (n_E,n_H) where both low-quality and high-quality hotels make positive profits. Define the first-stage profit function as:

$$\Pi(n) \equiv \max_{n_E,n_H : n_E+n_H=n} \min[\pi_E(n_E,n_H)\ ,\ \pi_H(n_E,n_H)] \tag{5.30}$$

Then, the equilibrium number of hotels in the first stage is the value n^* that satisfies two conditions: (1) every active firm wants to be in the market, that is, $\Pi(n^*) \geq 0$; and

(2) every inactive firm prefers to be out of the market, that is, $\Pi(n^*+1) < 0$. If the profit functions π_E and π_H are strictly decreasing functions in the number of active firms (n_E, n_H), then $\Pi(n)$ is also a strictly decreasing function, and the equilibrium number of stores in the first stage, n^*, is unique.

In the second stage, active hotels choose simultaneously their type or quality level. In this second stage, an equilibrium is a pair (n_E^*, n_H^*) such that every firm chooses the type that maximizes its profit given the choices of the other firms. That is, low quality firms are not better off by switching to high quality, and vice versa:

$$\begin{aligned} \pi_E(n_E^*, n_H^*) &\geq \pi_H(n_E^* - 1, n_H^* + 1) \\ \pi_H(n_E^*, n_E^*) &\geq \pi_E(n_E^* + 1, n_E^* - 1) \end{aligned} \tag{5.31}$$

Mazzeo shows that the equilibrium pair (n_E^*, n_H^*) in this second stage is also unique.

Using these equilibrium conditions, it is possible to obtain a closed form expression for the (quadrangle) region in the space of the unobservables $(\varepsilon_E, \varepsilon_H)$ that generate a particular value of the equilibrium pair (n_E^*, n_H^*). Let $R_\varepsilon(n_E, n_H; s, x)$ be the quadrangle region in $\mathbb{R}^2$ associated with the pair (n_E, n_H) given exogenous market characteristics (s, x) , and let $F(\varepsilon_E, \varepsilon_H)$ be the CDF of the unobservable variables. Then, we have that:

$$\Pr(n_E^* = n_E, n_H^* = n_H | s, x) = \int 1\{(\varepsilon_E, \varepsilon_H) \in R_\varepsilon(n_E, n_H; s, x)\, dF(\varepsilon_E, \varepsilon_H) \tag{5.32}$$

In the empirical application, Mazzeo finds that hotels have strong incentives to differentiate from their rivals to avoid nose-to-nose competition.

Ellickson and Misra (2008) estimate a game of incomplete information for the US supermarket industry where supermarkets choose the type of 'pricing strategy': 'everyday low price' (EDLP) versus 'high-low' pricing. The choice of pricing strategy can be seen as a form of horizontal product differentiation. The authors find evidence of strategic complementarity between supermarkets' pricing strategies: firms competing in the same market tend to adopt the same pricing strategy not only because they face the same type of consumers but also because there are positive synergies in the adoption of the same strategy.

From an empirical point of view, Ellickson and Misra's result is more controversial than Mazzeo's finding of firms' incentives to differentiate from each other. In particular, the existence of unobservables that are positively correlated across firms but are not fully accounted for in the econometric model, may generate a spurious estimate of positive spillovers in the adoption of the same strategy.

Vitorino (2012) estimates a game of store entry in shopping centers that allows for incomplete information, positive spillover effects among stores, and also unobserved market heterogeneity for the researcher that is common knowledge to firms. Her empirical results show that, after controlling for unobserved market heterogeneity, firms face business stealing effects but also significant incentives to collocate, and that the relative magnitude of these two effects varies substantially across store types.

5.4.4 Firm heterogeneity

The assumption that all potential entrants and incumbents are homogeneous in their variable profits and entry costs is very convenient and facilitates the estimation, but it

is also very unrealistic in many applications. A potentially very important factor in the determination of market structure is that firms, potential entrants, are ex-ante heterogeneous. In many applications we want to take into account this heterogeneity. Allowing for firm heterogeneity introduces two important issues in these models: endogenous explanatory variables, and multiple equilibria. We will comment on different approaches that have been used to deal with these issues.

Consider an industry with N potential entrants. For instance, the airline industry. These potential entrants decide whether to be active or not in a market. We observe M different realizations of this entry game. These realizations can be different geographic markets (different routes or city pairs, for instance, Toronto-New York, Montreal-Washington, etc) or different periods of time. We index firms with $i \in \{1,2,...,N\}$ and markets with $m \in \{1,2,...,M\}$.

Let $a_{im} \in \{0,1\}$ be the binary indicator of the event "firm i is active in market m". For a given market m, the N firms choose simultaneously whether to be active or not in the market. When making its decision, a firm wants to maximize its profit.

Once firms have decided whether to be active or not in the market, active firms compete in prices or in quantities and firms' profits are realized. For the moment, we do not make explicit the specific form of competition in this second part of the game, or the structure of demand and variable costs. We take as given an "indirect profit function" that depends on exogenous market and firm characteristics and on the number and the identity of the active firms in the market. This indirect profit function comes from a model of price or quantity competition, but at this point we do not make that model explicit here. Also, we consider that the researcher does not have access to data on firms' prices and quantities such that demand and variable cost parameters in the profit function cannot be estimated from demand, and/or Bertrand/Cournot best response functions.

The (indirect) profit function of an incumbent firm depends on market and firm characteristics affecting demand and costs, and on the entry decisions of the other potential entrants:

$$\Pi_{im} = \begin{cases} \Pi_i\left(x_{im},\ \varepsilon_{im}, a_{-im}\right) & \text{if} \quad a_{im} = 1 \\ \\ 0 & \text{if} \quad a_{im} = 0 \end{cases} \tag{5.33}$$

where x_{im} and ε_{im} are vectors of exogenous market and firm characteristics, and $a_{-im} \equiv \{a_{jm} : j \neq i\}$. The vector x_{im} is observable to the researcher while ε_{im} is unobserved to the researcher. For the moment we assume that $x_m \equiv \{x_{1m}, x_{2m}, ..., x_{Nm}\}$ and $\varepsilon_m \equiv \{\varepsilon_{1m}, \varepsilon_{2m}, ..., \varepsilon_{Nm}\}$ are common knowledge for all players.

For instance, in the example of the airline industry, the vector x_{im} may include market characteristics such as population and socioeconomic characteristics in the two cities that affect demand, characteristics of the airports such as measures of congestion (that affect costs), and firm characteristics such as the number of other connections that the airline has in the two airports (that affect operating costs due to economies of scale and scope).

The N firms choose simultaneously $\{a_{1m}, a_{2m}, ..., a_{Nm}\}$ and the assumptions of Nash equilibrium hold. A Nash equilibrium in this entry game is an N-tuple $a_m^* = (a_{1m}^*, a_{2m}^*, ..., a_{Nm}^*)$ such that for any player i:

$$a_{im}^* = 1\left\{\ \Pi_i\left(x_{im},\ \varepsilon_{im}, a_{-im}^*\right) \geq 0\ \right\} \tag{5.34}$$

where $1\{.\}$ is the indicator function.

Given a dataset with information on $\{a_{im}, x_{im}\}$ for every firm in the M markets, we want to use this model to learn about the structure of the profit function Π_i. In these applications, we are particularly interested in the effect of other firms' entry decisions on a firm's profit. For instance, how Southwest's entry in the Chicago-Boston market affectsthe profit of American Airlines.

For the sake of concreteness, consider the following specification of the profit function:

$$\Pi_{im} = x_{im}\,\beta_i - \sum_{j \neq i} a_{jm}\,\delta_{ij} + \varepsilon_{im} \tag{5.35}$$

where x_{im} is a $1 \times K$ vector of observable market and firm characteristics; β_i is a $K \times 1$ vector of parameters; $\delta_i = \{\delta_{ij} : j \neq i\}$ is a $(N-1) \times 1$ vector of parameters, with δ_{ij} being the effect of firm $j's$ entry on firm $i's$ profit; ε_{im} is a zero mean random variable that is observable to the players but unobservable to the econometrician.

We assume that ε_{im} is independent of x_m, and it is $i.i.d.$ over m, and independent across i. If x_{im} includes a constant term, then without loss of generality $\mathbb{E}(\varepsilon_{im}) = 0$. Define $\sigma_i^2 \equiv Var(\varepsilon_{im})$. Then, we also assume that the probability distribution of $\varepsilon_{im}/\sigma_i$ is known to the researcher. For instance, $\varepsilon_{im}/\sigma_i$ has a standard normal distribution.

The econometric model can be described as a system of N simultaneous equations where the endogenous variables are the entry dummy variables:

$$a_{im} = 1\left\{ x_{im}\,\beta_i - \sum_{j \neq i} a_{jm}\,\delta_{ij} + \varepsilon_{im} \geq 0 \right\} \tag{5.36}$$

We want to estimate the vector of parameters $\theta = \{\beta_i/\sigma_i, \delta_i/\sigma_i : i = 1, 2, ..., N\}$.

There are two main econometric issues in the estimation of this model: (1) endogenous explanatory variables, a_{jm}; and (2) multiple equilibria.

Endogeneity of other players' actions

In the system of structural equations in (5.36), the actions of the other players, $\{a_{jm} : j \neq i\}$ are endogenous in an econometric sense. That is, a_{jm} is correlated with the unobserved term ε_{im}, and ignoring this correlation can lead to serious biases in our estimates of the parameters β_i and δ_i.

There two sources of endogeneity or correlation between a_{jm} and ε_{im}: (i) simultaneity; and (ii) correlation between ε_{im} and ε_{jm}. It is interesting to distinguish between these two sources of endogeneity because they bias the parameter δ_{ij} in opposite directions.

(i) Simultaneity. An equilibrium of the model is a reduced form equation where we represent the action of each player as a function of only exogenous variables in x_m and ε_m. In this reduced form, a_{jm} depends on ε_{im}. It is possible to show that this dependence is negative: keeping all the other exogenous variables constant, if ε_{im} is small enough then $a_{jm} = 0$, and if ε_{im} is large enough then $a_{jm} = 1$. Suppose that our estimator of δ_{ij} ignores this dependence. Then, the negative dependence between a_{jm} and ε_{im} contributes to generate a upward bias in the estimator of δ_{ij}.

That is, we will spuriously over-estimate the negative effect of Southwest on the profit of American Airlines because Southwest tends to enter in markets where AA has low values of ε_{im}.

(ii) Positively correlated unobservables. It is reasonable to expect that ε_{im} and ε_{jm} are positively correlated. This is because both ε_{im} and ε_{jm} contain unobserved market

characteristics that affect in a similar way, or at least in the same direction, all the firms in the same market. Some markets are more profitable than others for every firm, and part of this market heterogeneity is observable to firms but unobservable to us as researchers. The positive correlation between ε_{im} and ε_{jm} generates also a positive dependence between a_{jm} and ε_{im}.

For instance, suppose that $\varepsilon_{im} = \omega_m + u_{im}$, where ω_m represents the common market effect, and u_{im} is independent across firms. Then, keeping x_m and the unobserved u variables constant, if ω_m is small enough then ε_{im} is small and $a_{jm} = 0$, and if ω_m is large enough then ε_{im} is large and $a_{jm} = 1$. Suppose that our estimator of δ_{ij} ignores this dependence. Then, the positive dependence between a_{jm} and ε_{im} contributes to generate a downward bias in the estimator of δ_{ij}. In fact, the estimate of δ_{ij} could have the wrong sign, that is, could be negative instead of positive.

Therefore, we could spuriously find that American Airlines benefits from the operation of Continental in the same market because we tend to observe that these firms are always active in the same markets. This positive correlation between a_{im} and a_{jm} can be completely driven by the positive correlation between ε_{im} and ε_{jm}.

These two sources of endogeneity generate biases of opposite sign in δ_{ij}. There is evidence from different empirical applications that the bias due to unobserved market effects is much more important than the simultaneity bias. For instance, among others, Orhun (2013) in the US supermarket industry, Collard-Wexler (2013) in the US cement industry, Aguirregabiria and Mira (2007) in several retail industries in Chile, Igami and Yang (2016) in the Canadian fast-food restaurant industry, and Aguirregabiria and Ho (2012) in the US airline industry.

How do we deal with this endogeneity problem? The intuition for the identification in this model is similar to the identification using standard instrumental variables (IV) and control function (CF) estimation methods.

IV approach. There are exogenous firm characteristics in x_{jm} that affect the action of firm j but do not have a direct effect on the action of firm i: that is, observable characteristics with $\beta_j \neq 0$ but $\beta_i = 0$.

CF approach. There is an observable variable C_{it} that "proxies" or "controls for" the endogenous part of ε_{im} such that if we include C_{it} in the equation for firm i then the new error term in that equation and a_{jm} become independent (conditional on C_{it}).

The method of instrumental variables is the most common approach to deal with endogeneity in linear models. However, IV or GMM cannot be applied to estimate discrete choice models with endogenous variables. Control function approaches: Rivers and Vuong (1988), Vytlacil and Yildiz (2007). These approaches have not been extended yet to deal with models with multiple equilibria.

An alternative approach is Maximum likelihood. If we derive the probability distribution of the dummy endogenous variables conditional on the exogenous variables (that is, the reduced form of the model), we can use these probabilities to estimate the model by maximum likelihood.

$$\ell(\theta) = \sum_{m=1}^{M} \ln \Pr(a_{1m}, a_{2m}, ..., a_{Nm} \mid x_m,\ \theta) \tag{5.37}$$

This is the approach that has been most commonly used in this literature. However, we will have to deal with the problem of multiple equilibria.

Multiple equilibria

Consider the model with two players and assume that $\delta_1 \geq 0$ and $\delta_2 \geq 0$.

$$\begin{aligned} a_1 &= 1\{\, x_1\beta_1 - \delta_1\, a_2 + \varepsilon_1 \geq 0 \,\} \\ a_2 &= 1\{\, x_2\beta_2 - \delta_2\, a_1 + \varepsilon_2 \geq 0 \,\} \end{aligned} \tag{5.38}$$

The reduced form of the model is a representation of the endogenous variables (a_1, a_2) only in terms of exogenous variables and parameters. This is the reduced form of this model:

$$\begin{aligned} \{x_1\beta_1 + \varepsilon_1 < 0\} \;\&\; \{x_2\beta_2 + \varepsilon_2 < 0\} &\Rightarrow (a_1, a_2) = (0,0) \\ \{x_1\beta_1 - \delta_1 + \varepsilon_1 \geq 0\} \,\&\, \{x_2\beta_2 - \delta_2 + \varepsilon_2 \geq 0\} &\Rightarrow (a_1, a_2) = (1,1) \\ \{x_1\beta_1 - \delta_1 + \varepsilon_1 < 0\} \;\&\; \{x_2\beta_2 + \varepsilon_2 \geq 0\} &\Rightarrow (a_1, a_2) = (0,1) \\ \{x_1\beta_1 + \varepsilon_1 \geq 0\} \;\&\; \{x_2\beta_2 - \delta_2 + \varepsilon_2 < 0\} &\Rightarrow (a_1, a_2) = (1,0) \end{aligned} \tag{5.39}$$

The graphical representation in the space $(\varepsilon_1, \varepsilon_2)$ is in Figure 5.1.

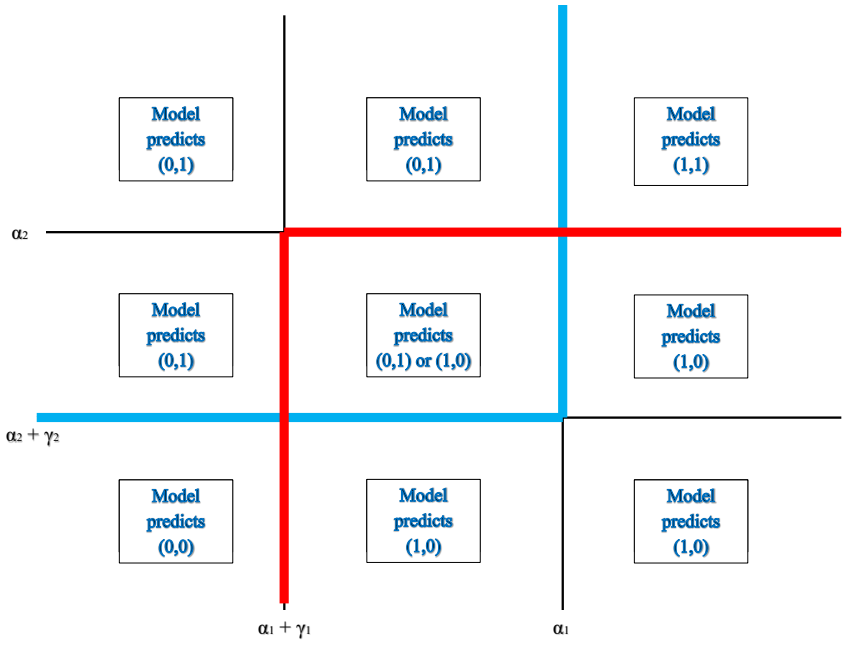


Figure 5.1: Outcomes in the Region of the Unobservables

Note that when:

$$\{0 \leq x_1\beta_1 + \varepsilon_1 < \delta_1\} \text{ and } \{0 \leq x_2\beta_2 + \varepsilon_2 < \delta_2\} \tag{5.40}$$

we have two Nash equilibria: $(a_1, a_2) = (0,1)$ and $(a_1, a_2) = (1,0)$. For this range of values of $(\varepsilon_1, \varepsilon_2)$, the reduced form (that is, the equilibrium) is not uniquely determined.

Therefore, we can not uniquely determine the probability $\Pr(a_{1m}, a_{2m}|x_m;\theta)$ that we need to estimate the model by maximum likelihood. We know $\Pr(1,1|\theta)$, and $\Pr(0,0|\theta)$, but we only have lower and upper bounds for $\Pr(0,1|\theta)$ and $\Pr(1,0|\theta)$.

The problem of indeterminacy of the probabilities of different outcomes becomes even more serious in empirical games with more than 2 players or/and more than two choice alternatives.

There have been different approaches to deal with the problem of multiple equilibria. Some authors have imposed additional structure in the model to guarantee equilibrium uniqueness or at least uniqueness of some observable outcome, for instance, number of entrants). A second group of studies do not impose additional structure and use methods such as moment inequalities or pseudo maximum likelihood to estimate structural parameters. The main motivation of this second group of studies is that identification and multiple equilibria are different problems and we do not need equilibrium uniqueness to identify parameters. We discuss these methods below.

5.4.5 Incomplete information

Model and basic assumptions

Consider a market with N potential entrants. If firm i does not operate in market m ($a_{im} = 0$), its profit is zero. If the firm is active in the market ($a_{im} = 1$), the profit is:

$$\Pi_{im} = \Pi_i(x_m, a_{-im}) - \varepsilon_{im} \tag{5.41}$$

For instance,

$$\Pi_{im} = x_{im}\,\beta_i - \varepsilon_{im} - \sum\nolimits_{j\neq i} \delta_{ij}\, a_{jm} \tag{5.42}$$

where β_i and δ_i are parameters. These parameters and the vector $s_m = (s_{1m}, s_{2m}, ..., s_{Nm})$ contain the variables which are common knowledge for all players. Now ε_{im} is private information of firm i. For the moment, we assume that private information variables are independent of s_m, and independently distributed over firms with distribution functions $G_i(\varepsilon_{im})$. The distribution function G_i is strictly increasing in $\mathbb{R}$. The information of player i is (s_m, ε_{im}).

A player's strategy depends on the variables in her information set. Let $\alpha \equiv \{\alpha_i(s_m, \varepsilon_{im}) : i = 1, 2, ..., N\}$ be a set of strategy functions, one for each player, such that $\alpha_i : S \times \mathbb{R} \to \{0, 1\}$. The actual profit Π_{im} is unknown to player i because the private information of the other players is unknown to player i. Players maximize expected profits:

$$\pi_i(s_m, \varepsilon_{im}, \alpha_{-i}) = s_{im}\,\beta_i - \varepsilon_{im} - \sum\nolimits_{j\neq i} \delta_{ij} \left[\int 1\left\{\alpha_j(s_m, \varepsilon_{jm}) = 1\right\} dG_j(\varepsilon_{jm}) \right] \tag{5.43}$$

or:

$$\begin{aligned} \pi_i(s_m, \varepsilon_{im}, \alpha_{-i}) &= s_{im}\,\beta_i - \varepsilon_{im} - \textstyle\sum_{j\neq i} \delta_{ij}\, P_j^{\alpha}(s_m) \\ &= s_{im}\,\beta_i - \varepsilon_{im} - P_{-i}^{\alpha}(s_m)'\,\delta_i \end{aligned} \tag{5.44}$$

where $P_j^{\alpha}(s_m) \equiv \int 1\left\{\alpha_j(s_m, \varepsilon_{jm}) = 1\right\} dG_j(\varepsilon_{jm})$ is player j's probability of entry if she behaves according to her strategy in α.

Suppose that players other than i play their respective strategies in α. What is player i's best response? Let $b_i(s_m, \varepsilon_{im}, \alpha_{-i})$ be player i's best response function. This function is:

$$\begin{aligned} b_i(s_m, \varepsilon_{im}, \alpha_{-i}) &= 1\{\ \pi_i(s_m, \varepsilon_{im}, \alpha_{-i}) \geq 0\ \} \\ &= 1\left\{\ \varepsilon_{im} \leq s_{im}\ \beta_i - P^{\alpha}_{-i}(s_m)'\delta_i\ \right\} \end{aligned} \tag{5.45}$$

Associated with the best response function b_i (in the space of strategies), we can define a *best response probability function* in the space of probabilities as:

$$\begin{aligned} \Psi_i(s_m, P^{\alpha}_{-i}) &= \int 1\left\{\ b_i(s_m, \varepsilon_{im}, \alpha_{-i}) = 1\ \right\} dG_i(\varepsilon_{im}) \\ &= \int 1\left\{\ \varepsilon_{im} \leq s_{im}\ \beta_i - P^{\alpha}_{-i}(s_m)'\delta_i\ \right\} dG_i(\varepsilon_{im}) \\ &\quad G_i\left(s_{im}\ \beta_i - P^{\alpha}_{-i}(s_m)'\delta_i\right) \end{aligned} \tag{5.46}$$

A *Bayesian Nash equilibrium* (BNE) in this model is a set of strategy functions α^* such that, for any player i and any value of (s_m, ε_{im}), we have that:

$$\alpha^*_i(s_m, \varepsilon_{im}) = b_i(s_m, \varepsilon_{im}, \alpha^*_{-i}) \tag{5.47}$$

Associated with the set of strategies α^* we can define a set of choice probability functions $P^* = \{P^*_i(s_m) : i = 1, 2, ..., N\}$ such that $P^*_i(s_m) \equiv \int 1\{\alpha^*_i(s_m, \varepsilon_{im}) = 1\} dG_i(\varepsilon_{im})$. Note that these equilibrium choice probabilities are such that, for any player i and any value of s_m:

$$\begin{aligned} P^*_i(s_m) &= \Psi_i(s_m, P^*_{-i}) \\ &= G_i\left(s_{im}\ \beta_i - P^*_{-i}(s_m)'\delta_i\right) \end{aligned} \tag{5.48}$$

Therefore, we can define a BNE in terms of strategy functions α^* or in terms of choice probabilties P^*. There is a one-to-one relationship between α^* and P^*. Given α^*, it is clear that there is only one set of choice probabilities P^* defined as $P^*_i(s_m) \equiv \int I\{\alpha^*_i(s_m, \varepsilon_{im}) = 1\} dG_i(\varepsilon_{im})$. And given P^*, there is only one set of strategies α^* that is a BNE and is consistent with P^*. These strategy functions are:

$$\alpha^*_i(s_m, \varepsilon_{im}) = 1\left\{\ \varepsilon_{im} \leq s_{im}\ \beta_i - P^*_{-i}(s_m)'\delta_i\ \right\} \tag{5.49}$$

Suppose that the distribution of ε_{im} is known up to some scale parameter σ_i. For instance, suppose that $\varepsilon_{im} \sim iid\ N(0,1)$. Then, we have that equilibrium choice probabilities in market m solve the fixed point mapping in probability space:

$$P^*_i(s_m) = \Phi\left(s_{im}\ \frac{\beta_i}{\sigma_i} - P^{\alpha}_{-i}(s_m)'\frac{\delta_i}{\sigma_i}\right) \tag{5.50}$$

For notational simplicity we will use β_i and δ_i to represent β_i/σ_i and δ_i/σ_i, respectively.

We use θ to represent the vector of structural parameters $\{\beta_i, \delta_i : i = 1, 2, ..., N\}$. To emphasize that equilibrium probabilities depend on θ we use $P(s_m, \theta) = \{P_i(s_m, \theta) : i = 1, 2, ..., N\}$ to represent a vector of equilibrium probabilities associated with the exogenous conditions (s_m, θ). In general, there are values of $(s_m; \theta)$ for which the model has multiple equilibria. This is very common in models where players are heterogeneous, but we can find also multiple symmetric equilibria in models with homogeneous players, especially if there is strategic complementarity (that is, $\delta_i < 0$) as in coordination games.

Data and identification

Suppose that we observe this game played in M independent markets. We observe players' actions and a subset of the common knowledge state variables, $x_{im} \subseteq s_{im}$. That is,

$$Data = \{x_{im}, a_{im} : m = 1,2,...,M;\ i = 1,2,...,N\} \tag{5.51}$$

The researcher does not observe private information variables. It is important to distinguish two cases: (Case I) No common knowledge unobservables, that is, $x_{im} = s_{im}$; (Case II) Common knowledge unobservables, that is, $s_{im} = (x_{im}, \omega_{im})$, where ω_{im} is unobservable.

Case I: No common knowledge unobservables. Suppose that we have a random sample of markets and we observe:

$$\{x_{im}, a_{im} : m = 1,2,...,M;\ i = 1,2,...,N\} \tag{5.52}$$

We can describe this type of dataset as *data with global players* as all the firms play the entry game in the M markets. Let $P^0 = \{P_i^0(x) : i = 1,2,..,N; x \in X\}$ be players' entry probabilities in the population under study. The population is an equilibrium of the model. That is, for any i and any $x \in X$:

$$P_i^0(x) = \Phi\left(x_i\ \beta_i - P_{-i}^0(x)'\delta_i\right) \tag{5.53}$$

From our sample, we can nonparametrically identify the population P^0, that is, $P_i^0(x) = \mathbb{E}(a_{im}|x_m = x)$. Given P^0 and the equilibrium conditions in (5.53), can we uniquely identify θ? Notice that we can write these equations as:

$$\Phi^{-1}\left(P_i^0(x_m)\right) = x_{im}\ \beta_i - P_{-i}^0(x_m)'\delta_i = Z_{im}\ \theta_i \tag{5.54}$$

Define $Y_{im} \equiv \Phi^{-1}\left(P_i^0(x_m)\right)$; $Z_{im} \equiv (x_{im}, P_{-i}^0(x_m))$; and $\theta_i^0 \equiv (\beta_i^0, \delta_i^0)$. Then,

$$Y_{im} = Z_{im}\ \theta_i \tag{5.55}$$

And we can also write this system as:

$$\mathbb{E}(Z_{im}'Y_{im}) = \mathbb{E}(Z_{im}'Z_{im})\ \theta_i \tag{5.56}$$

It is clear that θ_i is uniquely identified if $\mathbb{E}(Z_{im}'Z_{im})$ is a nonsigular matrix. Note that if x_{im} contains variables that vary both over markets and over players then we have exclusion restrictions that imply that $\mathbb{E}(Z_{im}'Z_{im})$ is a nonsigular matrix.

I some empirical applications, the dataset includes only local players. That is, firms that are potential entrants in only one local market. In this case, we have a random sample of markets and we observe:

$$\{x_m, n_m : m = 1,2,...,M\} \tag{5.57}$$

Let $P^0 = \{P^0(x) : x \in X\}$ be the entry probabilities in the population under study. The population is an equilibrium of the model, and therefore there is a θ such that for any $x \in X$:

$$P^0(x) = \Phi\left(x\ \beta - \delta\ H(P^0[x])\right) \tag{5.58}$$

From our sample, we can nonparametrically identify the population P^0. To see this, notice that: (1) we can identify the distribution for the number of firms: $\Pr(n_m = n|x_m = x)$; (2) the model implies that conditional on $x_m = x$ the number of firms follows a Binomial distribution with arguments N and $P^0(x)$, and therefore:

$$\Pr(n_m = n|x_m = x) = \binom{n}{N} P^0(x)^n \left(1 - P^0(x)\right)^{N-n}; \tag{5.59}$$

and (3) given the previous expression, we can obtain the $P^0(x)$ associated with $\Pr(n_m = n|x_m = x)$. Given P^0 and the equilibrium condition $P^0(x) = \Phi\left(x\,\beta - \delta\,H(P^0[x])\right)$, can we uniquely identify θ? Notice that we can write these equations as:

$$Y_m = x_m\,\beta - \delta\,H(P^0[x_m]) = Z_m\,\theta \tag{5.60}$$

where $Y_m \equiv \Phi^{-1}\left(P^0(x_m)\right)$; $\theta \equiv (\beta, \delta)$; and $Z_m \equiv (x_m, H(P^0[x_m]))$. And we can also write this system as:

$$\mathbb{E}(Z_m' Y_m) = \mathbb{E}(Z_m' Z_m)\,\theta \tag{5.61}$$

It is clear θ is uniquely identified if $\mathbb{E}(Z_m' Z_m)$ is a nonsingular matrix.

Case II: Common knowledge unobservables. Conditional on x_m, players' actions are still correlated across markets. This is evidence that ....

In applications where we do not observe the identity of the potential entrants, we consider a model without firm heterogeneity:

$$\Pi_{im} = x_m\,\beta - \delta\,h\left(1 + \sum_{j\neq i} a_{jm}\right) + \varepsilon_{im} \tag{5.62}$$

A symmetric Bayesian Nash equilibrium in this model is a probability of entry $P^*(x_m;\theta)$ that solves the fixed point problem:

$$P^*(x_m;\theta) = \Phi\left(x_m\,\beta - \delta\,H(P[x_t,\theta])\right) \tag{5.63}$$

where $H(P)$ is the expected value of $h\left(1 + \sum_{j\neq i} a_j\right)$ conditional on the information of firm i, and under the condition that the other firms behave according to their entry probabilities in P. That is,

$$H(P) = \sum_{a_{-i}} \left(\prod_{j\neq i} P_j^{a_j} \left[1 - P_j\right]^{1-a_j}\right) h\left(1 + \sum_{j\neq i} a_j\right) \tag{5.64}$$

and $\sum_{a_{-i}}$ represents the sum over all the possible actions of firms other than i.

Pseudo ML estimation

The goal is to estimate the vector of structural parameters θ^0 given a random sample $\{x_{im}, a_{im}\}$. Equilibrium probabilities are not uniquely determined for some values of the primitives. However, for any vector of probabilities P, the best response probability functions $\Phi\left(x_{im}\,\beta_i - \sum_{j\neq i}\delta_{ij}\,P_j(x_m)\right)$ are always well-defined. We define a pseudo likelihood function based on best responses to the population probabilities:

$$\begin{aligned} Q_M(\theta, P^0) &= \sum_{m=1}^{M}\sum_{i=1}^{N} a_{im}\,\ln\Phi\left(x_{im}\,\beta_i - \textstyle\sum_{j\neq i}\delta_{ij}\,P_j^0(x_m)\right) \\ &+ (1 - a_{im})\,\ln\Phi\left(-x_{im}\,\beta_i + \textstyle\sum_{j\neq i}\delta_{ij}\,P_j^0(x_m)\right) \end{aligned} \tag{5.65}$$

It is possible to show that θ uniquely maximizes $Q_\infty(\theta, P^0)$. The PML estimator of θ^0 maximizes $Q_M(\theta, \hat{P}^0)$, where $\hat{P}^0$ is a consistent nonparametric estimator of P^0. This estimator is consistent and asymptotically normal. Iterating in this procedure can provide efficiency gains both in finite samples and asymptotically (Aguirregabiria, 2004).

5.4.6 Entry and spatial competition

How do market power and profits of a retail firm depend on the location of its store(s) relative to the location of competitors? How important is spatial differentiation in explaining market power? These are important questions in the study of competition in retail markets. Seim (2006) studies these questions in the context of the video rental industry. Seim's work is the first study that endogenizes store locations and introduces spatial competition in a game of market entry.

Seim's model has important similarities with the static game with single-store firms and incomplete information that we have presented above. The main difference is that Seim's model does not include an explicit model of spatial consumer demand and price competition. Instead, she considers a 'semi-structural' specification of a store's profit that captures the idea that the profit of a store declines when competing stores get closer in geographic space. The specification seems consistent with the idea that consumers face transportation costs, and therefore spatial differentiation between stores can increase profits.

From a geographical point of view, a market in this model is a compact set in the two-dimension Euclidean space. There are L locations in the market where firms can operate stores. These locations are a set grid points where the grid can be as fine as we want. We index locations by ℓ that belongs to the set $\{1,2,...,L\}$.

There are N potential entrants in the market. Each firm makes two decisions: (1) whether to be active or not in the market; and (2) if it decides to be active, where to open its store. Note that Seim does not model multi-store firms. Aguirregabiria and Vicentini (2016) present an extension of Seim's model with multi-store firms, endogenous consumer behavior, and dynamics.

Let a_i represent the decisions of firm i, such that $a_i \in \{0,1,...,L\}$ and $a_i = 0$ represents "no entry", and $a_i = \ell > 0$ represents entry in location ℓ.

The profit of not being active in the market is normalized to zero. Let $\Pi_{i\ell}$ be the profit of firm i if it has a store in location ℓ. These profits depend on the store location decisions of the other firms. In particular, $\Pi_{i\ell}$ declines with the number of other stores "close to" location ℓ.

Of course, the specific meaning of being close to location ℓ is key for the implications of this model. This should depend on how consumers perceive as close substitutes stores in different locations. In principle, if we have data on quantities and prices for the different stores that are active in this city, we could estimate a demand system that would provide a measures of consumers' transportation costs and of the degree of substitution in demand between stores at different locations. Houde (2012) applies this approach to gasoline markets. However, for this industry we may not have information on prices and quantities at the store level. Fortunately, store location decisions may contain useful (and even better) information for identifying the degree of competition between stores at different locations.

Seim's specification of the profit function is "semi-structural": it does not model explicitly consumer behavior,but it is consistent with the idea that consumers face transportation costs, and therefore spatial differentiation (ceteris paribus) can increase profits.

For every location ℓ in the city, Seim defines B rings around the location: a first ring of radius d_1 (say half a mile); a second ring of radius $d_2 > d_1$ (say one mile), and so on.

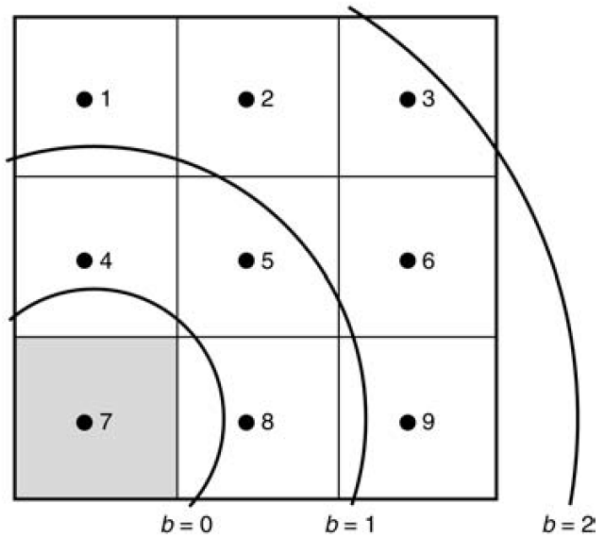


Figure 5.2: Seim, 2006): Defition of Local Markets

The profit of a store depends on the number of other stores located within each of the B rings. We expect that closer stores should have stronger negative effects on the store's profits. The profit function of an active store at location ℓ is:

$$\Pi_{i\ell} = x_\ell \, \beta + \sum_{b=1}^{B} \gamma_b \, N_{b\ell} + \xi_\ell + \varepsilon_{i\ell} \tag{5.66}$$

where β, γ_1, γ_2, ..., and γ_B are parameters; x_ℓ is a vector of observable exogenous characteristics that affect profits in location ℓ; $N_{b\ell}$ is the number of stores in ring b around location ℓ excluding i; ξ_ℓ represents exogenous characteristics of location ℓ that are unobserved to the researcher but common and observable to firms; and $\varepsilon_{i\ell}$ is a component of the profit of firm i in location ℓ that is private information to this firm. For the "no entry" choice, $\Pi_{i0} = \varepsilon_{i0}$.

Assumption. Let $\varepsilon_i = \{\varepsilon_{i\ell} : \ell = 0, 1, ..., L\}$ be the vector with the private information

variables of firm i at every possible location. ε_i is i.i.d. over firms and locations with a extreme value type 1 distribution.

The information of firm i is (x, ξ, ε_i), where x and ξ represent the vectors with x_ℓ and ξ_ℓ, respectively, at every location ℓ in the city. Firm i does not know the $\varepsilon's$ of other firms. Therefore, $N_{b\ell}$ is unknown to the firm. Firms only know the probability distribution of $N_{b\ell}$. Therefore, firms maximize expected profits. The expected profit of firm i is:

$$\Pi^e_{i\ell} = x_\ell \, \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell} + \xi_\ell + \varepsilon_{i\ell} \tag{5.67}$$

where $N^e_{b\ell}$ represents $\mathbb{E}(N_{b\ell}|x, \xi)$.

A firm's strategy depends on the variables in its information set. Let $\alpha_i(x, \xi, \varepsilon_i)$ be a strategy function for firm i such that $\alpha_i : X \times \mathbb{R}^2 \to \{0, 1, ..., L\}$. Given expectations $N^e_{b\ell}$, the best response strategy of firm i is:

$$\alpha_i(x, \xi, \varepsilon_i) = \arg \max_{\ell \in \{0,1,...,L\}} \left\{ x_\ell \, \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell} + \xi_\ell + \varepsilon_{i\ell} \right\} \tag{5.68}$$

Or similarly, $\alpha_i(x, \xi, \varepsilon_i) = \ell$ if and only if $x_\ell \, \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell} + \xi_\ell + \varepsilon_{i\ell}$ is greater that $x_{\ell'} \, \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell'} + \xi_{\ell'} + \varepsilon_{i\ell'}$ for any other location ℓ'.

From the point of view of other firms that do not know the private information of firm i but know the strategy function $\alpha_i(x, \xi, \varepsilon_i)$, the strategy of firm i can be described as a probability distribution: $P_i \equiv \{P_{i\ell} : \ell = 0, 1, ..., L\}$ where $P_{i\ell}$ is the probability that firm i chooses location ℓ when following her strategy $\alpha_i(x, \xi, \varepsilon_i)$. That is,

$$P_{i\ell} \equiv \int 1\{\alpha_i(x, \xi, \varepsilon_i) = \ell\} \, dF(\varepsilon_i) \tag{5.69}$$

where $F(\varepsilon_i)$ is the CDF of ε_i. By construction, $\sum_{\ell=0}^{L} P_{i\ell} = 1$.

Given expectations $N^e_{b\ell}$, we can also represent the best response strategy of firm i as a choice probability. A best response probability $P_{i\ell}$ is:

$$P_{i\ell} = \int 1 \left[\ell = \arg\max_{\ell'} \left\{ x_{\ell'} \, \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell'} + \xi_{\ell'} + \varepsilon_{i\ell'} \right\} \right] \, dF(\varepsilon_i) \tag{5.70}$$

And given the extreme value assumption on ε_i:

$$P_{i\ell} = \frac{\exp\left\{ x_\ell \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell} + \xi_\ell \right\}}{1 + \exp\left\{ x_{\ell'} \beta + \sum_{b=1}^{B} \gamma_b \, N^e_{b\ell'} + \xi_{\ell'} \right\}} \tag{5.71}$$

In this application, there is no information on firms' exogenous characteristics, and Seim assumes that the equilibrium is symmetric: $\alpha_i(x, \xi, \varepsilon_i) = \alpha(x, \xi, \varepsilon_i)$ and $P_{i\ell} = P_\ell$ for every firm i.

The expected number of firms in ring b around location ℓ, $N^e_{b\ell}$, is determined by the vector of entry probabilities $P \equiv \{P_{\ell'} : \ell' = 1, 2, ..., L\}$. That is:

$$N^e_{b\ell} = \sum_{\ell'=1}^{L} 1\{\ell' \text{belongs to ring } b \text{ around } \ell\} \, P_{\ell'} \, N \tag{5.72}$$

To emphasize this dependence we use the notation $N^e_{b\ell}(P)$.

Therefore, we can define a (symmetric) equilibrium in this game as a vector of probabilities $P \equiv \{P_\ell : \ell = 1,2,...,L\}$ that solve the following system of equilibrium conditions: for every $\ell = 1,2,...,L$:

$$P_\ell = \frac{\exp\left\{x_\ell \beta + \sum_{b=1}^{B} \gamma_b \, N_{b\ell}^e(P) + \xi_\ell\right\}}{1 + \exp\left\{x_{\ell'} \beta + \sum_{b=1}^{B} \gamma_b \, N_{b\ell'}^e(P) + \xi_{\ell'}\right\}} \tag{5.73}$$

By Brower's Theorem an equilibrium exists. The equilibrium may not be unique. Seim shows that if the γ parameters are not large and they decline fast enough with b, then the equilibrium is unique.

Let $\theta = \{N, \beta, \gamma_1, \gamma_2, ..., \gamma_B\}$ be the vector of parameters of the model. These parameters can be estimated even if we have data only from one city. Suppose that the data set is $\{x_\ell, n_\ell : \ell = 1,2,...,L\}$ for L different locations in a city, where L is large, and n_ℓ represents the number of stores in location ℓ. We want to use these data to estimate θ. We describe below the estimation with data from only one city. Later, we will see that the extension to data from more than one city is trivial.

Let x be the vector $\{x_\ell : \ell = 1,2,...,L\}$. All the analysis is conditional on x, which is a description of the "landscape" of observable socioeconomic characteristics in the city. Given x, we can think of $\{n_\ell : \ell = 1,2,...,L\}$ as *one realization of a spatial stochastic process*. In terms of the econometric analysis, this has similarities with time series econometrics in the sense that a time series is a single realization from a stochastic process. Despite having just one realization of a stochastic process, we can estimate consistently the parameters of that process as long as we make some stationarity assumptions.

This is the model considered by Seim (2006): there is city unobserved heterogeneity (her dataset includes multiple cities) but within a city there is no unobserved location heterogeneity.

Conditional on x, spatial correlation/dependence in the unobservable variables ξ_ℓ can generate dependence between the number of firms at different locations $\{n_\ell\}$. We start with the simpler case where there is no unobserved location heterogeneity: that is, $\xi_\ell = 0$ for every location ℓ.

Without unobserved location heterogeneity, and conditional on x, the variables n_ℓ are independently distributed, and n_ℓ is a random draw from a Binomial random variable with arguments $(N, P_\ell(x,\theta))$, where $P_\ell(x,\theta)$ are the equilibrium probabilities defined above where now we explicitly include (x,θ) as arguments.

$$n_\ell \sim i.i.d. \text{ over } \ell \text{ Binomial}(N, P_\ell(x,\theta)) \tag{5.74}$$

Therefore,

$$\begin{aligned} \Pr(n_1, n_2, ..., n_L \mid x, \theta) &= \prod_{\ell=1}^{L} \Pr(n_\ell \mid x, \theta) \\ &= \prod_{\ell=1}^{L} \frac{N!}{n_\ell (N - n_\ell)!} P_\ell(x,\theta)^{n_\ell} (1 - P_\ell(x,\theta))^{N - n_\ell} \end{aligned} \tag{5.75}$$

The log-likelihood function is:

$$\ell(\theta) = \sum_{\ell=1}^{L} \ln\left(\frac{N!}{(N-n_\ell)!}\right) + n_\ell \ln P_\ell(x,\theta) + (N - n_\ell) \ln(1 - P_\ell(x,\theta)) \tag{5.76}$$

The maximum likelihood estimator, $\hat{\theta}$, is the value of θ that maximizes this likelihood. The parameters of the model, including the number of potential entrants N, are identified. Partly, the identification comes from functional form assumptions. However, there are also exclusion restrictions that can provide identification even if some of these assumptions are relaxed. In particular, for the identification of β and γ_b, the model implies that $N^e_{b\ell}$ depends on socioeconomic characteristics at locations other than ℓ (that is, $x_{\ell'}$ for $\ell' \neq \ell$). Therefore, $N^e_{b\ell}$ has sample variability that is independent of x_ℓ and this implies that the effects of x_ℓ and $N^e_{b\ell}$ on a firm's profit can be identified even if we relax the linearity assumption.[1]

Now, let's consider the model where $\xi_\ell \neq 0$. A simple (but restrictive approach) is to assume that there is a number R of "regions" or districts in the city, where the number of regions R is small relative to the number of locations L, such that all the unobserved heterogeneity is between regions but there is no unobserved heterogeneity within regions. Under this assumption, we can control for unobserved heterogeneity by including region dummies. In fact, this case is equivalent to the previous case without unobserved location heterogeneity with the only difference being that the vector of observables x_ℓ now includes region dummies.

A more interesting case is when the unobserved heterogeneity is at the location level. We assume that $\xi = \{\xi_\ell : \ell = 1,2,...,L\}$ is independent of x and it is a random draw from a spatial stochastic process. The simplest process is when ξ_ℓ is $i.i.d.$ with a known distribution, say $N(0,\sigma^2_\xi)$ where the zero mean is without loss of generality. However, we can allow for spatial dependence in this unobservable. For instance, we may consider a Spatial autoregressive process (SAR):

$$\xi_\ell = \rho \ \bar{\xi}^C_\ell + u_\ell \tag{5.77}$$

where u_ℓ is $i.i.d.$ $N(0,\sigma^2_u)$, ρ is a parameter, and $\bar{\xi}^C_\ell$ is the mean value of ξ at the C locations closest to location ℓ, excluding location ℓ itself. To obtain a random draw of the vector ξ from this stochastic process it is convenient to write the process in vector form:

$$\xi = \rho \ \mathbf{W}^C \ \xi + u \tag{5.78}$$

where ξ and u are $L \times 1$ vectors, and $\mathbf{W}^C$ is a $L \times L$ weighting matrix such that every row, say row ℓ, has values $1/C$ at positions that correspond to locations close to location ℓ, and zeroes otherwise. Then, we can write $\xi = (I - \rho \ \mathbf{W}^C)^{-1} u$. First, we take independent draws from $N(0,\sigma^2_u)$ to generate the vector u, and then we pre-multiply that vector by $(I - \rho \ \mathbf{W}^C)^{-1}$ to obtain ξ.

Note that now the vector of structural parameters includes the parameters in the stochastic process of ξ, that is, σ_u and ρ.

Now, conditional on both x and ξ, the variables n_ℓ are independently distributed, and n_ℓ is a random draw from Binomial random variable with arguments $(N, P_\ell(x,\xi,\theta))$, where $P_\ell(x,\xi,\theta)$ are the equilibrium probabilities. Importantly, for different values of ξ

[1] Xu (2018) studies the asymptotics of this type of estimator. His model is a bit different to Seim's model because players and locations are interchangeable.

we have different equilibrium probabilities. Then,

$$\begin{aligned}\Pr(n_1,n_2,...,n_L \mid x,\theta) &= \int \Pr(n_1,n_2,...,n_L \mid x,\xi,\theta)\ dG(\xi) \\ &= \int \left[\prod_{\ell=1}^{L} \Pr(n_\ell \mid x,\xi,\theta)\right]\ dG(\xi) \\ &= \prod_{\ell=1}^{L} \frac{N!}{n_\ell(N-n_\ell)!} \\ &\quad \int \left[\prod_{\ell=1}^{L} P_\ell(x,\xi,\theta)^{n_\ell}(1-P_\ell(x,\xi,\theta))^{N-n_\ell}\right] dG(\xi)\end{aligned} \tag{5.79}$$

And the log-likelihood function is:

$$\ell(\theta) = \sum_{\ell=1}^{L} \ln\left(\frac{N!}{(N-n_\ell)!}\right) \tag{5.80}$$

$$\tag{5.81}$$

$$+\ln\left(\int \left[\prod_{\ell=1}^{L} P_\ell(x,\xi,\theta)^{n_\ell}(1-P_\ell(x,\xi,\theta))^{N-n_\ell}\right] dG(\xi)\right) \tag{5.82}$$

The maximum likelihood estimator is defined as usual.

In their empirical study on competition between big-box discount stores in the US (that is, Kmart, Target and Walmart), Zhu and Singh (2009) extend Seim's entry model by introducing firm heterogeneity. The model allows competition effects to be asymmetric across three different chains. For example, the model can incorporate a situation where the impact on the profits of Target from a Walmart store 10 miles away is stronger than the impact from a Kmart store located 5 miles away. The specification of the profit function of a store of chain i at location ℓ is:

$$\pi_{i\ell} = x_\ell\ \beta_i + \sum_{j\neq i}\sum_{b=1}^{B} \gamma_{bij}\ n_{b\ell j} + \xi_\ell + \varepsilon_{i\ell} \tag{5.83}$$

where $n_{b\ell j}$ represents the number of stores that chain j has within the $b-ring$ around location ℓ. Despite the paper studying competition between retail chains, it still makes similar simplifying assumptions as in Seim's model that ignores important aspects of competition between retail chains. In particular, the model ignores economies of density, and firms' concerns about cannibalization between stores of the same chain. It assumes that the entry decisions of a retail chain are made independently at each location. Under these assumptions, the equilibrium of the model can be described as a vector of $N*L$ entry probabilities, one for each firm and location, that solves the following fixed point problem:

$$P_{i\ell} = \frac{\exp\left\{x_\ell\beta_i + \sum_{j\neq i}\sum_{b=1}^{B}\gamma_{bij}\ N\left[\sum_{\ell'=1}^{L} D^b_{\ell\ell'}\ P_{j\ell'}\right] + \xi_\ell\right\}}{1+\sum_{\ell'=1}^{L}\exp\left\{x_{\ell'}\beta_i + \sum_{j\neq i}\sum_{b=1}^{B}\gamma_{bij}\ N\left[\sum_{\ell''=1}^{L} D^b_{\ell'\ell''}\ P_{j\ell''}\right] + \xi_{\ell'}\right\}} \tag{5.84}$$

The authors find substantial heterogeneity in the competition effects between these three big-box discount chains, and in the pattern of how these effects decline with distance. For instance, Walmart's supercenters have a very substantial impact even at a large distance.

Datta and Sudhir (2013) estimate an entry model of grocery stores that endogenizes both location and product type decisions. They are interested in evaluating the effects of zoning restrictions on market structure. Zoning often reduces firms' ability to avoid competition by locating remotely each other. Theory suggests that in such a market firms have a stronger incentive to differentiate their products. Their estimation results support this theoretical prediction. The authors also investigate different impacts of various types of zoning ('centralized zoning', 'neighborhood zoning', and 'outskirt zoning') on equilibrium market structure.

5.4.7 Multi-store firms

As we have mentioned above, economies of density and cannibalization are potentially important factors in store location decisions of retail chains. A realistic model of competition between retail chains should incorporate this type of spillover effects. Taking into account these effects requires a model of competition between multi-store firms similar to the one in section 2.1.2. The model takes into account the joint determination of a firm's entry decisions at different locations. A firm's entry decision is represented by the L-dimension vector $a_i \equiv \{a_{i\ell} : \ell = 1,2,\ldots,L\}$, with $a_{i\ell} \in \{0,1\}$, such that the set of possible actions contains 2^L elements. For instance, Jia (2008) studies competition between two chains (Walmart and Kmart) over 2065 locations (US counties). The number of possible decisions of a retail chain is 2^{2065}. Without further restrictions, computing firms' best responses is intractable.

In her paper, Jia therefore imposes restrictions on the specification of firms' profits that imply the supermodularity of the game and facilitate substantially the computation of an equilibrium. Suppose that we index the two firms as i and j. The profit function of a firm, say i, is $\Pi_i = V_i(a_i,a_j) - EC_i(a_i)$, where $V_i(a_i,a_j)$ is the variable profit function such that:

$$V_i(a_i,a_j) = \sum_{\ell=1}^{L} a_{i\ell} \left[x_\ell \beta_i + \gamma_{ij} a_{j\ell}\right] \tag{5.85}$$

x_ℓ is a vector of market/location characteristics. γ_{ij} is a parameter that represents the effect on the profit of firm i of competition from a store of chain j. $EC_i(a_i)$ is the entry cost function such that:

$$EC_i(a_i) = \sum_{\ell=1}^{L} a_{i\ell} \left[\theta_{i\ell}^{EC} - \frac{\theta^{ED}}{2} \sum_{\ell'=1}^{L} \frac{a_{i\ell'}}{d_{\ell\ell'}}\right] \tag{5.86}$$

$\theta_{i\ell}^{EC}$ is the entry cost that firm i would have in location ℓ in the absence of economies of density (that is, if it were a single-store firm); θ^{ED} is a parameter that represents the magnitude of the economies of density and is assumed to be positive; and $d_{\ell\ell'}$ is the distance between locations ℓ and ℓ'.

Jia further assumes that the entry cost $\theta_{i\ell}^{EC}$ consists of three components:

$$\theta_{i\ell}^{EC} = \theta_i^{EC} + (1-\rho)\,\xi_\ell + \varepsilon_{i\ell}, \tag{5.87}$$

where θ_i^{EC} is chain-fixed effects, ρ is a scale parameter, ξ_ℓ is a location random effect, and $\varepsilon_{i\ell}$ is a firm-location error term. Both ξ_ℓ and $\varepsilon_{i\ell}$ are i.i.d. draws from the standard normal distribution and known to all the players when making decisions. To capture

economies of density, the presence of stores from the same firm at other locations is weighted by the inverse of the distance between locations, $1/d_{\ell\ell'}$. This term is multiplied by one-half to avoid double counting in the total entry cost of the retail chain.

The specification of the profit function in equations (5.85) and () imposes some important restrictions. Under this specification, locations are interdependent only through economies of density. In particular, there are no cannibalization effects between stores of the same chain at different locations. Similarly, there is no spatial competition between stores of different chains at different locations. In particular, this specification ignores the spatial competition effects between Kmart, Target, and Walmart that Zhu and Singh (2009) find in their study. The specification also rules out cost savings that do not depend on store density such as lower wholesale prices owing to strong bargaining power of chain stores. The main motivation for these restrictions is to have a supermodular game that facilitates very substantially the computation of an equilibrium, even when the model has a large number of locations.

In a Nash equilibrium of this model, the entry decisions of a firm, say i, should satisfy the following L optimality conditions:

$$a_{i\ell} = 1\left\{ x_\ell\, \beta_i + \gamma_{ij}\, a_{j\ell} - \theta_{i\ell}^{EC} + \frac{\theta^{ED}}{2} \sum_{\ell'=1}^{L} \frac{a_{i\ell'}}{d_{\ell\ell'}} \geq 0 \right\} \tag{5.88}$$

These conditions can be interpreted as the best response of firm i in location ℓ given the other firm's entry decisions, and given also firm i's entry decisions at locations other than ℓ. We can write this system of conditions in a vector form as $a_i = br_i(a_i, a_j)$. Given a_j, a fixed point of the mapping $br_i(., a_j)$ is a (full) best response of firm i to the choice a_j by firm j. With $\theta^{ED} > 0$ (that is, economies of density), it is clear from equation (9.31) that the mapping br_i is increasing in a_i. By Topkis's theorem, this increasing property implies that: (1) the mapping has at least one fixed point solution; (2) if it has multiple fixed points they are ordered from the lowest to the largest; and (3) the smallest (largest) fixed point can be obtained by successive iterations in the mapping br_i using as starting value $a_i = 0$ ($a_i = 1$). Given these properties, Jia shows that the following algorithm provides the Nash equilibrium that is most profitable for firm i:

Step [firm i]: Given the lowest possible value for $a_j = 0$, that is, $a_i = (0, 0, \ldots 0)$, we apply successive iterations with respect to a_i in the fixed point mapping $br_i(., a_j = 0)$ starting at $a_i = (1, 1, \ldots 1)$. These iterations converge to the largest best response of firm i, that we denote by $a_i^{(1)} = BR_i^{(High)}(0)$.

Step [firm j]: Step [j]: Given $a_i^{(1)}$, we apply successive iterations with respect to a_j in the fixed point mapping $br_j(., a_i^{(1)})$ starting at $a_j = 0$. These iterations converge to the lowest best response of firm j, that we denote by $a_j^{(1)} = BR_j^{(Low)}(a_i^{(1)})$.

We keep iterating in Step [firm i] and Step [firm j] until convergence. At every iteration, say k, given $a_j^{(k-1)}$ we first apply (Step [i]) to obtain $a_i^{(k)} = BR_i^{(High)}(a_j^{(k-1)})$, and then we apply (Step [j]) to obtain $a_j^{(k)} = BR_j^{(Low)}(a_i^{(k)})$. The supermodularity of the game ensures the convergence of this process and the resulting fixed point is the Nash equilibrium that most favors firm i. Jia combines this solution algorithm with a simulation of unobservables to estimate the parameters of the model using the method of simulated moments (MSM).

In his empirical study of convenience stores in Okinawa Island of Japan, Nishida (2015) extends Jia's model in two directions. First, a firm is allowed to open multiple stores (up to four) in the same location. Second, the model explicitly incorporates some form of spatial competition: a store's revenue is affected not only by other stores in the same location but also by those in adjacent locations.

Although the approach used in these two studies is elegant and useful, its use in other applications is somewhat limited. First, supermodularity requires that the own network effect on profits is monotonic, that is, the effect is either always positive ($\theta^{ED} > 0$) or always negative ($\theta^{ED} < 0$). This condition rules out situations where the net effect of cannibalization and economies of density varies across markets. Second, the number of (strategic) players must be equal to two. For a game to be supermodular, players' strategies must be strategic complements. In a model of market entry, players' strategies are strategic substitutes. However, when the number of players is equal to two, any game of strategic substitutes can be transformed into one of strategic complements by changing the order of strategies of one player (for example, use zero for entry and one for no entry). This trick no longer works when we have more than two players.

Ellickson, Houghton, and Timmins (2013, hereafter EHT) propose an alternative estimation strategy and apply it to data of US discount store chains. Their estimation method is based on a set of inequalities that arise from the best response condition of a Nash equilibrium. Taking its opponents' decisions as given, a chain's profit associated with its observed entry decision must be larger than the profit of any alternative entry decision. EHT consider particular deviations that relocate one of the observed stores to another location.

Let a_i^* be the observed vector of entry decisions of firm i, and suppose that in this observed vector the firm has a store in location ℓ but not in location ℓ'. Consider the alternative (hypothetical) choice $a_i^{\ell \to \ell'}$ that is equal to a_i^* except that the store in location ℓ is closed and relocated to location ℓ'. Revealed preference implies that $\pi_i(a_i^*) \geq \pi_i(a_i^{\ell \to \ell'})$. EHT further simplify this inequality by assuming that there are no economies of scope or density (for example, $\theta^{ED} = 0$), and that there are no firm-location-specific factors unobservable to the researcher, that is, $\varepsilon_{i\ell} = 0$. Under these two assumptions, the inequality above can be written as the profit difference between two locations:

$$[x_\ell - x_{\ell'}]\beta_i + \sum_{j \neq i} \gamma_{ij} \left[a_{j\ell}^* - a_{j\ell'}^* \right] + [\xi_\ell - \xi_{\ell'}] \geq 0 \tag{5.89}$$

Now, consider another chain, say k, that has an observed choice a_k^* with a store in location ℓ' but not in location ℓ. For this chain, we consider the opposite (hypothetical) relocation decision from firm i above: the store in location ℓ' is closed and a new store is open in location ℓ. For this chain, revealed preference implies that

$$[x_{\ell'} - x_\ell]\beta_k + \sum_{j \neq k} \gamma_{kj} \left[a_{j\ell}^* - a_{j\ell'}^* \right] + [\xi_{\ell'} - \xi_\ell] \geq 0 \tag{5.90}$$

Summing up the inequalities for firms i and k, we generate an inequality that is free from location fixed effects ξ.

$$[x_{\ell'} - x_\ell] \ [\beta_i - \beta_k] + \sum_{j \neq i} \gamma_{ij} \left[a_{j\ell}^* - a_{j\ell'}^* \right] + \sum_{j \neq k} \gamma_{kj} \left[a_{j\ell}^* - a_{j\ell'}^* \right] \geq 0 \tag{5.91}$$

EHT construct a number of inequalities of this type and obtain estimates of the parameters of the model by using a smooth maximum score estimator (Manski, 1975; Horowitz, 1992; Fox, 2007).

Unlike the lattice theory approach of Jia and Nishida, the approach applied by EHT can accommodate more than two players, allows the researcher to be agnostic about equilibrium selections, and is robust to the presence of unobserved market heterogeneity. Their model, however, rules out any explicit interdependence between stores in different locations, including spatial competition, cannibalization and economies of density. Although incorporating such inter-locational interdependencies does not seem to cause any fundamental estimation issue, doing so can be difficult in practice as it considerably increases the amount of computation. Another possible downside of this approach is the restriction it imposes on unobservables. The only type of structural errors that this model includes are the variables ξ_ℓ that are common for all firms. Therefore, to accommodate observations that are incompatible with the inequalities in EHT model, the model requires non-structural errors, which may be interpreted as firms' optimization errors.

5.5 Estimation

The estimation of games of entry and spatial competition in retail markets should deal with some common issues in the econometrics of games and dynamic structural models. Here we do not try to present a detailed discussion of this econometric literature. Instead, we provide a brief description of the main issues, with an emphasis on aspects that are particularly relevant for empirical applications in retail industries.

5.5.1 Multiple Equilibria

Entry models with heterogeneous firms often generate more than one equilibrium for a given set of parameters. Multiple equilibria pose challenges to the researcher for two main reasons. First, standard maximum likelihood estimation no longer works because the likelihood of certain outcomes is not well defined without knowing the equilibrium selection mechanism. Second, without further assumptions, some predictions or counterfactual experiments using the estimated model are subject to an identification problem. These predictions depend on the type of equilibrium that is selected in a hypothetical scenario not included in the data.

Several approaches have been proposed to estimate an entry game with multiple equilibria. Which method works the best depends on assumptions imposed in the model, especially its information structure. In a game of complete information, there are at least four approaches. The simplest approach is to impose some particular equilibrium selection rule beforehand and estimate the model parameters under this rule. For instance, Jia (2008) estimates the model of competition between big-box chains using the equilibrium that is most preferable to K-mart. She also estimates the same model under alternative equilibrium selection rules to check for the robustness of some of her results. The second approach is to construct a likelihood function for some endogenous outcomes of the game that are common across all the equilibria. Bresnahan and Reiss (1991) estimate their model by exploiting the fact that, in their model, the total number of entrants is unique in all the equilibria.

A third approach is to make use of inequalities that are robust to multiple equilibria. One example is the profit inequality approach of EHT, which we described above. Another example is the method of moment inequality estimators proposed by Ciliberto and Tamer (2009). They characterize the lower and upper bounds of the probability of a certain outcome that are robust to any equilibrium selection rule. Estimation of structural parameters relies on the set of probability inequalities constructed from these bounds. In the first step, the researcher nonparametrically estimates the probabilities of equilibrium outcomes conditional on observables. The second step is to find a set of structural parameters such that the resulting probability inequalities are most consistent with the data. The application of Ciliberto and Tamer's approach to a spatial entry model may not be straightforward. In models of this class, the number of possible outcomes (that is, market structures) is often very large. For example, consider a local market consisting of ten sub-blocks. When two chains decide whether they enter into each of these sub-blocks, the total number of possible market structures is 2^{10}. Such a large number of possible outcomes makes it difficult to implement this approach for two reasons. The first stage estimate is likely to be very imprecise even when a sample size is reasonably large. The second stage estimation can be computationally intensive because one needs to check, for a given set of parameters, whether each possible outcome meets the equilibrium conditions or not.

A fourth approach proposed by Bajari, Hong, and Nekipelov (2010) consists in the specification of a flexible equilibrium selection mechanism and in the joint estimation of the parameters in this mechanism and the structural parameters in firms' profit functions. Together with standard exclusion restrictions for the identification of games, the key specification and identification assumption in this paper is that the equilibrium selection function depends only on firms' profits.

In empirical games of incomplete information, the standard way to deal with multiple equilibria is to use a two-step estimation method (Aguirregabiria and Mira 2007); Bajari, Hong, and Ryan 2010). In the first step, the researcher estimates the probabilities of firms' entry conditional on market observables (called policy functions) in a nonparametric way, for example, a sieves estimator. The second step is to find a set of structural parameters that are most consistent with the observed data and these estimated policy functions. A key assumption for the consistency of this approach is that, in the data, two markets with the same observable characteristics do not select different types of equilibria, that is, same equilibria conditional on observables. Without this assumption, the recovered policy function in the first stage would be a weighted sum of firms' policies under different equilibria, making the second-stage estimates inconsistent. Several authors have recently proposed extensions of this method to allow for multiplicity of equilibria in the data for markets with the same observable characteristics.

Identification and multiple equilibria

Tamer (2003) showed that all the parameters of the previous entry model with $N = 2$ is (point) identified under standard exclusion restrictions, and that multiple equilibria do not play any role in this identification result. Tamer's result can be extended to any number N of players, as long as we have the appropriate exclusion restrictions.

More generally, equilibrium uniqueness is neither a necessary nor a sufficient condition for the identification of a model (Jovanovic, 1989). To see this, consider a model

with a vector of structural parameters $\theta \in \Theta$, and define the mapping $C(\theta)$ from the set of parameters Θ to the set of measurable predictions of the model. For instance, $C(\theta)$ may contain the probability distribution of players' actions conditional on exogenous variables $\Pr(a_1, a_2, ..., a_N | x, \theta)$.

Multiple equilibria implies that the mapping C is a correspondence. A model is not point-identified if at the observed data (say $P^0 = \Pr(a_1, a_2, ..., a_N | x, \theta)$ for any vector of actions and $x's$) the inverse mapping C^{-1} is a correspondence. In general, C being a function (that is, equilibrium uniqueness) is neither a necessary nor a sufficient condition for C^{-1} being a function (that is, for point identification).

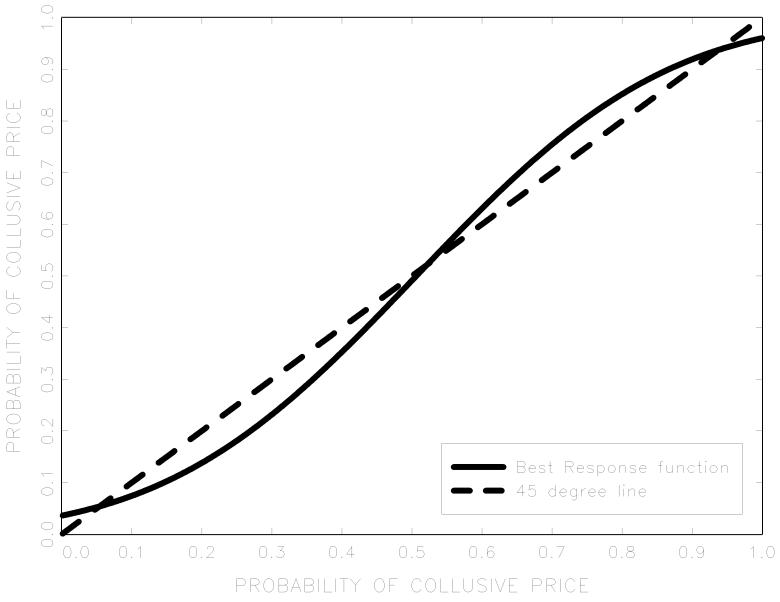


Figure 5.3: Multiple Equilibria

To illustrate the identification of a game with multiple equilibria, we start with a simple binary choice game with identical players and where the equilibrium probability P is implicitly defined as the solution to the condition $P = \Phi(-1.8 + \theta\ P)$, where θ is a structural parameter, and $\Phi(.)$ is the CDF of the standard normal. Suppose that the true value θ_0 is 3.5. It is possible to verify that the set of equilibria associated with θ_0 is $C(\theta_0) = \{\ P^{(A)}(\theta_0) = 0.054, P^{(B)}(\theta_0) = 0.551$, and $P^{(C)}(\theta_0) = 0.924\}$. The game has been played M times and we observe players' actions for each realization of the game $\{a_{im} : i, m\}$. Let P_0 be the population probability $\Pr(a_{im} = 1)$. Without further assumptions the probability P_0 can be estimated consistently from the data. For instance, a simple frequency estimator $\hat{P}_0 = (NM)^{-1} \sum_{i,m} a_{im}$ is a consistent estimator of P_0. Without further assumption, we do not know the relationship between population probability P_0 and the equilibrium probabilities in $C(\theta_0)$. If all the sample observations come from the same equilibrium, then P_0 should be one of the points in $C(\theta_0)$. However, if the observations come from different equilibria in $C(\theta_0)$, then P_0 is a mixture of

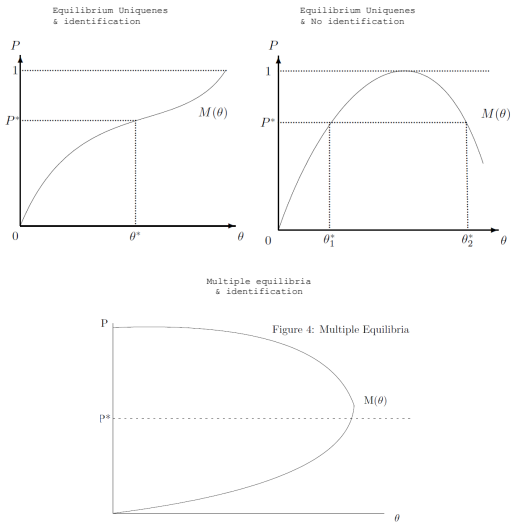


Figure 5.4: Multiple Equilibria versus Identification

the elements in $C(\theta_0)$. To obtain identification, we can assume that every observation in the sample comes from the same equilibrium. Under this condition, since P_0 is an equilibrium associated with θ_0, we know that $P_0 = \Phi(-1.8 + \theta_0 P_0)$. Given that Φ is an invertible function, we have that $\theta_0 = (\Phi^{-1}(P_0) + 1.8)/P_0$. Provided that P_0 is not zero, it is clear that θ_0 is point identified regardless of the existence of multiple equilibria in the model.

5.5.2 Unobserved market heterogeneity

Some market characteristics affecting firms' profits may not be observable to the researcher. For example, consider local attractions that spur the demand for hotels in a particular geographic location. Observing and controlling for all the relevant attractions are often impossible to the researcher. This demand effect implies that markets with such attractions should have more hotels than those without such attractions but with equivalent observable characteristics. Therefore, without accounting for this type of unobservables, researchers may wrongly conclude that competition boosts profits, or underestimate the negative effect of competition on profits.

Unobserved market heterogeneity usually appears as an additive term (ω_ℓ) in the firm's profit function ($\pi_{i\ell}$) where ω_ℓ is a random effect from a distribution known up to some parameters. The most common assumption (for example, Seim 2006; Zhu and Singh 2009; Datta and Sudhir 2013) is that these unobservables are common across locations in the same local market (that is, $\omega_\ell = \omega$ for all ℓ). Under this assumption, the magnitude of unobserved market heterogeneity matters in terms of whether the firm enters some location in this market, but not the location itself. Orhun (2013) relaxes this assumption by allowing unobserved heterogeneity to vary across locations in the same market.

In a game of complete information, accommodating unobserved market heterogeneity does not require a fundamental change in the estimation process. In a game of incomplete information, however, unobserved market heterogeneity introduces an additional challenge. Consistency of the two-step method requires that the initial nonparametric estimator of firms' entry probabilities in the first step should account for the presence of unobserved market heterogeneity. A possible solution is to use a finite mixture model. In this model, every market's ω_ℓ is drawn from a distribution with finite support. Aguirregabiria and Mira (2007) show how to accommodate such market-specific unobservables into their nested pseudo likelihood (NPL) algorithm. arcidiacono and Miller (2011) propose an expectation-maximization (EM) algorithm in a more general environment. An alternative way to deal with this problem is to use panel data with a reasonably long time horizon. In that way, we can incorporate market fixed effects as parameters to be estimated. This approach is popular when estimating a dynamic game (for example, Ryan 2012; Suzuki 2013). A necessary condition to implement this approach is that every market at least observes some entries during the sample period. Dropping markets with no entries from the sample may generate a selection bias.

5.5.3 Computation

The number of geographic locations, L, introduces two dimensionality problems in the computation of firms' best responses in games of entry with spatial competition. First, in a static game, a multi-store firm's set of possible actions includes all the possible spatial configurations of its store network. The number of alternatives in this set is equal to 2^L, and this number is extremely large even with modest values of L, such as a few hundred geographic locations. Without further assumptions, the computation of best responses becomes impractical. This is an important computational issue that has deterred some authors from accounting for multi-store retailers in their spatial competition models, for example, Seim (2006), or Zhu and Singh (2009), among many others. As we have described in section 2.2.5, two approaches that have been applied to deal with this issue are (1) to impose restrictions that guarantee supermodularity of the game (that is, only two players, no cannibalization effects), and (2) to avoid the exact computation of best responses and use instead inequality restrictions implied by these best responses.

Looking at the firms' decision problem as a sequential or dynamic problem helps also to deal with the dimensionality in the space of possible choices. In a given period of time (for example, year, quarter, or month), we typically observe that a retail chain makes small changes in its network of stores, that is, it opens only a few new stores, or closes only a few existing stores. Imposing these small changes as a restriction on the model implies a very dramatic reduction in the dimension of the action space such that the computation of best responses becomes practical, at least in a 'myopic' version of the sequential decision problem.

However, to fully take into account the sequential or dynamic nature of a firm's decision problem, we also need to acknowledge that firms are forward-looking. In the firm's dynamic programming problem, the set of possible states is equal to all the possible spatial configurations of a store network, and it has 2^L elements. Therefore, by going from a static model to a dynamic forward-looking model, we have just 'moved' the dimensionality problem from the action space into the state space. Recent papers propose different approaches to deal with this dimensionality problem in the state space.

arcidiacono et al. (2013) present a continuous-time dynamic game of spatial competition in a retail industry and propose an estimation method of this model. The continuous-time assumption eliminates the curse of dimensionality associated with integration over the state space. Aguirregabiria and Vicentini (2016) propose a method of spatial interpolation that exploits the information provided by the (indirect) variable profit function.

5.6 Further topics

Spillovers between different retail sectors. Existing applications of games of entry and spatial competition in retail markets concentrate on a single retail industry. However, there are also interesting spillover effects between different retail industries. Some of these spillovers are positive, such as good restaurants making a certain neighborhood more attractive for shopping. There are also negative spillovers effects through land prices. Retail sectors with high value per unit of space (for example, jewelry stores) are willing to pay higher land prices than supermarkets that have low markups and are intensive in the use of land. The consideration and measurement of these spillover effects are interesting in and of themselves, and they can help to explain the turnover and reallocation of industries in different parts of a city. Relatedly, endogenizing land prices would also open the possibility of using these models for the evaluation of specific public policies at the city level.

Richer datasets with store level information on prices, quantities, inventories. The identification and estimation of competition effects based mainly on data of store locations have been the rule more than the exception in this literature. This approach typically requires strong restrictions in the specification of demand and variable costs. The increasing availability of datasets with rich information on prices and quantities at the product and store level should create a new generation of empirical games of entry and spatial competition that relax these restrictions. Also, data on store characteristics such as product assortments or inventories will enable the introduction of these important decisions as endogenous variables in empirical models of competition between retail stores.

Measuring spatial pre-emption. So far, all the empirical approaches to measure the effects of spatial pre-emption are based on the comparison of firms' actual entry with firms' behavior in a counterfactual scenario characterized by a change in either (1) a structural parameter (for example, a store exit value), or (2) firms' beliefs (for example, a firm believes that other firms' entry decisions do not respond to this firm's entry behavior). These approaches suffer from the serious limitation in which they do not only capture the effect of pre-emption, but also other effects. The development of new approaches to measure the pure effect of pre-emption would be a methodological contribution with relevant implications in this literature.

Geography. Every local market is different in its shape and its road network. These differences may have important impacts on the resulting market structure. For example, the center of a local market may be a quite attractive location for retailers when all highways go through there. However, it may not be the case anymore when highways encircle the city center (for example, Beltway in Washington DC). These differences may affect retailers' location choices and the degree of competition in an equilibrium. The

development of empirical models of competition in retail markets that incorporate, in a systematic way, these idiosyncratic geographic features will be an important contribution in this literature.

6. Introduction to Dynamics

6.1 Introduction

Dynamics in demand and/or supply can be important aspects of competition in oligopoly markets. In many markets demand is dynamic in the sense that (a) consumers' current decisions affect their future utility, and (b) consumers' current decisions depend on expectations about the evolution of future prices (states). Some sources of dynamics in demand are consumer switching costs, habit formation, brand loyalty, learning, and storable or durable products. On the supply side, most firm investment decisions have implications on future profits. Some examples are market entry, investment in capacity, inventories, or equipment, or choice of product characteristics. Firms' production decisions also have dynamic implications if there is learning-by-doing. Similarly, menu costs, or other forms of price adjustment costs, imply that pricing decisions have dynamic effects.

Identifying the factors governing the dynamics is important to understanding competition and the evolution of market structure, and for the evaluation of public policy. To identify and understand these factors, we specify and estimate dynamic structural models of demand and supply in oligopoly industries. A dynamic structural model is a model of individual behavior where agents are forward looking and maximize expected intertemporal payoffs. The parameters are structural in the sense that they describe preferences and technological and institutional constraints. Under the *principle of revealed preference*, these parameters are estimated using longitudinal micro data on individuals' choices and outcomes over time.

We start with some examples and a brief discussion of applications of dynamic structural models of Industrial Organization. These examples illustrate why taking into account forward-looking behavior and dynamics in demand and supply is important for the empirical analysis of competition in oligopoly industries.

Example 1: Demand of storable goods

For a storable product, purchases in a given period (week, month) are not equal to consumption. When the price is low, consumers have incentives to buy a large amount to

store the product and consume it in the future. When the price is high, or the household has a large inventory of the product, the consumer does not buy, and instead consumes from her inventory. Dynamics arise because consumers' past purchases and consumption decisions impact their current inventory and therefore the benefits of purchasing today. Furthermore, consumers' expectations about future prices also impact the perceived trade-offs of buying today versus in the future.

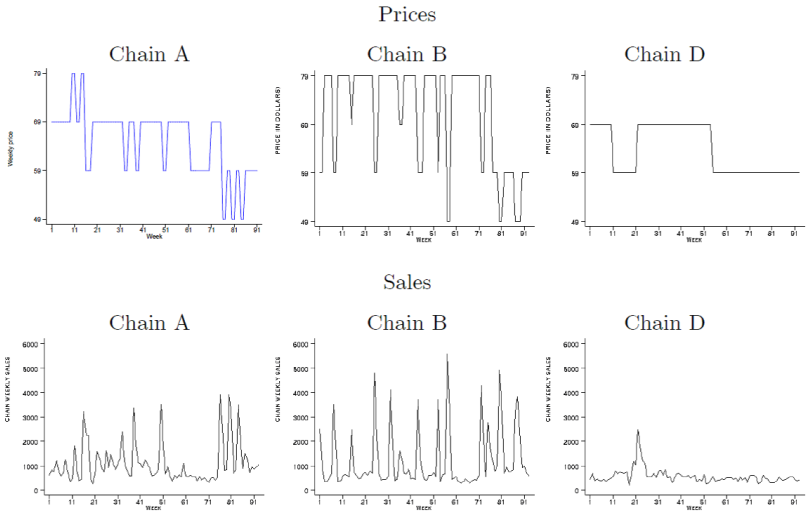


Figure 6.1: Price promotions and sales of a storable good

What are the implications of ignoring consumer dynamic behavior when we estimate the demand of differentiated storable products? An important implication is that we can get serious biases in the estimates of price demand elasticities. In particular, we can incorrectly interpret a short-run intertemporal substitution as a long-run substitution between brands (or stores).

To illustrate this issue, it is useful to consider a specific example. Figure 6.1 presents weekly times series data of prices and sales of canned tuna in a supermarket store. The time series of prices is characterized by "High-Low" pricing, which is quite common in many supermarkets. The price fluctuates between a high regular price and a low promotion price. The promotion price is infrequent and lasts only a few days, after which the price returns to its "regular" level. Sales of this storable product respond to this type of dynamics in prices.

As we can see in figure 6.1, most sales are concentrated in the very few days with low prices. Apparently, the short-run response of sales to these temporary price reductions is very large: the typical discount of a sales promotion is between 10% and 20%, and the increase in sales is around 300%. In a static demand model, this type of response would suggest that the price elasticity of demand of the product is very large. In particular, with these data, the estimation of a static demand model provides estimates of own-price elasticities greater than 8. The static model interprets the large response of sales to

a price reduction in terms of consumers' substitution between brands (and to some extent between supermarkets too). Based on these estimates of demand elasticities, a model of competition would imply that price-cost margins are very small and firms (both supermarkets and brand manufacturers) have very little market power. A large degree of substitution between brands implies that product differentiation is small and market power is low.

This interpretation ignores dynamics in consumer purchasing decisions, and can therefore be seriously wrong. Most of the short-run response of sales to a temporary price reduction is not substitution between brands or stores, but intertemporal substitution in households' purchases. The temporary price reduction induces consumers to buy and store today, and to buy less in the future. The long-run substitution effect is much smaller, and it is this long-run effect that is relevant in measuring firms' market power.

In order to distinguish between short-run and long-run responses to price changes, we need to specify and estimate a dynamic model of demand of differentiated products. In this type of models, consumers are forward looking and take into account their expectations about future prices as well as storage costs. Examples of applications estimating dynamic structural models of demand of differentiated storable products are Erdem, Imai, and Keane (2003), Hendel and Nevo (2006), and Wang (2015),

Example 2: Demand of a new durable product.

The price of a new durable product typically declines over time during the months after the introduction of the product. Different factors may explain this price decline, for instance, intertemporal price discrimination, increasing competition, exogenous cost decline, or endogenous cost decline due to learning-by-doing. As in the case of the "high-low" pricing of storable goods, explaining these pricing dynamics requires one to take into account dynamics in supply. For the moment, we concentrate here on demand. If consumers are forward-looking, they expect that the price will be lower in the future, and this generates an incentive to wait and buy the good in the future.

A static model that ignores dynamics in demand of durable goods can introduce two different types of biases in the estimates of the distribution of consumers' willingness to pay, and therefore of demand. The first source of bias comes from the failure to recognize that market size is declining over time. The demand curve moves downwards over time because high willingness-to-pay consumers but the product at early periods and leave the market. A second source of bias comes from ignoring consumers' forward-looking behavior. The estimation of a static model that ignores this forward-looking behavior implies that consumers' willingness-to-pay can be contaminated by consumers' willingness to wait because of the expectation of future lower prices.

We can illustrate the first source of bias using a simple example. Consider a market with an initial mass of 100 consumers and a uniform distribution of willingness to pay over the the unit interval. Consumers are myopic and buy the product if the price is below their willingness to pay. Once consumers buy the product they are out of the market forever. Time is discrete and indexed by $t \in \{1,2,...\}$. Every period t, the aggregate demand is:

$$q_t = s_t \ \Pr(v_t \geq p_t) \ = \ s_t \ [1 - F_t(p_t)] \tag{6.1}$$

where q_t and p_t are quantity and price, respectively, s_t is the mass of consumers who

remain in market at period t, and F_t is the distribution function of willingness to pay for consumers who remain in the market at period t.

Suppose that we observe a declining sequence of prices equal to $p_1 = 0.9$, $p_2 = 0.8$, $p_3 = 0.7$, and so on. Given this price sequence, it is easy to show that the demand curve changes over as follows:

$$\begin{aligned} \textit{Period } \text{t=1: } q_1 &= 100\ (1-p_1) \\ \textit{Period } \text{t=2: } q_2 &= 90\ \left(\tfrac{0.9-p_2}{0.9}\right) = 100\ (0.9-p_2) \\ \textit{Period } \text{t=3: } q_3 &= 80\ \left(\tfrac{0.8-p_2}{0.8}\right) = 100\ (0.8-p_3) \\ &\ldots \end{aligned} \tag{6.2}$$

Therefore, the sequence of realized quantities is constant over time: $q_1 = q_2 = q_3 = ... = 10$. A static demand model – with market size constant over time – leads the researcher to conclude that consumers are not sensitive to price, since the same quantity is sold as prices decline. The estimate of the price elasticity would be zero. This example illustrates how ignoring dynamics in demand of durable goods can lead to serious biases in the estimates of the price sensitivity of demand.

We can also illustrate the second source of bias – from ignoring consumer forward-looking behavior – using a simple variation of the previous example. Suppose that consumers are one-period forward-looking: they compare the utility of purchasing at period t versus waiting one period and purchase at $t+1$. Consumers have a time discount factor β and perfect foresight about next period price. Accordingly, a consumer with valuation v purchases the product at period t if $v - p_t \geq \beta(v - p_{t+1})$, or equivalently, if $v \geq (p_t - \beta p_{t+1})/(1-\beta)$. This consumer behavior implies the following aggregate demand at period t:

$$q_t = s_t \left[1 - F_t\left(\frac{p_t - \beta p_{t+1}}{1-\beta}\right)\right] \tag{6.3}$$

A static demand model that ignores forward-looking behavior estimates a demand equation $q_t = s_t\ [1 - F_t(p_t)]$, omitting next period price p_{t+1} and discount factor β. This misspecification can lead to substantial biases in the estimation of the probability distribution of consumer valuations.

Examples of empirical applications estimating dynamic structural models of demand of differentiated products are Esteban and Shum (2007), Carranza (2010), Gowrisankaran and Rysman (2012), and Melnikov (2013).

Example 3: Product repositioning in differentiated product markets.

A common assumption in static demand models of differentiated product is that product characteristics, other than prices, are exogenous. However, in many industries, product characteristics are important strategic variables. Firms modify these characteristics in response to changes in demand, costs, regulation, or mergers.

Ignoring the endogeneity of product characteristics has several implications. First, it can bias the estimated demand parameters. A dynamic game that acknowledges the endogeneity of some product characteristics and exploits the dynamic structure of the model to generate valid moment conditions can deal with this problem.

A second important limitation of a static model of firm behavior is that it cannot recover the costs of repositioning product characteristics. As a result, the static model cannot address important empirical questions such as the effect of a merger on product repositioning. That is, the evaluation of the effects of a merger using a static model should assume that the product characteristics (other than prices) of the new merging firm would remain the same as before the merger. This is at odds both with the predictions of theoretical models and with informal empirical evidence. Theoretical models of horizontal mergers show that product repositioning is an important source of value for a merging firm. Informal evidence shows that soon after a merger firms implement significant changes in their product portfolio.

Sweeting (2013) and Aguirregabiria and Ho (2012) are two examples of empirical applications that endogenize product attributes using a dynamic game of competition in a differentiated product industry. Sweeting (2013) estimates a dynamic game of oligopoly competition in the US commercial radio industry. The model endogenizes the choice of radio stations format (genre), and estimates product repositioning costs. Aguirregabiria and Ho (2012) propose and estimate a dynamic game of airline network competition where the number of direct connections that an airline has in an airport is an endogenous product characteristic.

Example 4: Dynamics of market structure

Ryan (2012) and Kasahara (2009) provide excellent examples of how ignoring supply-side dynamics and firms' forward-looking behavior can lead to misleading results.

Ryan (2012) studies the effects of the 1990 Amendments to the Clean Air Act on the US cement industry. This environmental regulation added new categories of regulated emissions, and introduced the requirement of an environmental certification that cement plants have to pass before starting their operation. Ryan estimates a dynamic game of competition where the sources of dynamics are sunk entry costs and adjustment costs associated with changes in installed capacity. The estimated model shows that the new regulation had negligible effects on variable production costs but it increased significantly the sunk cost of opening a new cement plant. A static analysis, that ignores the effects of the policy on firms' entry-exit decisions, would conclude that the regulation had negligible effects on firms profits and consumer welfare. In contrast, the dynamic analysis shows that the increase in sunk-entry costs caused a reduction in the number of plants, which in turn implied higher markups and a decline in consumer welfare.

Kasahara (2009) proposes and estimates a dynamic model of firm investment in equipment and uses it to evaluate the effect of an important increase in import tariffs in Chile during the 1980s. The increase in tariffs had a substantial effect on the price of imported equipment and it may have a significant effect on firms' investment. An important feature of this policy is that the government announced that it was a temporary increase and that tariffs would go back to their original levels after a few years. Kasahara shows that the temporary aspect of this policy exacerbated its negative effects on firm investment. Given that firms anticipated the future decline in both import tariffs and the price of capital, a significant fraction of firms decided not to invest and waited until the reduction of tariffs. This waiting and inaction would not appear if the policy change were perceived as permanent. Kasahara shows that the Chilean economy would have recovered faster from the economic crisis of 1982-83 if the increase in tariffs would

have been perceived as permanent.

Example 5: Dynamics of prices in a retail market

The significant cross-sectional dispersion of prices is a well-known stylized fact in retail markets. Retailing firms selling the same product, and operating in the same (narrowly defined) geographic market and at the same period of time, charge prices that differ by significant amounts, for instance, 10% price differentials or even larger. This empirical evidence has been well established for gas stations and supermarkets, among other retail industries.

Interestingly, the price differentials between firms, and the ranking of firms in terms of prices, have very low persistence over time. A gas station that charges a price 5% below the average in a given week may be charging a price 5% above the average the following week. Using a more graphical description, we can say that a firm's price follows a cyclical pattern, and the price cycles of the different firms in the market are not synchronized. Understanding price dispersion and the dynamics of price dispersion is very important to understand not only competition and market power but also the construction of price indexes.

Different explanations have been suggested to explain this empirical evidence. Some explanations have to do with dynamic pricing behavior or "state dependence" in prices.

For instance, one explanation is based on the relationship between firm inventory and optimal price. In many retail industries with storable products, we observe that firms' orders to suppliers are infrequent. For instance, for products such as laundry detergent, a supermarket's ordering frequency can be lower than one order per month. A simple and plausible explanation of this infrequency is that there are fixed or lump-sum costs of placing an order that do not depend on the size of the order, or at least that do not increase proportionally with the size of the order. Then, inventories follow a so called (S,s) cycle: inventories increase by a large amount up to a maximum threshold when an order is placed, and decline gradually until a minimum value is reached, at which time a new order is placed. Given these dynamics of inventories, it is simple to show that the optimal price of the firm should also follow a cycle. The price drops to a minimum when a new order is placed and then increases over time up to a maximum just before the next order when the price drops again.

Aguirregabiria (1999) shows this joint pattern of prices and inventories for many products in a supermarket chain. Specifically, I show that these types of inventory-dependence price dynamics can explain more than 20% of the time series variability of prices in the data.

6.2 Firms' investment decisions

Some important firm investment decisions are discrete or at the *extensive margin*. Market entry and exit, machine replacement, or adoption of a new technology are examples of discrete investment investment decisions. Starting with the seminal work by Pakes (1986) on patent renewal and Rust (1987) on machine replacement, models and methods for dynamic discrete choice structural models have been applied to study these investment decisions. In this section, we review models and applications that abstract from dynamic oligopoly competition or assume explicitly that firms operate in either competitive or monopolistic markets.

Let $a_{it} \in \mathscr{A} = \{0, 1, ..., J\}$ be the discrete variable that represents the investment decision of firm i at period t. The profit function is:

$$\Pi_{it} = p_{it}\, Y\left(a_{it}, k_{it}, \omega_{it}; \theta_y\right) - C\left(a_{it}, r_{it}; \theta_c\right) + \varepsilon_{it}(a_{it}) \tag{6.4}$$

p_{it} represents output price. The term $y_{it} = Y\left(a_{it}, k_{it}, \omega_{it}; \theta_y\right)$ is a production function that depends on investment a_{it}, predetermined capital stock k_{it}, total factor productivity ω_{it}, and the structural parameters θ_y. The term $C\left(a_{it}, r_{it}; \theta_c\right)$ is the investment cost, where r_{it} is the price of new capital and θ_c is a vector of structural parameters.

The vector of variables $\varepsilon_{it} = \{\varepsilon_{it}(a) : a \in \mathscr{A}\}$ represents a component of the investment cost that is unobservable to the researcher. These unobservables have mean zero and typically they are assumed $i.i.d.$ across plants and over time. Capital stock k_{it} depreciates at an exogenous rate δ and increases when new investments are made according to the standard transition rule,

$$k_{it+1} \;=\; (1-\delta)\,(k_{it} + a_{it}). \tag{6.5}$$

Total factor productivity ω_{it} and the price of capital r_{it} are exogenous state variables. They evolve over time according to first order Markov processes.

Every period t, the manager observes the state variables $x_{it} \equiv (k_{it}, p_{it}, r_{it}, \omega_{it})$, and ε_{it}, and decides her investment in order to maximize expected and discounted profits $\mathbb{E}_t\left(\sum_{j=0}^{\infty} \beta^j\, \Pi_{i,t+j}\right)$, where $\beta \in (0,1)$ is the discount factor. The optimal decision rule for investment is:

$$a_{it} = \arg\max_{a \in \mathscr{A}} \{\; v(a, x_{it}) + \varepsilon_{it}(a) \;\}, \tag{6.6}$$

where $v(a, x_{it})$ is the *conditional choice value function*, that is the unique solution to a Bellman equation that we describe in detail in section 6.2.2.

Given the distribution of the unobservables $\varepsilon_{it}\}$, the observable exogenous state variables, and the vector of structural parameters $\theta = (\theta_y, \theta_c, \delta)$, this model implies a probability for the observed path of output and investment of a firm, $\{y_{it}, a_{it} : t = 1, 2, ..., T\}$. A standard approach to estimate the parameters of this model is by (Conditional) Maximum Likelihood. Rust (1987) proposed the Nested Fixed Point algorithm (NFXP) for the computation of this estimator. Hotz and Miller (1993) propose a two-step *Conditional Choice Probabilities* (CCP) estimator that avoids computing a solution of the dynamic programming problem. Aguirregabiria and Mira (2002) propose the Nested Pseudo Likelihood algorithm (NPL) that is a recursive extension of the CCP method that returns the maximum likelihood estimates at a substantially lower computing time than NFXP. Section 6.2.3 describes in detail these estimation methods.

Rust (1987) developed this model and applied it to estimate the costs of bus engine replacement and maintenance in the Madison-Wisconsin metropolitan bus company. Since Rust's seminal work, this model has been applied to different datasets on discrete choice investment problems. Das (1992) studies the decision to operate, hold idle, or retire a kiln by plants in the U.S. cement industry. Kennet (1993) analyzes airlines' replacement decisions of aircraft engines and identifies significant changes in the decision rule after the deregulation of the US airline industry in 1978. Rust and Rothwell (1995) consider the refuelling decisions of nuclear power plants. They use their model to evaluate the impact on maintenance costs, profits, and firm behavior of a change in

the safety regulation by the US Nuclear Regulatory Commission. Cooper, Haltiwanger, and Power (1999) and Cooper and Haltiwanger (2006) show that this model provides provides better fit and more plausible expanation for the dynamics of investment in US manufacturing plants. Das, Roberts, and Tybout (2007) investigate why the decision to export by Colombian manufacturing plants is very persistent over time. The authors disentangle the effects of sunk costs, prior exporting experience, and serially correlated unobserved heterogeneity. Kasahara (2009) studies the effect of import tariffs on capital investment decisions by Chilean manufacturing plants. He shows that the temporary feature of a tariff increase in the mid-1980s exacerbated firms' zero-investment response. Rota (2004) and Aguirregabiria and Alonso-Borrego (2014) estimate dynamic discrete choice models of labor demand and use them to measure the magnitude of labor adjustment costs for permanent workers in Italy and Spain, respectively, and the effects of labor market reforms. Holmes (2011) studies the geographic expansion of Wal-Mart stores during the period 1971-2005. He estimates a dynamic model of entry and store location by a multi-store firm that incorporates economies of density and cannibalization between Wal-Mart stores. Holmes finds that Wal-Mart obtains large savings in distribution costs by having a dense store network.

6.2.1 Model

Suppose that we have panel data of N plants operating in the same industry with information on output, investment, and capital stock over T periods of time.

$$\text{Data} = \{\, y_{it}, a_{it}, k_{it} : i = 1,2,...,N \text{ and } t = 1,2,...,T \,\} \tag{6.7}$$

Suppose that the investment data is characterized by infrequent and lumpy investments. That is, a_{it} contains a large proportion of zeroes (no investment), and when investment is positive the investment-to-capital ratio a_{it}/k_{it} is quite large. For instance, for some industries and samples we can find that the proportion of zeroes is above 60% (even with annual data!) and the average investment-to-capital ratio conditional on positive investment is above 20%.

A possible explanation for this type of dynamics in firms' investment is that there are significant indivisibilities in the purchases of new capital, and fixed or lump-sum costs associated with purchasing and installing new capital. Machine replacement models are models of investment that emphasize the existence of these indivisibilities and lump-sum costs of investment.

The profit function has the structure in equation 6.4, and capital stock follows the transition rule in equation 6.5. A key feature in models of machine replacement is the indivisibility in the investment decision. In the standard machine replacement model, the firm decides between zero investment ($a_{it} = 0$) and the replacement of the old capital ($k_{it} = 0$) by a "new machine" that implies a fixed amount of new capital k^*. Therefore, $a_{it} \in \{0, \ , k^* - k_{it}\ \}$, and the transition rule for capital stock can be written as:

$$k_{it+1} = (1-\delta)\ [(1-a_{it})\ k_{it} + a_{it}\ k^*] \tag{6.8}$$

This implies that capital stock can take a finite number of possible values: $(1-\delta)k^*$ one period after replacement; $(1-\delta)k^*$ one period after replacement; $(1-\delta)^2 k^*$ two periods after replacement; ... $(1-\delta)^d k^*$ d periods after replacement. This implies the

following linear relationship between the logarithm of capital stock and variable d_{it} that represents the duration (or number of periods) since the last machine replacement by plant i, or the age of capital:

$$\ln(k_{it}) = \delta_0 + \delta_1 \, d_{it} \tag{6.9}$$

with $\delta_0 \equiv \ln(k^*)$ and $\delta_1 \equiv \ln(1-\delta)$. Therefore, capital stock k_{it} and age of capital d_{it} are equivalent state variables. The transition rule for the age of capital is by very simple: d_{it+1} is equal to 1 if there was replacement at period t, and it is equal to $d_{it}+1$ if there was not replacement. That is,

$$d_{it+1} \;=\; (1-a_{it}) \; d_{it} \; + 1 \tag{6.10}$$

These assumptions on the values of investment and capital seem natural in applications where the investment decision is actually a machine replacement decision, as in the papers by Rust (1987), Das (1992), Kennet (1994), or Rust and Rothwell (1995), among others. However, this framework may be restrictive when we look at less specific investment decisions, such as investment in equipment as in the papers byCooper, Haltiwanger, and Power (1999), Cooper and Haltiwanger (2006), and Kasahara (2009). In these other papers, investment in the data is very lumpy, which is an implication of a model of machine replacement, but firms in the sample use multiple types of equipment, have very different sizes and their capital stocks are very different. These papers consider that investment is either zero or a constant proportion of the installed capital, that is, $a_{it} \in \{0 \, , \, q \, k_{it}\}$ where q is a constant, for instance, $q = 25\%$. Here we maintain the most standard assumption of machine replacement models.

The production function is:

$$y_{it} = Y\left(a_{it}, k_{it}, \omega_{it}\right) = \exp\left\{\theta_0^y + \omega_{it}\right\} \; \left[k_{it} \; + a_{it} \; (k^* - k_{it})\right]^{\theta_1^y} \tag{6.11}$$

where θ_0^y and θ_1^y are parameters, and ω_{it} is log-TFP. The cost function has two components: the investment or replacement cost (when $a_{it} = 1$, which is equal to $\theta^r \, k^* + \varepsilon_{it}$; and the maintenance cost (when $a_{it} = 0$), that increases with the age of capital (or equivalently, declines with the capital stock) according to the function $\theta_1^m \, k_{it} + \theta_2^m \, k_{it}^2$. heterogeneity in replacement costs. Therefore, the profit function is:

$$\Pi_{it} = \begin{cases} p_{it} \; \exp\left\{\theta_0^y + \omega_{it}\right\} \; \left[k_{it}\right]^{\theta_1^y} - \theta_1^m \, k_{it} + \theta_2^m \, k_{it}^2 & \text{if} \quad a_{it} = 0 \\ \\ p_{it} \; \exp\left\{\theta_0^y + \omega_{it}\right\} \; \left[k^*\right]^{\theta_1^y} - \theta^r \, k^* - \varepsilon_{it} & \text{if} \quad a_{it} = 1 \end{cases} \tag{6.12}$$

Every period t, the firm observes the state variables $s_{it} = (p_{it}, k_{it}, \omega_{it}, \varepsilon_{it})$ and then it decides its investment in order to maximize its expected value:

$$\mathbb{E}_t \left(\sum_{j=0}^{\infty} \beta^j \, \Pi_{i,t+j} \right) \tag{6.13}$$

where $\beta \in (0,1)$ is the discount factor. The main trade-off in this machine replacement decision is simple. On the one hand, the productivity/efficiency of a machine declines over time and therefore the firm prefers younger machines. However, using younger machines requires frequent replacement and replacing a machine is costly.

The firm has uncertainty about future realizations of output price, log-TFP, and the stochastic component of the replacement cost. To complete the model we have to specify the stochastic processes of these state variables. Output price and log-TFP follow a Markov process with transition density function $f_{p\omega}(p_{it+1}, \omega_{it+1}|p_{it}, \omega_{it})$. The the shock in replacement cost ε_{it} is i.i.d. with density function f_ε.

Let $V(s_{it}, \varepsilon_{it})$ be the value function. This value function is the solution to the *Bellman equation*:

$$V(s_{it}) = \max_{a_{it}\in\{0,1\}} \left\{ \Pi(a_{it}, s_{it}) + \beta \int V(s_{it+1}) f_s(s_{it+1}|a_{it}, s_{it})\, ds_{it+1} \right\} \tag{6.14}$$

where $f_s(s_{it+1}|a_{it}, s_{it})$ is the transition probability of the vector of state variables, that based on the previous assumptions has the following form:

$$f_s(s_{it+1}|a_{it}, s_{it}) = 1\{d_{it+1} = (1 - a_{it})\, d_{it} + 1\}\ f_{p\omega}(p_{it+1}, \omega_{it+1}|p_{it}, \omega_{it})\ f_\varepsilon(\varepsilon_{it+1}) \tag{6.15}$$

where $1\{.\}$ is the indicator function. We can also represent the Bellman equation as:

$$V(s_{it}) = \max\{\ v(0, x_{it})\ ;\ v(1, x_{it}) - \varepsilon_{it}\ \} \tag{6.16}$$

where $x_{it} = (p_{it}, k_{it}, \omega_{it})$ and $v(0, x_{it})$ and $v(1, x_{it})$ are the *conditional choice value functions*. For $a \in \{0, 1\}$:

$$v(a, x_{it}) \equiv \pi(a_{it}, x_{it}) + \beta \int V(s_{it+1})\ f_s(s_{it+1}|a_{it}, s_{it})\ ds_{it+1}, \tag{6.17}$$

and $\pi(a_{it}, x_{it})$ being the part of the profit function that does not depend on $varepsilon_{it}$.

Given the value function, $V(.)$, the optimal decision rule gives us the optimal investment decision as a function of the state variables. We use $\alpha(.)$ to represent the optimal decision rule such that $a_{it} = \alpha(s_{it})$. Given the infinite horizon of the model and the time-homogeneity of the profit function and the transition probability functions, Blackwell's Theorem establishes that the value function and the optimal decision rule are time-invariant (Blackwell, 1965).

We use the vector θ to represent all the parameters in this model. It is convenient to distinguish several components in this vector, $\theta = (\theta_\pi, \theta_{f_x}, \theta_\varepsilon, \beta)$, where: θ_π contains the parameters in the profit function, θ_0^y, θ_1^y, θ_r, θ_1^m, and θ_0^y; θ_{f_x} contains the parameters in the transition probability function of the x state variables; and θ_ε contains the parameters in the distribution of ε.

It is straightforward to extend this binary choice model to a multinomial choice model. That is, the investment variable a_{it} can take a $J+1$ possible values $0, 1, 2, ..., J$. The profit function, the transition rule of the capital stock, the value function, the Bellman equation, and the conditional choice value functions have the same structure. We only extend the definition of the stochastic component of the investment cost to include J elements: $\varepsilon_{it} = (\varepsilon_{it}(1).\varepsilon_{it}(2), ..., \varepsilon_{it}(J))$.

6.2.2 Solving the dynamic programming problem

The Bellman equation is a functional equation that maps value functions into value functions. That is, given a value function $V : \mathbb{S} \to \mathbb{R}$, the right-hand side of the Bellman

equation provides a new value function, say V'. We represent the right-hand side of the Bellman equation for state s as $\Gamma(V)(s)$, and we use $\Gamma(V)$ to represent the mapping for every value of s, that is, $\Gamma(V) \equiv \{\Gamma(V)(s) : s \in \mathbb{S}\}$. Using this notation, we can present the Bellman equation in a compact form as the following functional equation:

$$V = \Gamma(V) \tag{6.18}$$

Mapping Γ is denoted the *Bellman operator* or *Bellman mapping*. Therefore, the value function V is a fixed point of the Bellman mapping.

Provided $\beta \in (0,1)$, the Bellman mapping is a contraction. That is, for any two functions V and V', the distance between $\Gamma(V')$ and $\Gamma(V)$ is smaller that the distance between V' and V. More precesily,

$$\|V' - V\| \leq \|\Gamma(V') - \Gamma(V)\| \tag{6.19}$$

where the distance operator $\|.\|$ is the supremum or L_∞ distance, and it is defined as $\|f\| = max_{s\in\mathbb{S}}|f(s)|$.

By the *contraction mapping Theoren* (Banach, 1922), this contraction property of the Bellman operator has two important implications for the solution of the dynamic programming problem. First, the solution to the Bellman equation – that is, the value function V and the corresponding optimal decision rule α – is unique. And second, this solution can be obtained using successive iterations in the Bellman operator. That is, given an arbitrary initial value for the value function, say V^0, at every iteration $n \geq 1$, we obtain a new value function as:

$$V^{n+1} = \Gamma(V^{n+1}) \tag{6.20}$$

Regardless of the initial value V^0, this algorithm always converges to the unique solution: $\lim_{n\to\infty} V^n = V$.

////////////////////////// *** REVISADO HASTA AQUI *** //////////////////////////

*** Explain challenges in solution of DP problem with continuous variables: Curse of dimensionality; infinite dimension; and numerical integration. *** Explain three tricks in the literature to deal with these problems: (1) x variables are discrete; (2) use the integrated Bellman equation (integrated over epsilon); and (3) consider a distribution of epsilon that implies a closed form expression for the integrated Bellman operator. //////////////////////////

For given values of structural parameters and functions, $\{\alpha_0, \alpha_1, r, f_C, \sigma_\varepsilon\}$, and of the individual effects η_i^Y and η_i^C, we can solve the DP problem of firm i by simply using successive approximations to the value function, that is, iterations in the Bellman equation.

In models where some of the state variables are not serially correlated, it is computationally very convenient (and also convenient for the estimation of the model) to define versions of the value function and the Bellman equation that are integrated over the non-serially correlated variables. In our model, ε is not serially correlated. The *integrated value function* of firm i is:

$$\bar{V}_i(K_{it}, C_t) \equiv \int V_i(K_{it}, C_t, \varepsilon_{it})\, df_\varepsilon(\varepsilon_{it})$$

And the integrated Bellman equation is:

$$\bar{V}_i(K_{it}, C_t) = \int \max\{\ v_i(0; K_{it}, C_t)\ ;\ v_i(1; K_{it}, C_t) - \varepsilon_{it}\ \}\, df_\varepsilon(\varepsilon_{it})$$

The main advantage of using the integrated value function is that it has a lower dimensionality than the original value function.

Given the extreme value distribution of ε_{it}, the integrated Bellman equation is:

$$\bar{V}_i(K_{it}, C_t) = \sigma_\varepsilon\ \ln\left[\exp\left\{\frac{v_i(0; K_{it}, C_t)}{\sigma_\varepsilon}\right\} + \exp\left\{\frac{v_i(1; K_{it}, C_t)}{\sigma_\varepsilon}\right\}\right]$$

where

$$v_i(0; K_{it}, C_t) \equiv \exp\left\{\alpha_0 + \eta_i^Y\right\}\ K_{it}^{\alpha_1} + \beta \int \bar{V}_i((1-\delta)K_{it}, C_{t+1})\ f_C(C_{t+1}|C_t)$$

$$v_i(1; K_{it}, C_t) \equiv \exp\left\{\alpha_0 + \eta_i^Y\right\}\ K_{it}^{\alpha_1} - C_t\ I^* - r(K_{it}) - \eta_i^C + \beta \int \bar{V}_i((1-\delta)K^*, C_{t+1})\ f_C(C_{t+1}|C_t)$$

The optimal decision rule of this dynamic programming (DP) problem is:

$$a_{it} = 1\{\ \varepsilon_{it} \leq v_i(1; K_{it}, C_t) - v_i(0; K_{it}, C_t)\ \}$$

Suppose that the price of new capital, C_t, has a discrete and finite range of variation: $C_t \in \{c^1, c^2, ..., c^L\}$. Then, the value function $\bar{V}_i$ can be represented as a $M \times 1$ vector in the Euclidean space, where $M = T * L$ and the T is the number of possible values for the capital stock. Let $\mathbf{V}_i$ be that vector. The integrated Bellman equation in matrix form is:

$$\mathbf{V}_i = \sigma_\varepsilon \ln\left(\exp\left\{\frac{\Pi_i(0) + \beta\ \mathbf{F}(0)\ \mathbf{V}_i}{\sigma_\varepsilon}\right\} + \exp\left\{\frac{\Pi_i(1) + \beta\ \mathbf{F}(1)\ \mathbf{V}_i}{\sigma_\varepsilon}\right\}\right)$$

where $\Pi_i(0)$ and $\Pi_i(1)$ are the $M \times 1$ vectors of one-period profits when $a_{it} = 0$ and $a_{it} = 1$, respectively. $\mathbf{F}(0)$ and $\mathbf{F}(1)$ are $M \times M$ transition probability matrices of (K_{it}, C_t) conditional on $a_{it} = 0$ and $a_{it} = 1$, respectively.

Given this equation, the vector $\mathbf{V}_i$ can be obtained by using value function iterations in the Bellman equation. Let $\mathbf{V}_i^0$ be an arbitrary initial value for the vector $\mathbf{V}_i$. For instance, $\mathbf{V}_i^0$ could be a $M \times 1$ vector of zeroes. Then, at iteration $k = 1, 2, ...$ we obtain:

$$\mathbf{V}_i^k = \sigma_\varepsilon \ln\left(\exp\left\{\frac{\Pi_i(0) + \beta\ \mathbf{F}(0)\ \mathbf{V}_i^{k-1}}{\sigma_\varepsilon}\right\} + \exp\left\{\frac{\Pi_i(1) + \beta\ \mathbf{F}(1)\ \mathbf{V}_i^{k-1}}{\sigma_\varepsilon}\right\}\right)$$

Since the (integrated) Bellman equation is a contraction mapping, this algorithm always converges (regardless of the initial $\mathbf{V}_i^0$) and it converges to the unique fixed point. Exact convergence requires infinite iterations. Therefore, we stop the algorithm when the distance (for instance, Euclidean distance) between $\mathbf{V}_i^k$ and $\mathbf{V}_i^{k-1}$ is smaller than some small constant, for instance, 10^{-6}.

An alternative algorithm to solve the DP problem is the **Policy Iteration algorithm**. Define the *Conditional Choice Probability (CCP) function* $P_i(K_{it}, C_t)$ as:

$$\begin{aligned} P_i(K_{it}, C_t) &\equiv \Pr(\ \varepsilon_{it} \leq v_i(1; K_{it}, C_t) - v_i(0; K_{it}, C_t)\ ) \\ &= \frac{\exp\left\{\dfrac{v_i(1; K_{it}, C_t) - v_i(0; K_{it}, C_t)}{\sigma_\varepsilon}\right\}}{1 + \exp\left\{\dfrac{v_i(1; K_{it}, C_t) - v_i(0; K_{it}, C_t)}{\sigma_\varepsilon}\right\}} \end{aligned}$$

Given that (K_{it}, C_t) are discrete variables, we can describe the CCP function P_i as a $M \times 1$ vector of probabilities $\mathbf{P}_i$. The expression for the CCP in vector form is:

$$\mathbf{P}_i = \frac{\exp\left\{\frac{\Pi_i(1)-\Pi_i(0)+\beta\ [\mathbf{F}(1)-\mathbf{F}(0)]\ \mathbf{V}_i}{\sigma_\varepsilon}\right\}}{1+\exp\left\{\frac{\Pi_i(1)-\Pi_i(0)+\beta\ [\mathbf{F}(1)-\mathbf{F}(0)]\ \mathbf{V}_i}{\sigma_\varepsilon}\right\}}$$

Suppose that the firm behaves according to the probabilities in $\mathbf{P}_i$. Let $\mathbf{V}_i^{\mathbf{P}}$ be the vector of values if the firm behaves according to $\mathbf{P}$. That is $\mathbf{V}_i^{\mathbf{P}}$ is the expected discounted sum of current and future profits if the firm behaves according to $\mathbf{P}_i$. Ignoring for the moment the expected future $\varepsilon's$, we have that:

$$\mathbf{V}_i^{\mathbf{P}} = (1-\mathbf{P}_i) * \left[\Pi_i(0) + \beta\ \mathbf{F}(0)\mathbf{V}_i^{\mathbf{P}}\right] + \mathbf{P}_i * \left[\Pi_i(1) + \beta\ \mathbf{F}(1)\mathbf{V}_i^{\mathbf{P}}\right]$$

And solving for $\mathbf{V}_i^{\mathbf{P}}$:

$$\mathbf{V}_i^{\mathbf{P}} = \left(I - \beta\ \mathbf{F}_i^{\mathbf{P}}\right)^{-1} ((1-\mathbf{P}_i) * \Pi_i(0) + \mathbf{P}_i * \Pi_i(1))$$

where $\mathbf{F}_i^{\mathbf{P}} = (1-\mathbf{P}_i) * \mathbf{F}(0) + \mathbf{P}_i * \mathbf{F}(1)$.

Taking into account this expression for $\mathbf{V}_i^{\mathbf{P}}$, we have that the optimal CCP $\mathbf{P}_i$ is such that:

$$\mathbf{P}_i = \frac{\exp\left\{\frac{\tilde{\Pi}_i + \beta\ \tilde{\mathbf{F}}\ \left(I-\beta\ \mathbf{F}_i^{\mathbf{P}}\right)^{-1}((1-\mathbf{P}_i)*\Pi_i(0)+\mathbf{P}_i*\Pi_i(1))}{\sigma_\varepsilon}\right\}}{1+\exp\left\{\frac{\tilde{\Pi}_i + \beta\ \tilde{\mathbf{F}}\ \left(I-\beta\ \mathbf{F}_i^{\mathbf{P}}\right)^{-1}((1-\mathbf{P}_i)*\Pi_i(0)+\mathbf{P}_i*\Pi_i(1))}{\sigma_\varepsilon}\right\}}$$

where $\tilde{\Pi}_i \equiv \Pi_{\mathbf{i}}(1) - \Pi_i(0)$, and $\tilde{\mathbf{F}} \equiv \mathbf{F}(1) - \mathbf{F}(0)$. This equation defines a fixed point mapping in $\mathbf{P}_i$. This fixed point mapping is called the Policy Iteration mapping. This is also a contraction mapping. The optimal $\mathbf{P}_i$ is its unique fixed point.

Therefore we compute $\mathbf{P}_i$ by iterating in this mapping. Let $\mathbf{P}_i^0$ be an arbitrary initial value for the vector $\mathbf{P}_i$. For instance, $\mathbf{P}_i^0$ could be a $M \times 1$ vector of zeroes. Then, at each iteration $k = 1, 2, ...$ we do the following two-step procedure:

Valuation step:

$$\mathbf{V}_i^k = \left(I - \beta\ \mathbf{F}^{\mathbf{P}_i^{k-1}}\right)^{-1} \left((1-\mathbf{P}_i^{k-1}) * \Pi_i(0) + \mathbf{P}_i^{k-1} * \Pi_i(1)\right)$$

Policy step:

$$\mathbf{P}_i^k = \frac{\exp\left\{\frac{\tilde{\Pi}_i + \beta\ \tilde{\mathbf{F}}\ \mathbf{V}_i^k}{\sigma_\varepsilon}\right\}}{1+\exp\left\{\frac{\tilde{\Pi}_i + \beta\ \tilde{\mathbf{F}}\ \mathbf{V}_i^k}{\sigma_\varepsilon}\right\}}$$

Policy iterations are more costly than Value function iterations (especially due to the matrix inversion in the valuation step). However, the policy iteration algorithm requires a much lower number of iterations, especially with β is close to one. Rust (1987,1994) proposes a hybrid algorithm: start with a few value function iterations and then switch to policy iterations.

6.2.3 Estimation

The primitives of the model are: (a) The parameters in the production function; (b) the replacement cost function r; (c) the probability distribution of firm heterogeneity F_η; (d) the dispersion parameter σ_ε; and (e) the discount factor β. Let θ represent the vector of structural parameters. We are interested in the estimation of θ.

Here we describe the Maximum Likelihood estimation of these parameters. Conditional on the observed history of the price of capital and on the initial condition for the capital stock, we have that:

$$\Pr(Data \mid C, K_{i1}, \theta) = \prod_{i=1}^{N} \Pr(a_{i1},Y_{i1},..., a_{iT},Y_{iT} \mid C, K_{i1}, \theta)$$

The probability $\Pr(a_{i1},Y_{i1},..., a_{iT},Y_{iT} \mid C, K_{i1}, \theta)$ is the contribution of firm i to the likelihood function. Conditional on the individual heterogeneity, $\eta_i \equiv (\eta_i^Y, \eta_i^C)$, we have that:

$$\begin{aligned}\Pr(a_{i1},Y_{i1},..., a_{iT},Y_{iT} \mid C, K_{i1}, \eta_i, \theta) &= \prod_{t=1}^{T} \Pr(a_{it},Y_{it} \mid C_t, K_{it}, \eta_i, \theta) \\ &= \prod_{t=1}^{T} \Pr(Y_{it} \mid a_{it}, C_t, K_{it}, \eta_i, \theta) \Pr(a_{it} \mid C_t, K_{it}, \eta_i, \theta)\end{aligned}$$

where $\Pr(a_{it} \mid C_t, K_{it}, \eta_i, \theta)$ is the CCP function:

$$\Pr(a_{it} \mid C_t, K_{it}, \eta_i, \theta) = P_i(K_{it},C_t, \theta)^{a_{it}} \left[1 - P_i(K_{it},C_t, \theta)\right]^{1-a_{it}}$$

and $\Pr(Y_{it} \mid a_{it}, C_t, K_{it}, \eta_i, \theta)$ comes from the production function, $Y_{it} = \exp\{\alpha_0 + \eta_i^Y\} [(1-a_{it}) K_{it} + a_{it} K^*]^{\alpha_1}$. In logarithms, the production function is:

$$\ln Y_{it} = \alpha_0 + \alpha_1 (1-a_{it}) \ln K_{it} + \kappa a_{it} + \eta_i^Y + e_{it}$$

where κ is a parameter that represents $\alpha_1 \ln K^*$, and e_{it} is a measurement error in output, that we assume i.i.d. $N(0, \sigma_e^2)$, and independent of ε_{it}. Therefore,

$$\Pr(Y_{it} \mid a_{it}, C_t, K_{it}, \eta_i, \theta) = \phi\left(\frac{\ln Y_{it} - \alpha_0 - \alpha_1 (1-a_{it}) \ln K_{it} - \kappa a_{it} - \eta_i^Y}{\sigma_e}\right)$$

where $\phi(.)$ is the density function of the standard normal distribution.

Putting all these pieces together, we have that the log-likelihood function of the model is $\ell(\theta) = \sum_{i=1}^{N} \ell_i(\theta)$ where $\ell_i(\theta) \equiv \ln \Pr(a_{i1},Y_{i1},..., a_{iT},Y_{iT} \mid C, K_{i1}, \theta)$ such that:

$$\ell_i(\theta) = \ln\left(\sum_{\eta \in \Omega} F_\eta(\eta) \left[\begin{array}{c} \prod_{t=1}^{T} \phi\left(\frac{\ln Y_{it} - \alpha_0 - \alpha_1 (1-a_{it}) \ln K_{it} - \kappa a_{it} - \eta^Y}{\sigma_e}\right) \\ \\ P_i(K_{it},C_t, \eta, \theta)^{a_{it}} \left[1 - P_i(K_{it},C_t, \eta, \theta)\right]^{1-a_{it}} \end{array}\right]\right) \quad (6.21)$$

Given this likelihood, we can estimate the parameter vector using Maximum Likelihood (ML).

The NFXP algorithm is a gradient iterative search method to obtain the MLE of the structural parameters.

This algorithm nests a BHHH method (outer algorithm), that searches for a root of the likelihood equations, with a value function or policy iteration method (inner algorithm), that solves the DP problem for each trial value of the structural parameters. The algorithm is initialized with an arbitrary vector $\hat{\theta}_0$.

A BHHH iteration is defined as:

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \left(\sum_{i=1}^{N} \nabla \ell_i(\hat{\theta}_k) \nabla \ell_i(\hat{\theta}_k)' \right)^{-1} \left(\sum_{i=1}^{N} \nabla \ell_i(\hat{\theta}_k) \right)$$

where $\nabla \ell_i(\theta)$ is the gradient in θ of the log-likelihood function for individual i. In a partial likelihood context, the score $\nabla \ell_i(\theta)$ is:

$$\nabla \ell_i(\theta) = \sum_{t=1}^{T_i} \nabla \log P(a_{it}|x_{it}, \theta)$$

To obtain this score we have to solve the DP problem.

In our machine replacement model:

$$\ell(\theta) \quad = \quad \sum_{i=1}^{N} \sum_{t=1}^{T_i} a_{it} \ \log P(x_{it}, \theta) + (1 - a_{it}) \ \log(1 - P(x_{it}, \theta))$$

with:

$$\mathbf{P}(\theta) \quad = \quad F_{\tilde{\varepsilon}} \left(\begin{array}{c} [\theta_{Y0} + \theta_{Y1}\mathbf{X} + \beta \ \mathbf{F}_x(0)\mathbf{V}(\theta)] \\ -[\theta_{Y0} - \theta_{R0} - \theta_{Y1}\mathbf{X} + \beta \ \mathbf{F}_x(1)\mathbf{V}(\theta)] \end{array} \right)$$

The NFXP algorithm works as follows. At each iteration we can distinguish three main tasks or steps.

Step 1: Inner iteration: DP solution. Given $\hat{\theta}_0$, we obtain the vector $\bar{\mathbf{V}}(\hat{\theta}_0)$ by using either successive iterations or policy iterations.
Step 2: Construction of scores. Then, given $\hat{\theta}_0$ and $\bar{\mathbf{V}}(\hat{\theta}_0)$ we construct the choice probabilities

$$\mathbf{P}(\hat{\theta}_0) \quad = \quad F_{\tilde{\varepsilon}} \left(\begin{array}{c} \left[\theta_{Y0} + \theta_{Y1}\mathbf{X} + \beta \ \mathbf{F}_x(0)\mathbf{V}(\hat{\theta}_0)\right] \\ -\left[\theta_{Y0} - \theta_{R0} - \theta_{Y1}\mathbf{X} + \beta \ \mathbf{F}_x(1)\mathbf{V}(\hat{\theta}_0)\right] \end{array} \right)$$

the Jacobian $\dfrac{\partial \bar{\mathbf{V}}(\hat{\theta}_0)'}{\partial \theta}$ and the scores $\nabla \ell_i(\hat{\theta}_0)$
Step 3: BHHH iteration. We use the scores $\nabla \ell_i(\hat{\theta}_0)$ to make a new BHHH iteration in order to obtain $\hat{\theta}_1$.

$$\hat{\theta}_1 = \hat{\theta}_0 + \left(\sum_{i=1}^{N} \nabla \ell_i(\hat{\theta}_0) \nabla \ell_i(\hat{\theta}_0)' \right) \left(\sum_{i=1}^{N} \nabla \ell_i(\hat{\theta}_0) \right)$$

Then, we replace $\hat{\theta}_0$ by $\hat{\theta}_1$ and go back to step 1.
* We repeat steps 1 to 3 until convergence: that is, until the distance between $\hat{\theta}_1$ and $\hat{\theta}_0$ is smaller than a pre-specified convergence constant.

The main advantages of the NFXP algorithm are its conceptual simplicity and, more importantly, that it provides the MLE which is the most efficient estimator asymptotically under the assumptions of the model.

The main limitation of this algorithm is its computational cost. In particular, the DP problem should be solved for each trial value of the structural parameters.

6.3 Patent Renewal Models

• **What is the value of a patent? How to measure it?**
• The valuation of patents is very important for: merger and acquisition decisions; using patents as collateral for loans; value of innovations; value of patent protection.
• Very few patents are traded, and there is substantial selection. We cannot use a "hedonic" approach.
• The number of citations of a patent is a very imperfect measure of patent value.
• Multiple patents are used in the production of multiple products, and in generating new patents. A "production function approach" seems also unfeasible.

6.3.1 Pakes (1986)

• Pakes (1986) proposes to use information on patent renewal fees together with a *Revealed Preference approach* to estimate the value of a patent.
• Every year, a patent holder should pay a renewal fee to keep her patent.
• If the patent holder decides to renew, it is because her expected value of holding the patent is greater than the renewal fee (that is publicly known).
• Therefore, observed decisions on patent renewal / non renewal contain information on the value of a patent.

Model: Basic Framework
• Consider a patent holder who has to decide whether to renew her patent or not. We index patents by i.
• This decision should be taken at ages $t = 1, 2, ..., T$ where $T < \infty$ is the regulated term of a patent (for instance, 20 years in US, Europe, or Canada).
• Patent regulation also establishes a sequence of **Maintenance Fees** $\{c_t : t = 1, 2, ..., T\}$. This sequence of renewal fees is deterministic such that a patent owner knows with certainty future renewal fees.
• The schedule $\{c_t : t = 1, 2, ..., T\}$ is typically increasing in patent age t, and it may increase from a few hundred dollars to a few thousand dollars.

• A patent generates a sequence of profits $\{\pi_{it} : t = 1, 2, ..., T\}$.
• At age t, a patent holder knows current profit π_{it} but has uncertainty about future profits $\pi_{i,t+1}$, $\pi_{i,t+2}$, ...
• The evolution of profits depends on the following factors:
(1) the initial "quality" of the idea/patent;

(2) innovations (new patents) which are substitutes for the patent and therefore, depreciate its value or even make it obsolete;

(3) innovations (new patents) which are complements of the patent and therefore, increase its value.

Stochastic process of patent profits

• Pakes proposes the following stochastic process for profits, that tries to capture the three forces mentioned above.

• A patent profit at the first period is a random draw from a log-normal distribution with parameters μ_1 and σ_1:

$$\ln(\pi_{i1}) \sim N(\mu_1, \sigma_1^2)$$

• After the first year, profit evolves according to the following formula:

$$\pi_{i,t+1} = \tau_{i,t+1} \max\left\{ \delta \pi_{it} ; \xi_{i,t+1} \right\}$$

• $\delta \in (0,1)$ is the depreciation rate. In the absence of unexpected shocks, the value of the patent depreciates according to the rule: $\pi_{i,t+1} = \delta \pi_{it}$.

• $\tau_{i,t+1} \in \{0,1\}$ is a binary variable that represents the patent becoming obsolete (that is, zero value) due to competing innovations. The probability of this event is a decreasing function of profit at the previous year:

$$\Pr(\tau_{i,t+1} = 0 \mid \pi_{it}, t) = \exp\{-\lambda \pi_{it}\}$$

• The larger the profit of the patent at age t, the smaller the probability of it becoming obsolete.

• Variable $\xi_{i,t+1}$ represents innovations which are complements of the patent and increase its profitability.

• $\xi_{i,t+1}$ has an exponential distribution with mean γ and standard deviation $\phi^t \sigma$:

$$p(\xi_{i,t+1} \mid \pi_{it}, t) = \frac{1}{\phi^t \sigma} \exp\left\{ -\frac{\gamma + \xi_{i,t+1}}{\phi^t \sigma} \right\}$$

• If $\phi < 1$, the variance of $\xi_{i,t+1}$ declines over time (and the $\mathbb{E}(\max\left\{ x ; \xi_{i,t+1} \right\})$ value declines as well).

• If $\phi > 1$, the variance of $\xi_{i,t+1}$ increases over time (and the $\mathbb{E}(\max\left\{ x ; \xi_{i,t+1} \right\})$ value increases as well).

• Under this specification, profits $\{\pi_{it}\}$ follow a non-homogeneous Markov process with initial density $\pi_{i1} \sim \ln N(\mu_1, \sigma_1^2)$, and transition density function:

$$f_\varepsilon(\pi_{it+1} | \pi_{it}, t) = \begin{cases} \exp\{-\lambda \pi_{it}\} & \text{if} \quad \pi_{it+1} = 0 \\ \Pr(\xi_{it+1} < \delta \pi_{it} \mid \pi_{it}, t) & \text{if} \quad \pi_{it+1} = \delta \pi_{it} \\ \dfrac{1}{\phi^t \sigma} \exp\left\{ -\dfrac{\gamma + \pi_{it+1}}{\phi^t \sigma} \right\} & \text{if} \quad \pi_{it+1} > \delta \pi_{it} \end{cases}$$

• The vector of structural parameters is $\theta = (\lambda, \delta, \gamma, \phi, \sigma, \mu_1, \sigma_1)$.

Model: Dynamic Decision Model

• $V_t(\pi)$ is the value of an active patent of age t and current profit π.
• Let $a_{it} \in \{0,1\}$ be the decision variable that represents the event "the patent owner decides to renew the patent at age t".
• The value function is implicitly defined by the Bellman equation:

$$V_t(\pi_{it}) = \max\left\{ 0 \; ; \; \pi_{it} - c_t + \beta \int V_{t+1}(\pi_{i,t+1}) \; f_\varepsilon(d\pi_{i,t+1} \mid \pi_{it}, t) \right\}$$

with $V_t(\pi_{it}) = 0$ for any $t \geq T+1$.
• The value of not renewal ($a_{it} = 0$) is zero. The value of renewal ($a_{it} = 1$) is the current profit $\pi_{it} - c_t$ plus the expected and discounted future value.

Model: Solution (Backwards induction)
• We can use backwards induction to solve for the sequence of value functions $\{V_t\}$ and optimal decision rules $\{\alpha_t\}$:
• Starting at age $t = T$, for any profit π:

$$V_T(\pi) = \max\{ 0 \; ; \; \pi - c_T \}$$

and

$$\alpha_T(\pi) = 1\{ \; \pi - c_T \geq 0 \; \}$$

• Then, for age $t < T$, and for any profit π:

$$V_t(\pi) = \max\left\{ 0 \; ; \; \pi - c_t + \beta \int V_{t+1}(\pi') \; f_\varepsilon(d\pi'|\pi, t) \right\}$$

and

$$\alpha_t(\pi) = 1\left\{ \; \pi - c_t + \beta \int V_{t+1}(\pi') \; f_\varepsilon(d\pi'|\pi, t) \geq 0 \right\}$$

Solution - A useful result.
• Given the form of $f_\varepsilon(\pi'|\pi,t)$, the future and discounted expected value, $\beta \int V_{t+1}(\pi')$ $f_\varepsilon(d\pi'|\pi,t)$, is increasing in current π.
• This implies that the solution of the DP problem can be described as a **sequence of threshold values for profits** $\{\pi_t^* : t = 1,2,...,T\}$ such that the optimal decision rule is:

$$\alpha_t(\pi) = 1\{ \; \pi \geq \pi_t^* \; \}$$

• π_t^* is the level of current profits that leaves the owner indifferent between renewing the patent or not: $V_t(\pi_t^*) = 0$.
• These threshold values are obtained using backwards induction:
• At period $t = T$:

$$\pi_T^* = c_T$$

• At period $t < T$, π_t^* is the unique solution to the equation:

$$\pi_t^* - c_t + \mathbb{E}\left(\sum_{s=t+1}^{T} \beta^{s-t} \; \max\{ \; 0 \; ; \; \pi_{t+1} - \pi_{t+1}^* \; \} \mid \pi_t = \pi_t^* \right) = 0$$

• Solving for a sequence of threshold values is much simpler that solving for a sequence of value functions.

Data

• Sample of N patents with complete (uncensored) durations $\{d_i : i = 1,2,...N\}$, where $d_i \in \{1,2,...,T+1\}$ is patent i's duration or age at its last renewal period.

• The information in this sample can be summarized by the empirical distribution of $\{d_i\}$:

$$\widehat{p}(t) = \frac{1}{N}\sum_{i=1}^{N} 1\{d_i = t\}$$

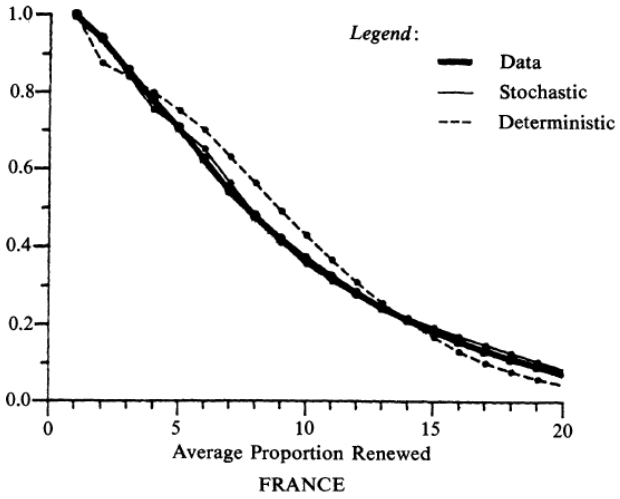


Figure 6.2: Pakes (1986)- Empirical Distribution - France

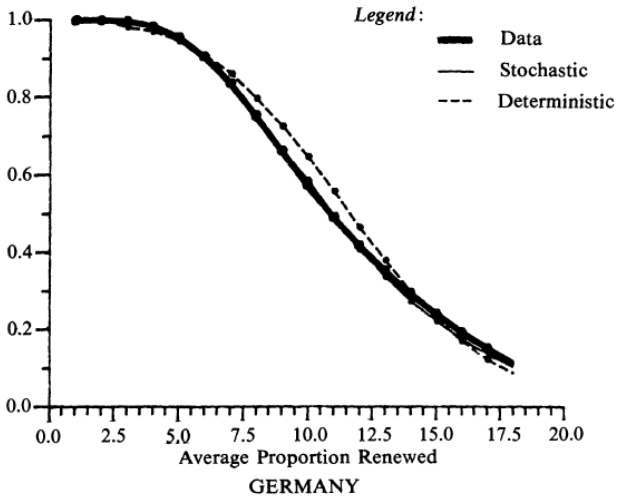


Figure 6.3: Pakes (1986)- Empirical Distribution - Germany

Estimation: Likelihood • The log-likelihood function of this model and data is:

$$\begin{aligned}\ell(\theta) &= \sum_{i=1}^{N}\sum_{t=1}^{T+1} 1\{d_i = t\}\ \ln \Pr(d_i = t|\theta) \\ &= N\sum_{t=1}^{T+1} \widehat{p}(t)\ \ln P(t|\theta)\end{aligned}$$

where:

$$\begin{aligned}P(t|\theta) &= \Pr(\pi_s \geq \pi_s^* \text{ for } s \leq t-1, \text{and } \pi_t < \pi_t^* \mid \theta) \\ &= \int_{\pi_1^*}^{\infty} \ldots \int_{\pi_{t-1}^*}^{\infty} \int_{0}^{\pi_t^*} dF(\pi_1, \ldots, \pi_{t-1}, \pi_t)\end{aligned}$$

• Computing $P(t|\theta)$ involves solving an integral of dimension t. For t greater than 4 or 5, it is computationally very costly to obtain the exact value of these probabilities. Instead, we approximate these probabilities using Monte Carlo simulation.

Estimation: Simulation of Probabilities

• For a given value of θ, let $\{\pi_t^{sim}(\theta) : t = 1,2,...,T\}$ be a simulated history of profits for patent i.

• Suppose that, for a given value of θ, we simulate R **independent** profit histories. Let $\{\pi_{rt}^{sim}(\theta) : t = 1,2,...,T;\ r = 1,2,...,R\}$ be these histories.

• Then, we can approximate the probability $P(t|\theta)$ using the following simulator:

$$\tilde{P}_R(t|\theta) = \frac{1}{R}\sum_{r=1}^{R} 1\{\pi_{rs}^{sim}(\theta) \geq \pi_s^* \text{ for } s \leq t-1, \text{and } \pi_{rt}^{sim} < \pi_t^*\}$$

• $\tilde{P}_R(t|\theta)$ is a *raw frequency simulator*. It has the following properties (Note that these are properties of a simulator, not of an estimator. $\tilde{P}_R(t|\theta)$ does not depend on the data).

(1) Unbiased: $\mathbb{E}\left(\tilde{P}_R(t|\theta)\right) = P(t|\theta)$

(2) $Var(\tilde{P}_R(t|\theta)) = P(t|\theta)(1-P(t|\theta))/R$

(3) Consistent as $R \to \infty$.

• It is possible to obtain better simulators (with lower variance) by using importance-sampling simulation. This is relevant because the bias and variance of simulated-based estimators depend on the variance (and bias) of the simulator.

• Furthermore, when $P(t|\theta)$ is small, the simulator $\tilde{P}_R(t|\theta)$ can be zero even when R is large, and this creates problems for ML estimation.

• A simple solution to this problem is to consider the following simulator which is based on the raw-frequency simulated probabilities $\tilde{P}_R(1|\theta)$, $\tilde{P}_R(2|\theta)$, ... $\tilde{P}_R(T+1|\theta)$:

$$P_R^*(t|\theta) = \frac{\exp\left\{\dfrac{\tilde{P}_R(t|\theta)}{\eta}\right\}}{\sum_{s=1}^{T+1}\exp\left\{\dfrac{\tilde{P}_R(s|\theta)}{\eta}\right\}}$$

where $\eta > 0$ is a smoothing parameter.

• The simulator P_R^* is biased. However, if $\eta \to 0$ as $R \to \infty$, then P_R^* is consistent, it has lower variance than $\tilde{P}_R$, and it is always strictly positive.

Simulation-Based Estimation.

• The estimator of θ (Simulated Method of Moments estimator) is the value that solves the system of T equations: for $t = 1,2,...T$:

$$\frac{1}{N}\sum_{i=1}^{N}\left[1\{d_i = t\} - \tilde{P}_{R,i}(t|\theta)\right] = 0$$

where the subindex i in the simulator $\tilde{P}_{R,i}(t|\theta)$ indicates that for each patent i in the sample we draw R independent histories and compute independent simulators.

• **Effect of simulation error.** Note that $\tilde{P}_{R,i}(t|\theta)$ is unbiased such that $\tilde{P}_{R,i}(t|\theta) = P(t|\theta) + e_i(t,\theta)$, where $e_i(t,\theta)$ is the simulation error. Since the simulation errors are

independent random draws:

$$\frac{1}{N}\sum_{i=1}^{N} e_i(t,\theta) \to_p 0 \quad \text{and} \quad \frac{1}{\sqrt{N}}\sum_{i=1}^{N} e_i(t,\theta) \to_d N(0,V_R)$$

The estimator is consistent an asymptotically normal for any R. The variance of the estimator declines with R.

Identification

• Since there are only 20 different values for the renewal fees $\{c_t\}$ we can at most identify 20 different points in the probability distribution of patent values.
• The estimated distribution at other points is the result of interpolation or extrapolation based on the functional form assumptions on the stochastic process for profits.
• It is important to note that the identification of the distribution of patent values is NOT up to scale but in dollar values.
• For a given patent of age t, all we can say is that: if $a_{it} = 0$, then $V_{it} < V(\pi_t^*)$; and if $a_{it} = 1$, then $V_{it} \geq V(\pi_t^*)$.

Empirical Questions • The estimated model can be used to address important empirical questions.
• **Valuation of the stock of patents.** Pakes uses the estimated model to obtain the value of the stock of patents in a country.
• According to the estimated model, the value of the stock of patents in 1963 was $315 million in France, $385 million in UK, and $511 million in Germany.
• Combining these figures with data on R&D investments in these countries, Pakes calculates rates of return of 15.6%, 11.0% and 13.8%, which look quite reasonable.

Empirical Questions.
• **Factual policies.** The estimated model shows that a very important part of the observed between-country differences in patent renewal can be explained by differences in policy parameters (that is, renewal fees and maximum length).
• **Counterfactual policy experiments.** The estimated model can be used to evaluate the effects of policy changes (in renewal fees and/or in maximum length) that are not observed in the data.

6.3.2 lanjow_1999 (lanjow_1999)

Estimates the private value of patent protection for four technology areas—computers, textiles, combustion engines, and pharmaceuticals - using new patent data for West Germany, 1953-1988. The model takes into account that patentees must pay not only renewal fees to keep their patents but also legal expenses to enforce them. The dynamic structural model takes into account the potential need to prosecute infringement. Results show that the aggregate value of protection generated per year is in the order of 10% of related R&D expenditure.

6.3.3 Trade of patents: Serrano (2018)

The sale of patents is an incentive to invest in R&D, especially for small firms. This market can generate social gains by reallocating patent rights from innovators to firms

that may be more effective in using, commercializing, or enforcing these rights. There are also potential social costs, if the acquiring firms can exercise more market power. Serrano (2018) investigates the value of trading patents by estimating a structural model that includes renewal and trading decisions.

Data: Panel of patents granted to U.S. small firms (no more than 500 employees) in the period 1988-1997 (15% of patents granted to firms). In the U.S. patent system, the patent holder needs to pay renewal fees to maintain the patent only at ages 5, 9, and 13 years. Fee increases with age: $c_{13} > c_9 > c_5$. **serrano_2000** (**serrano_2000**) constructs the dataset with renewals and transfers/sales. Working sample: 54,840 patents from 10 granting cohorts (1988 to 1997), followed from granting period until 2001 or until non-renewal.

Renewal and trading frequencies. Probability that a patent is traded (between renewal dates): - higher if previously untraded. - decreases with age. Probability of patent expiration (at renewal dates): - lower for previously traded. - increase over time.

Renewal and trading frequencies

Model: Key features. The transfer/sale of a patent involves a transaction cost. This transaction cost creates a selection effect: patents with higher per period returns are more likely to be traded. This selection effect explains the observed pattern that previously traded patents are: - more likely to be traded; - less likely to expire.

Returns. At age t, a patent has: - an **internal return** for the current patent owner, x_t; - a potential **external return** for the best alternative user, y_t. There is an "improvement factor", g_t^e, that relates external and internal returns: $y_t = g_t^e\ x_t$, where g_t^e is $i.i.d.$ with a truncated (at zero) exponential distribution: $\gamma^e \equiv \Pr(g_t^e = 0)$, and σ^e is the mean of the exponential. Initial (internal) returns: $\log(x_1) \sim N(\mu, \sigma_R^2)$. Next period returns:

$$x_{t+1} = \begin{cases} g_t^i\ x_t & \text{if not traded at age } t \\ g_t^i\ y_t & \text{if traded at age } t \end{cases}$$

g_t^i is a random variable with a truncated (at zero) exponential distribution: $\gamma^i \equiv \Pr(g_t^i = 0)$, and σ_t^i is the mean of this exponential, and $\sigma_t^i = \phi^t\ \sigma_0^i$, with $\phi \in (0,1)$. This implies that x_{t+1} follows a first order Markov process. Remember that there is a lump-sum transaction cost, τ. It is assumed to be paid by the buyer.

Model: Renewal and Sale decisions. Let $V_t(x_t, y_t)$ be the value of a patent with age t, current internal and external returns x_t and y_t, resp.

$$V_t(x_t, y_t) = \max\left\{0,\ V_t^K(x_t, y_t),\ V_t^S(x_t, y_t)\right\}$$

$V_t^K(x_t, y_t) =$ value of keeping; $V_t^S(x_t, y_t) =$ value of selling. And for $t \leq T = 17$:

$$\begin{aligned} V_t^K(x_t, y_t) &= x_t - c_t + \beta\ \mathbb{E}\left[V_{t+1}(x_{t+1}, y_{t+1}) \mid x_t, y_t, a_t = K\right] \\ V_t^S(x_t, y_t) &= x_t - c_t - \tau + \beta\ \mathbb{E}\left[V_{t+1}(x_{t+1}, y_{t+1}) \mid x_t, y_t, a_t = S\right] \end{aligned}$$

with $V_{T+1}^K = V_{T+1}^S = 0$.

Model: Optimal decision rule.

Lemma 1: $V_t(x_t, y_t)$ is weakly increasing in x_t and y_t, and weakly decreasing in t. Proposition 1. There are two threshold values: $x_t^*(\theta)$ that depends on age and structural

A-1: Percentage of Active Small Business Patents Traded and

Age	All	Not Previously Traded	Previously Traded (Years since last trade)	
			Any Year	One year
A. Probability that an active patent is traded				
2	2.99	2.85	7.47	7.47
7	2.81	2.46	4.79	6.63
11	2.51	2.13	3.77	2.55
B. Probability that an active patents is allowed to expire				
5	17.2	17.7	12.7	6.2
9	25.6	26.6	21.4	11.6
13	25.5	26.6	22.5	14.1

Figure 6.4: Serrano - Table of Frequencies

parameters, and $g_t^*(x,\theta)$, that depends on age, internal return, and parameters, such that the optimal decision rule a_t is:

$$a_t = \begin{cases} S & \text{if} \quad g_t^e \geq g_t^*(x_t,\theta) \\ K & \text{if} \quad g_t^e < g_t^*(x_t,\theta) \ \text{ and } \ x_t \geq x_t^*(\theta) \\ 0 & \text{if} \quad g_t^e < g_t^*(x_t,\theta) \ \text{ and } \ x_t < x_t^*(\theta) \end{cases}$$

Identification and Estimation. Method: Simulated method of moments. Moments describing the history of trading and renewal decisions of patent owners. (1) probability that an active patent is traded at different ages conditional on having been previously traded, and conditional on not having been previously traded. (2) probability that an active patent is allowed to expire at different renewal dates conditional on having been

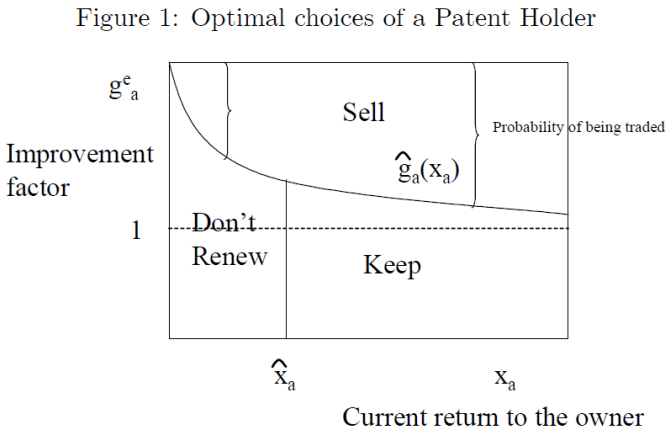


Figure 6.5: Serrano - Decision Rule

previously traded, and conditional on not having been previously traded. A total of 186 moments.

Parameter estimates. Transaction cost: $5,850, about one-third of the average return at age 1 (8% of the average value at age 1). On average, internal growth of returns is greater than external.

Evaluating the value of the market for patents. The possibility of trading patents has two types of effects on the value of the pool of patents: - a direct causal effect due to the reallocation to an owner with higher returns; - a selection effect, through the renewal decisions (renewal decision is different with and without the possibility of trading).

Serrano measures these two sources of value.

Evaluating the value of the market for patents. (1) Total effect on the value of patents: - 50% of the total value of patents. - Only 23% of patents are sold, but the value of a

Table 1: Parameter Estimates

Description (Parameter)	Estimate[a]
A. Patent initial returns	
Mean parameter of the Lognormal Initial Distribution (μ)	8.4179 ($4.0*10^{-2}$)
Std. Deviation parameter of the Lognormal Initial Distribution (σ_R)	1.6911 ($1.2*10^{-2}$)
B. Internal growth of returns	
Depreciation factor (δ)	0.8917 ($5.2*10^{-3}$)
Not obsolescence (γ^i)	0.9673 ($6.5*10^{-3}$)
Internal Growth of Returns (σ^i)	0.4450 ($3.6*10^{-3}$)
Upside opportunities (ϕ)	0.5941 ($6.1*10^{-3}$)
C. Market for patents and transaction costs	
Transaction cost (τ)	5,850.1 (50.49)
Mean External Growth of Returns (σ^e)	0.3745 ($2.9*10^{-3}$)
Proportion of unsuccessful transfers (γ^e)	0.0385 ($4*10^{-4}$)
Random transfers (ε)	0.0059 ($4*10^{-4}$)
Size of sample	54,840
Simulations in the estimation	164,520
MSE[b]	$3.784*10^{-4}$

Figure 6.6: Serrano - Parameter Estimates

traded patent is 3 times higher than untraded patent ($173,668 vs. $54,960). (2) Direct gains from trade (from reallocation) - accounts for 10% of the total value of the traded patents. - The distribution of the gains from trade is very skewed.

Counterfactual: Reducing transaction cost . Lowering transaction cost by 50% (from $5,850 to $2,925) raises the proportion of patents traded by 6 percentage points: from 23.1% to 29.6%. It also boosts the gains from trade (reallocation) by an additional 8.7%. Overall, it increases the total value of the patent market by 3%.

6.4 Dynamic pricing

Retail firms selling the same product and operating in the same narrowly defined geographic market can charge prices that differ by significant amounts. Cross-sectional dispersion of prices has been well established in different retail markets such as gas

stations or supermarkets, among others. Recent empirical papers show that temporary sales account for approximately half of all price changes of retail products in the U.S. (see **hosken_reiffen_2004** ,**hosken_reiffen_2004**, **nakamura_ steinsson_2008** ,**nakamura_ steinsson_2008**, or **midrigan_2011** ,**midrigan_2011**). Sales promotions can also account for a substantial part of cross-sectional price dispersion. Therefore, understanding the determinants of temporary sales is important to understand price stickiness, price dispersion, and firms' market power and competition.

varian_1980 (**varian_1980**) presents a model of price competition in an homogeneous product market, with two types of consumers: consumers that are informed about the price, and consumer that are not. Informed customers always buy in the store with the lowest price. Uninformed consumers choose a store at random and buy there as long as the price of the store is not above their reservation price. The model does not have an equilibrium in pure strategies. In mixed strategies, there is a unique symmetric equilibrium characterized by a U-shape density function on the price interval between the marginal cost and the reservation price of uninformed consumers. According to this equilibrium, the price charged by a store changes randomly over time between a "low" and a "high" price.

Though Varian's model can explain some important empirical features in the cross-section and time series of prices in retail markets, it cannot explain the time dependence of sales promotions that have been reported in many empirical studies (for instance, Slade ,1998, Aguirregabiria ,1999, or Pesendorfer ,2002, among others). The probability of a sales promotion increases with the duration since the last sale. Several studies have proposed and estimated dynamic structural models of retail pricing that can explain price dispersion, sales promotions and their state dependence. These studies also provide estimates of the magnitude and structure of firms' price adjustments costs.

Slade (1998) proposes a model where the demand of a product in a store depends on a stock of goodwill that accumulates over time when the store charges low prices, and erodes when the price is high. The model also incorporates menu costs of changing prices. The optimal pricing policy consists of a cycle between a low price (or sales promotion) and a high price. Slade estimates this model using weekly scanner data of prices and quantities of saltine crackers in four supermarket chains. The estimated model fits well the joint dynamics of prices and quantities. Her estimates of the cost of adjusting prices are approximately 4% of revenue.

Aguirregabiria (1999) studies the relationship between inventories and prices in supermarkets. The cost of placing orders to wholesalers has a fixed component. Retailers have also menu costs of changing prices, face substantial demand uncertainty, and have stockouts. Aguirregabiria proposes a model of price and inventory decisions that incorporates these features. In the optimal decision rule of this model, inventories follow an (S,s) cycle, and prices have a "high-low" cyclical pattern. When a new order is placed, the probability of a stockout declines, expected demand becomes more elastic, and the optimal price drops to a minimum. When inventories decline between two orders, the probability of a stockout increases, expected sales become more inelastic, and the optimal price eventually increases and stays high until the next order. Aguirregabiria estimates this model using data on inventories, prices, and sales from the warehouse of a supermarket chain. The estimated model fits well the joint cyclical pattern of prices and inventories in the data and can explain temporary sales. The estimated values

for the fixed ordering cost and the menu cost are 3.1% and 0.7% of monthly revenue, respectively. According to the estimated model, almost 50% of sales promotions are associated to the dynamics of inventories.

Pesendorfer (2002) studies the dynamics of consumer demand as a factor explaining sales promotions of storable products. He proposes a model of demand of a storable product with two types of consumers: store-loyal consumers and shoppers. The equilibrium of the model predicts that the probability that a store has a sale increases with the duration since the last sale both in that store and in other stores. The implied pattern of prices consists of an extended period of high prices followed by a short period of low prices. He tests the predictions of the model using supermarket scanner data for ketchup products. The effects of the duration variables are significant and have the predicted sign. Though this evidence suggests that demand accumulation could be important in the decision to conduct a sale, it is also consistent with models in Slade (1998) and Aguirregabiria (1999). As far as we know, there is no empirical study that has tried to disentangle the relative contribution of consumer inventories, firm inventories, and goodwill to explain temporary sales promotions.

kano_2013 (**kano_2013**) makes an interesting point on the estimation of menu costs in oligopoly markets. Dynamic price competition in oligopoly markets implies a positive strategic interaction between the prices of different firms. This strategic interaction may be an important source of price inertia even when menu costs are small. If a firm experiences an idiosyncratic increase in its marginal cost, it may prefer not to increase its price if the competitor maintains a constant price. A model of monopolistic competition that ignores strategic interactions among firms may spuriously overestimate menu costs. Kano estimates a dynamic pricing model that accounts for these strategic interactions and finds that they account for a substantial part of price rigidity.

6.4.1 Aguirregabiria (1999)

The significant cross-sectional dispersion of prices is a well-known stylized fact in retail markets. Retailing firms selling the same product, and operating in the same (narrowly defined) geographic market and at the same period of time, charge prices that differ by significant amounts, for instance, 10% price differentials or even larger. This empirical evidence has been well established for gas stations and supermarkets, among other retail industries. Interestingly, the price differentials between firms, and the ranking of firms in terms of prices, have very low persistence over time. A gas station that charges a price 5% below the average in a given week may be charging a price 5% above the average the next week. Using a more graphical description we can say that a firm's price follows a cyclical pattern, and the price cycles of the different firms in the market are not synchronized. Understanding price dispersion and the dynamics of price dispersion is very important to understand not only competition and market power but also for the construction of price indexes.

Different explanations have been suggested to explain this empirical evidence. Some explanations have to do with dynamic pricing behavior or "state dependence" in prices.

For instance, one explanation is based on the relationship between firm inventory and optimal price. In many retail industries with storable products, we observe that firms' orders to suppliers are infrequent. For instance, for products such as laundry detergent, a supermarket's ordering frequency can be lower than one order per month. A simple

and plausible explanation of this infrequency is that there are fixed or lump-sum costs of making an order that do not depend on the size of the order, or at least that do not increase proportionally with the size of the order. Then, inventories follow a so called (S,s) cycle: inventories increase by a large amount up to a maximum threshold when an order is placed, and decline gradually until a minimum value is reached, at which time a new order is placed. Given these dynamics of inventories, it is simple to show that the optimal price of the firm should also follow a cycle. The price drops to a minimum when a new order is placed and then increases over time up to a maximum just before the next order when the price drops again. Aguirregabiria (1999) shows this joint pattern of prices and inventories for many products in a supermarket chain. Specifically, I show that these types of inventory-dependence price dynamics can explain more than 20% of the time series variability of prices in the data.

Temporary sales and inventories. Recent empirical papers show that temporary sales account for approximately half of all price changes of retail products in the U.S.: **hosken_reiffen_2004** (**hosken_reiffen_2004**); **nakamura_steinsson_2008** (**nakamura_steinsson_2008**) **midrigan_2011** (**midrigan_2011**). Understanding the determinants of temporary sales is important to understand price stickiness and price dispersion, and it has important implications on the effects of monetary policy. It has also important implications in the study of firms' market power and competition. Different empirical models of sales promotions: Slade (1998) [Endogenous consumer loyalty], Aguirregabiria (1999) [Inventories], Pesendorfer (2002) [Intertemporal price discrimination], and **kano_2013** (**kano_2013**).

This paper studies how retail inventories, and in particular (S,s) inventory behavior, can explain both price dispersion and sales promotions in retail markets. Three factors are key for the explanation provided in this paper:

(1) Fixed (lump-sum) ordering costs, that generates (S,s) inventory behavior.

(2) Demand uncertainty.

(3) Sticky prices (Menu costs) that, together with demand uncertainty, creates a positive probability of excess demand (stockout).

Model: Basic framework

Consider a retail firm selling a product. We index products by i. Every period (month) t the firm chooses the retail price and the quantity of the product to order to manufacturers/wholesalers. **Monthly sales** are the minimum of supply and demand:

$$y_{it} = \min\{\, d_{it} \;;\; s_{it} + q_{it} \,\}$$

y_{it} = sales in physical units; d_{it} = demand; s_{it} = inventories at the beginning of month t; q_{it} = orders (and deliveries) during month t.

Demand and Expected sales. The firm has uncertainty about **current demand**:

$$d_{it} = d_{it}^{e} \, \exp(\xi_{it})$$

d_{it}^{e} = expected demand; ξ_{it} = zero mean demand shock unknown to the firm at t. Therefore, **expected sales** are:

$$y_{it}^{e} = \int \min\{d_{it}^{e} \, \exp(\xi) \;;\; s_{it} + q_{it} \,\} \, dF_{\xi}(\xi)$$

Assume monopolistic competition. **Expected Demand** depends on the own price, p_{it}, and a demand shock ω_{it}. The functional form is isoelastic:

$$d_{it}^e = \exp\{\gamma_0 - \gamma_1 \ln(p_{it}) + \omega_{it}\}$$

where γ_0 and $\gamma_1 > 0$ are parameters.

Price elasticity of expected sales. **Demand uncertainty** has important implications for the relationship between prices and inventories. The price elasticity of expected sales is a function of the **supply-to-expected-demand ratio** $(s_{it}+q_{it})/d_{it}^e$:

$$\begin{aligned}\eta_{y^e|p} \equiv \frac{-\partial y^e}{\partial p}\frac{p}{y^e} &= -\left[\int I\{d^e \exp(\xi) \; ; \; s+q\} \, dF_\xi(\xi)\right] \frac{\partial d^e}{\partial p}\frac{p}{y^e} \\ &= \gamma_1 \, F_\xi\left(\log\left[\frac{s+q}{d^e}\right]\right)\frac{d^e}{y^e}\end{aligned}$$

And we have that:

$$\eta_{y^e|p} \longrightarrow \begin{cases} \gamma_1 & \text{as} \quad (s+q)/d^e \longrightarrow \infty \\ 0 & \text{as} \quad (s+q)/d^e \longrightarrow 0 \end{cases}$$

Price elasticity of expected sales

$$\eta_{y^e|p} = \gamma_1 \, F_\xi\left(\log\left[\frac{s+q}{d^e}\right]\right)\frac{d^e}{y^e}$$

[FIGURE: $\eta_{y^e|p}$ increasing in $\dfrac{s+q}{d^e}$, with asymptote at γ_1]

When the supply-to-expected-demand ratio is large, the probability of a stockout is very small and $y^e \simeq d^e$, so the elasticity of expected sales is just the elasticity of demand. However, when the supply-to-expected-demand ratio is small, the probability of a stockout is large and the elasticity of expected sales can be much lower than the elasticity of demand.

Markup and inventories (myopic case). This has potentially important implications for the optimal price of an oligopolistic firm. To give some intuition, consider the pricing decision of the monopolistic firm without forward-looking behavior. That optimal price is:

$$\frac{p-c}{p} = \frac{1}{\eta_{y^e|p}}$$

OR

$$\frac{p-c}{c} = \frac{1}{\eta_{y^e|p}-1}$$

Variability over time in the supply-to-expected-demand ratio can generate significant fluctuations in price-cost margins. It can also explain temporary sales promotions. That can be the case under (S,s) inventory behavior.

Empirical Application

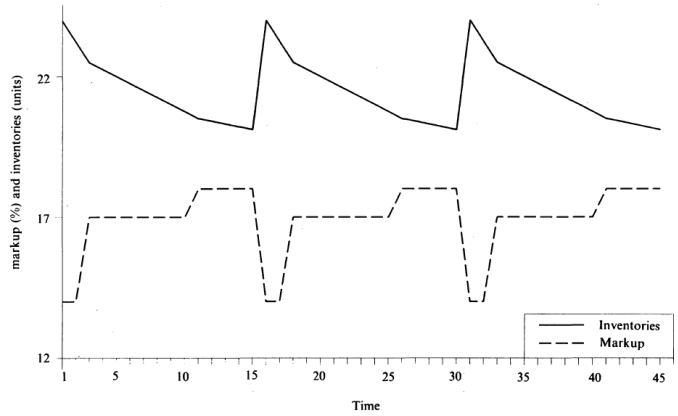


Figure 6.7: Cyclical Inventories and Prices

The paper investigates this hypothesis using data from a supermarket chain, with rich information on prices, sales, inventories, orders, and wholesale prices for many different products. Reduced form estimations present evidence that supports the hypothesis:

(1) Prices depend negatively and very significantly on the level of inventories.

(2) Inventories of many products follow (S,s) cycles.

(3) Price cost margins increase at the beginning of an (S,s) cycle, and decline monotonically during the cycle.

I estimate the parameters in the profit function (demand parameters, ordering costs, inventory holding costs) and use the estimated model to analyze how much of price variation and temporary sales promotions can be explained by firm inventories.

Profit function. **Expected current profits** are equal to expected revenue, minus

ordering costs, inventory holding costs and price adjustment costs:

$$\pi_{it} = p_{it}\, y^{e}_{it} - OC_{it} - IC_{it} - PAC_{it}$$

OC_{it} = ordering costs; IC_{it} = inventory holding costs; PAC_{it} = price adjustment (menu) costs. **Ordering costs:**

$$OC_{it} = \begin{cases} 0 & \text{if} \quad q_{it} = 0 \\ \\ F_{oc} + \varepsilon^{oc}_{it} - c_{it}\, q_{it} & \text{if} \quad q_{it} > 0 \end{cases}$$

F_{oc} = fixed (lump-sum) ordering cost. Parameter; ε^{oc}_{it} = zero mean shock in the fixed ordering cost; c_{it} = wholesale price. **Inventory holding costs:**

$$IC_{it} = \alpha\, s_{it}$$

Menu costs:

$$PAC_{it} = \begin{cases} 0 & \text{if} \quad p_{it} = p_{i,t-1} \\ \\ F^{(+)}_{mc} + \varepsilon^{mc(+)}_{it} & \text{if} \quad p_{it} > p_{i,t-1} \\ \\ F^{(-)}_{mc} + \varepsilon^{mc(-)}_{it} & \text{if} \quad p_{it} < p_{i,t-1} \end{cases}$$

$F^{(+)}_{mc}$ and $F^{(-)}_{mc}$ are price adjustment cost parameters; $\varepsilon^{mc(+)}_{it}$ and $\varepsilon^{mc(-)}_{it}$ are zero mean shocks in menu costs

State variables. The state variables of this DP problem are:

$$\left\{ \underbrace{s_{it},\, c_{it},\, p_{i,t-1},\, \omega_{it}}_{x_{it}},\ \underbrace{\varepsilon^{oc}_{it},\, \varepsilon^{mc(+)}_{it},\, \varepsilon^{mc(+)}_{it}}_{\varepsilon_{it}} \right\}$$

The decision variables are q_{it} and $\Delta p_{it} \equiv p_{it} - p_{i,t-1}$. We use a_{it} to denote $(q_{it}, \Delta p_{it})$. Let $V(x_{it}, \varepsilon_{it})$ be the value of the firm associated with product i. This value function solves the Bellman equation:

$$V(x_{it}, \varepsilon_{it}) = \max_{a_{it}} \left\{ \begin{array}{c} \pi(a_{it}, x_{it}, \varepsilon_{it}) \\ \\ +\beta \int V(x_{i,t+1}, \varepsilon_{i,t+1})\, dF(x_{i,t+1}, \varepsilon_{i,t+1} | a_{it}, x_{it}, \varepsilon_{it}) \end{array} \right\}$$

Discrete Decision variables. Most of the variability of q_{it} and Δp_{it} in the data is discrete. For simplicity, we assume that these variables have a discrete support.

$$q_{it} \quad \in \quad \{0\,,\ \kappa_i\}$$

$$\Delta p_{it} \quad \in \quad \{0\,,\ \delta^{(+)}_i,\ \delta^{(-)}_i\}$$

where $\kappa_i > 0$, $\delta^{(+)}_i > 0$, and $\delta^{(-)}_i < 0$ are parameters. Therefore, the set of choice alternatives at every period t is:

$$a_{it} \in A = \left\{ \ (0,0),\ (0,\delta^{(+)}_i),\ (0,\delta^{(-)}_i),\ (\kappa_i,0),\ (\kappa_i,\delta^{(+)}_i),\ (\kappa_i,\delta^{(-)}_i) \ \right\}$$

The transition rules for the state variables are:

$$\begin{array}{rcl} s_{i,t+1} & = & s_{it} + q_{it} - y_{it} \\ p_{it} & = & p_{i,t-1} + \Delta p_{it} \\ c_{i,t+1} & \sim & AR(1) \\ \omega_{i,t+1} & \sim & AR(1) \\ \varepsilon_{it} & \sim & i.i.d. \end{array}$$

Integrated Bellman Equation. The components of ε_{it} are independently and extreme value distributed with dispersion parameter σ_ε. Therefore, as in Rust (1987), the integrated value function $\bar{V}(x_{it})$ is the unique fixed point of the integrated Bellman equation:

$$\bar{V}(x_{it}) = \sigma_\varepsilon \ln \left(\sum_{a \in A} \exp \left\{ \frac{v(a,x_{it})}{\sigma_\varepsilon} \right\} \right)$$

where:

$$v(a,x_{it}) = \bar{\pi}(a,x_{it}) + \beta \sum_{x_{i,t+1}} \bar{V}(x_{i,t+1}) \, f_x(x_{i,t+1}|a,x_{it})$$

Discrete choice profit function

• $\bar{\pi}(a,x_{it})$ is the part of current profit which does not depend on ε_{it}:

$$\bar{\pi}(a,x_{it}) = \begin{cases} R_{it}(0,0) - \alpha \, s_{it} & \text{if} \quad a = (0,0) \\ R_{it}(0,\delta_i^{(+)}) - \alpha \, s_{it} - F_{mc}^{(+)} & \text{if} \quad a = (0,\delta_i^{(+)}) \\ R_{it}(0,\delta_i^{(-)}) - \alpha \, s_{it} - F_{mc}^{(-)} & \text{if} \quad a = (0,\delta_i^{(-)}) \\ R_{it}(\kappa_i,0) - \alpha \, s_{it} - F_{oc} - c_{it}\kappa_i & \text{if} \quad a = (\kappa_i,0) \\ R_{it}(\kappa_i,\delta_i^{(+)}) - \alpha \, s_{it} - F_{oc} - c_{it}\kappa_i - F_{mc}^{(+)} & \text{if} \quad a = (\kappa_i,\delta_i^{(+)}) \\ R_{it}(\kappa_i,\delta_i^{(-)}) - \alpha \, s_{it} - F_{oc} - c_{it}\kappa_i - F_{mc}^{(-)} & \text{if} \quad a = (\kappa_i,\delta_i^{(-)}) \end{cases}$$

where $R_{it}(.,.)$ is the expected revenue function.

Some predictions of the model. Fixed ordering cost F_{oc} generates infrequent orders: $(\mathbf{S},\mathbf{s})$ **inventory policy**. (S,s) inventory behavior, together with demand uncertainty (that is, optimal prices depend on the supply-to-expected demand ratio) generate a cyclical pattern in the price elasticity of sales. Prices decline significantly when an order is placed (sales promotion). This price decline and the consequent inventory reduction generate a price increase. Then, as inventories decline between two orders, prices tend to increase.

Data. Data from the central warehouse of a supermarket chain in the Basque Country (Spain). Monthly data: period January 1990 to May 1992. Estimation of Structural Parameters. Counterfactual Experiments

7. Dynamic Consumer Demand

7.1 Introduction

Consumers can stockpile a storable good when prices are low and use the stock for future consumption. This stockpiling behavior can introduce significant differences between short-run and long-run responses of demand to price changes. Also, the response of demand to a price change depends on consumers' expectations/beliefs about how permanent that price change is. For instance, if a price reduction is perceived by consumers as very transitory (for instance, a sales promotion), then a significant proportion of consumers may choose to increase purchases today, stockpile the product and reduce their purchases during future periods when the price will be higher. If the price reduction is perceived as permanent, this intertemporal substitution of consumer purchases will be much lower or even zero.

Ignoring consumers' stockpiling and forward-looking behavior can introduce serious biases in our estimates of own- and cross- price demand elasticities. These biases can be particularly serious when the time series of prices is characterized by "High-Low" pricing. The price fluctuates between a (high) regular price and a (low) promotion price. The promotion price is infrequent and last only few days, after which the price returns to its "regular" level. Most sales are concentrated in the very few days of promotion prices.

Static demand models assume that all the substitution is either between brands or product expansion. They rule out intertemporal substitution. This can imply serious biases in the estimated demand elasticities. With High-Low pricing, we expect the static model to over-estimate the own-price elasticity. The bias in the estimated elasticities also implies a bias in the estimated Price Cost Margins (PCM). We expect PCMs to be underestimated. These biases have serious implications on policy analysis, such as merger analysis and antitrust cases.

Here we discuss two papers that have estimated dynamic structural models of demand of differentiated products using consumer level data (scanner data): Hendel and Nevo (2006) and **erdem_keane_2003** (**erdem_keane_2003**). These papers extend

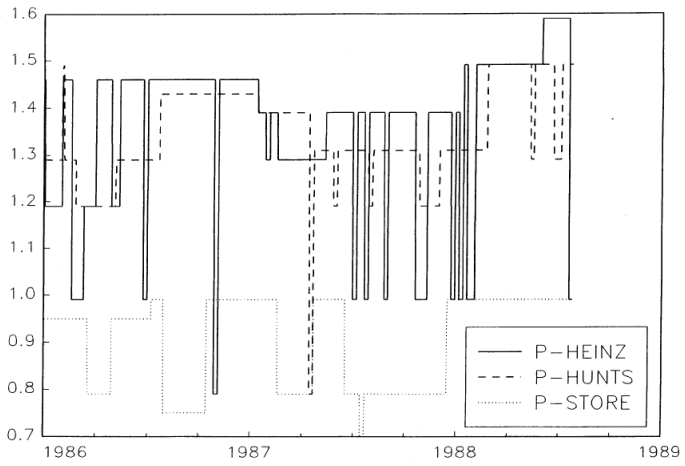


Figure 7.1: Pesendorfer (2002): Hi-Lo Pricing - Ketchup

microeconometric discrete choice models of product differentiation to a dynamic setting, and contain useful methodological contributions. Their empirical results show that ignoring the dynamics of demand can lead to serious biases. The papers also illustrate how the use of **micro level data on household choices** (in contrast to only aggregate data on market shares) is key for credible identification of the dynamics of differentiated product demand.

7.2 Data and descriptive evidence

In the following section, the researcher has access to consumer level data. Such data is widely available from several data collection companies and recently researchers in several countries have been able to gain access to such data for academic use. The data include the history of shopping behavior of a consumer over a period of one to

three years. The researcher knows whether a store was visited, and what product (brand and size) was purchased and at what price. From the view point of the model, the key information that is not observed is consumer inventory and consumption decisions.

Hendel and Nevo use consumer-level scanner data from Dominicks, a supermarket chain that operates in the Chicago area. The dataset comes from 9 supermarket stores and it covers the period June 1991 to June 1993. Purchases and price information are available in real (continuous) time but for the analysis in the paper it is aggregated at the weekly frequency.

The dataset has two components: store-level and household-level data. **Store level data:** For each detailed product (brand–size) in each store and in each week, we observe the (average) price charged, (aggregate) quantity sold, and promotional activities. **Household level data:** For a sample of households, we observe the purchases of households at the 9 supermarket stores: supermarket visits and total expenditure in each visit; purchases (units and value) of detailed products (brand-size) in 24 different product categories (for instance, laundry detergent, milk, etc). The paper specifically studies demand of laundry detergent products.

Table I in the paper presents summary statistics on household demographics, purchases, and store visits.

Table II in the paper presents the market shares of the main brands of laundry detergent in the data. The market is significantly concentrated, especially the market for Powder laundry detergent where the concentration ratios are $CR1 = 40\%$, $CR2 = 55\%$, and $CR3 = 65\%$. For most brands, the proportion of sales under a promotion price is important. However, this proportion varies importantly between brands, showing that different brands have different patterns of prices.

Descriptive evidence. H&N present descriptive evidence which is consistent with household inventory holding. See also Hendel and Nevo (2006). Though household purchase histories are observable, household inventories and consumption are unobservable. Therefore, empirical evidence on the importance of household inventory holding is indirect.

(a) Time duration since previous sale promotion has a positive effect on the aggregate quantity purchased.

(b) Indirect measures of storage costs (for instance, house size) are negatively correlated with households' propensity to buy on sale.

7.3 Model

7.3.1 Basic Assumptions

Consider a differentiated product, laundry detergent, with J different brands. Every week a household has some level of inventories of the product (that may be zero) and chooses (a) how much to consume from its inventory; and (b) how much to purchase (if any) of the product, and the brand to purchase.

An important simplifying assumption in Hendel-Nevo model is that consumers care about brand choice when they purchase the product, but not when they consume or store it. We explain below the computational advantages of this assumption. Of course, the assumption imposes some restrictions on the intertemporal substitution between brands,

and we will discuss this point too. **erdem_keane_2003** (**erdem_keane_2003**) do not impose that restriction.

The subindex t represents time, the subindex j represents a brand, and the subindex h represents a consumer or household. A household's current utility function is:

$$u_h(c_{ht}, v_{ht}) - C_h(i_{h,t+1}) + m_{ht}$$

$u_h(c_{ht}, v_{ht})$ is the utility from consumption of the storable product, with c_{ht} being consumption and v_{ht} being a shock in the utility of consumption:

$$u_h(c_{ht}, v_{ht}) = \gamma_h \ \ln(c_{ht} + v_{ht})$$

$C_h(i_{h,t+1})$ is the inventory holding cost, where $i_{h,t+1}$ is the level of inventory at the end of period t, after consumption and new purchases:

$$C_h(i_{h,t+1}) = \delta_{1h} \ i_{h,t+1} + \delta_{2h} \ i_{h,t+1}^2$$

m_{ht} is the indirect utility function from consumption of the composite good (outside good) plus the utility from brand choice (that is, the utility function in a static discrete model of differentiated products):

$$m_{ht} = \sum_{j=1}^{J} \sum_{x=0}^{X} d_{hjxt} \left(\beta_h \ a_{jxt} - \alpha_h \ p_{jxt} + \xi_{jxt} + \varepsilon_{hjxt} \right)$$

where $j \in \{1, 2, ..., J\}$ is the brand index, and $x \in \{0, 1, 2, ..., X\}$ is the index of quantity choice, where the maximum possible size is X units. In this application $X = 4$. Brands with different sizes are standardized such that the same measurement unit is used in x. The variable $d_{hjxt} \in \{0, 1\}$ is a binary indicator for the event "household purchases x units of brand j at week t". p_{jxt} is the price of x units of brand j at period t. Note that the models allows for nonlinear pricing, that is, for some brands and weeks p_{jxt} and $x * p_{j1t}$ can take different values. This is potentially important because the price data shows a significant degree of nonlinear pricing. a_{jxt} is a vector of product characteristics other than price that is observable to the researcher. In this application, the most important variables in a_{jxt} are those that represent store-level advertising, for instance, display of the product in the store, etc. The variable ξ_{jxt} is a random variable that is unobservable to the researcher and that represents all the product characteristics which are known to consumers but not in the set of observable variables in the data.

α_h and β_h represent the marginal utility of income and the marginal utility of product attributes in a_{jxt}, respectively. As it is well-known in the empirical literature of demand of differentiated products, it is important to allow for heterogeneity in these marginal utilities in order to have demand systems with flexible and realistic own and cross elasticities or substitution patterns. Allowing for this heterogeneity is much simpler with consumer level data on product choices than with aggregate level data on product market shares. In particular, micro level datasets can include information on a rich set of household socioeconomic characteristics such as income, family size, age, education, gender, occupation, house-type, etc, that can be included as observable variables that determine the marginal utilities α_h and β_h. This is the approach in Hendel and Nevo's paper.

Finally, ε_{hjxt} is a consumer idiosyncratic shock that is independently and identically distributed over (h, j, x, t) with an extreme value type 1 distribution. This is the typical logit error that is included in most discrete models of demand of differentiated products. Note that while ε_{hjxt} varies over individuals, ξ_{jxt} does not.

Let $\mathbf{p}_t$ be the vector of product characteristics, observable or unobservable, for all the brands and sizes at period t:

$$\mathbf{p}_t \equiv \left\{ p_{jxt}, a_{jxt}, \xi_{jxt} : j = 1,2,...,J \text{ and } x = 1,2,...,X \right\}$$

Every week t, the household knows her level of inventories, i_{ht}, observes product attributes $\mathbf{p}_t$, and its idiosyncratic shocks in preferences, v_{ht} and ε_{ht}. Given this information, the household decides her consumption of the storable product, c_{ht}, and how much to purchase of which product, $d_{ht} = \{d_{hjxt}\}$. The household makes this decision in order to maximize her expected and discounted stream of current and future utilities,

$$\mathbb{E}_t \left(\sum_{s=0}^{\infty} \delta^s \left[u_h(c_{ht+s}, v_{ht+s}) - C_h(i_{h,t+s+1}) + m_{ht+s} \right] \right)$$

where δ is the discount factor.

The vector of state variables of this DP problem is $\{i_{ht}, v_{ht}, \varepsilon_{ht}, \mathbf{p}_t\}$. The decision variables are c_{ht} and d_{ht}. To complete the model we need to make some assumptions on the stochastic processes of the state variables. The idiosyncratic shocks v_{ht} and ε_{ht} are assumed iid over time. The vector of product attributes $\mathbf{p}_t$ follows a Markov process. Finally, consumer inventories i_{ht} has the obvious transition rule:

$$i_{h,t+1} = i_{h,t+1} - c_{ht} + \left(\sum_{j=1}^{J} \sum_{x=0}^{X} d_{hjxt}\, x \right)$$

where $\sum_{j=1}^{J} \sum_{x=0}^{X} d_{hjxt}\, x$ represents the units of the product purchased by household h at period t.

Let $V_h(\mathbf{s}_{ht})$ be the value function of a household, where $\mathbf{s}_{ht}$ is the vector of state variables $(i_{ht}, v_{ht}, \varepsilon_{ht}, \mathbf{p}_t)$. A household decision problem can be represented using the Bellman equation:

$$V_h(\mathbf{s}_{ht}) = \max_{\{c_{ht}, d_{ht}\}} \left[u_h(c_{ht}, v_{ht}) - C_h(i_{h,t+1}) + m_{ht} + \delta\, \mathbb{E}\left(V_h(\mathbf{s}_{ht+1}) \mid \mathbf{s}_{ht},\, c_{ht},\, d_{ht} \right) \right]$$

where the expectation $\mathbb{E}(. \mid \mathbf{s}_{ht},\, c_{ht},\, d_{ht})$ is taken over the distribution of $\mathbf{s}_{ht+1}$ conditional on $(\mathbf{s}_{ht}, c_{ht}, d_{ht})$. The solution of this DP problem implies optimal decision rules for consumption and purchasing decisions: $c_{ht} = c_h^*(\mathbf{s}_{ht})$ and $d_{ht} = d_h^*(\mathbf{s}_{ht})$ where $c_h^*(.)$ and $d_h^*$are the decision rules. Note that they are household specific because there is time-invariant household heterogeneity in the marginal utility of product attributes (α_h and β_h), in the utility of consumption of the storable good u_h, and in inventory holding costs, C_h.

The optimal decision rules $c_h^*(.)$ and d_h^*depend also on the structural parameters of the model: the parameters in the utility function, and in the transition probabilities of the state variables. In principle, we could use the equations $c_{ht} = c_h^*(\mathbf{s}_{ht})$, $d_{ht} = d_h^*(\mathbf{s}_{ht})$, and our data on (some) decisions and state variables to estimate the parameters of the model. To apply this revealed preference approach, there are three main issues we have to deal with.

First, the dimension of the state space of $\mathbf{s}_{ht}$ is extremely large. In most applications of demand of differentiated products, there are dozens (or even more than a hundred) products. Therefore, the vector of product attributes $\mathbf{p}_t$ contains more than a hundred continuous state variables. Solving a DP problem with this state space, or even approximating the solution with enough accuracy using Monte Carlo simulation methods, is computationally very demanding even with the most sophisticated computer equipment. We will see how Hendel and Nevo propose and implement a method to reduce the dimension of the state space. The method is based on some assumptions that we discuss below.

Second, though we have good data on households' purchasing histories, information on households' consumption and inventories of storable goods is very rare. In this application, consumption and inventories, c_{ht} and i_{ht}, are unobservable to the researchers. Not observing inventories is particularly challenging. This is the key state variable in a dynamic demand model for the demand of a storable good. We will discuss below the approach used by Hendel and Nevo to deal with this issue, and also the approach used by **erdem_keane_2003** (**erdem_keane_2003**).

Third, as usual in the estimation of a model of demand, we should deal with the endogeneity of prices. Of course, this problem is not specific to a dynamic demand model. However, dealing with this problem may not be independent of the other issues mentioned above.

7.3.2 Reducing the dimension of the state space

Given that the state variables $(v_{ht}, \varepsilon_{ht})$ are independently distributed over time, it is convenient to reduce the dimension of this DP problem by using a value function that is integrated over these iid random variables. The integrated value function is defined as:

$$\bar{V}_h(i_{ht}, \mathbf{p}_t) \equiv \int V_h(\mathbf{s}_{ht})\, dF_\varepsilon(\varepsilon_{ht})\, dF_v(v_{ht})$$

where F_ε and F_v are the CDFs of ε_{ht} and v_{ht}, respectively. Associated with this integrated value function is an integrated Bellman equation. Given the distributional assumptions on the shocks ε_{ht} and v_{ht}, the integrated Bellman equation is:

$$\bar{V}_h(i_{ht}, \mathbf{p}_t) = \max_{c_{ht}, d_{ht}} \int \ln \left(\sum_{j=1}^{J} \exp \left\{ \begin{array}{l} u_h(c_h, v_{ht}) - C_i(i_{ht+1}) + m_{ht} \\ \\ +\delta\, \mathbb{E}[\bar{V}_h(i_{ht+1}, \mathbf{p}_{t+1}) \mid i_{ht}, \mathbf{p}_t, c_{ht}, d_{ht}] \end{array} \right\} \right) dF_v(v_{ht}).$$

This Bellman equation is also a contraction mapping in the value function. The main computational cost in the computation of the functions $\bar{V}_h$ comes from the dimension of the vector of product attributes $\mathbf{p}_t$. We now explore ways to reduce this cost.

First, note that the assumption of a single aggregate inventory for all products, instead of one inventory for each brand, $\{i_{hjt}\}$, already reduces importantly the dimension of the state space. This assumption not only reduces the state space but, as we see below, it also allows us to modify the dynamic problem, which can significantly aid in the estimation of the model.

Taken literally, this assumption implies that there is no differentiation in consumption: the product is homogenous in use. Note, that through ξ_{jxt} and ε_{ijxt} the model allows differentiation in purchase, as is standard in the IO literature. It is well known that this

differentiation is needed to explain purchasing behavior. This seemingly creates a tension in the model: products are differentiated at purchase but not in consumption. Before explaining how this tension is resolved we note that the tension is not only in the model but potentially in reality as well. Many products seem to be highly differentiated at the time of purchase but it is hard to imagine that they are differentiated in consumption. For example, households tend to be extremely loyal to the laundry detergent brand they purchase – a typical household buys only 2-3 brands of detergent over a very long horizon – yet it is hard to imagine that the usage and consumption are very different for different brands.

A possible interpretation of the model that is consistent with product differentiation in consumption is that the variables ξ_{jxt} not only capture instantaneous utility at period t but also the discounted value of consuming the x units of brand j. This is a valid interpretation if brand-specific utility in consumption is additive such that it does not affect the marginal utility of consumption.

This assumption has some implications that simplify importantly the structure of the model. It implies that the optimal consumption does not depend on which brand is purchased, only on the size. And relatedly, it implies that the brand choice can be treated as a static decision problem.

We can distinguish two components in the choice d_{ht}: the quantity choice, x_{ht}, and the brand choice j_{ht}. Given $x_{ht} = x$, the optimal brand choice is:

$$j_{ht} = \arg \max_{j \in \{1,2,\ldots,J\}} \left\{ \beta_h \, a_{jxt} - \alpha_h \, p_{jxt} + \xi_{jxt} + \varepsilon_{hjxt} \right\}$$

Then, given our assumption about the distribution of ε_{hjxt}, the component m_{ht} of the utility function can be written as $m_{ht} = \sum_{x=0}^{X} \omega_h(x, \mathbf{p}_t) + e_{ht}$ where $\omega_{ht}(x, \mathbf{p}_t)$ is the inclusive value:

$$\begin{aligned} \omega_h(x, \mathbf{p}_t) &\equiv \mathbb{E}\left(\max_{j \in \{1,2,\ldots,J\}} \left\{ \beta_h \, a_{jxt} - \alpha_h \, p_{jxt} + \xi_{jxt} + \varepsilon_{hjxt} \right\} \mid x_{ht} = x, \ \mathbf{p}_t \right) \\ &= \ln\left(\sum_{j=1}^{J} \exp\left\{ \beta_h \, a_{jxt} - \alpha_h \, p_{jxt} + \xi_{jxt} \right\} \right) \end{aligned}$$

and e_{ht} does not depend on size x (or on inventories and consumption), and therefore we can ignore this variable for the dynamic decisions on size and consumption.

Therefore, the dynamic decision problem becomes:

$$\bar{V}_h(i_{ht}, \mathbf{p}_t) = \max_{c_{ht}, x_{ht}} \int \left\{ u_h(c_{ht}, v_{ht}) - C_i(i_{ht+1}) + \omega_h(x, \mathbf{p}_t) + \delta \, \mathbb{E}\left[\bar{V}_h(i_{ht+1}, \mathbf{p}_{t+1}) \mid i_{ht+1}, \mathbf{p}_t \right] \right\} dF_v(v_{ht})$$

In words, the problem can now be seen as a choice between sizes, each with a utility given by the size-specific inclusive value (and extreme value shock). The dimension of the state space is still large and includes all product attributes, because we need these attributes to compute the evolution of the inclusive value. However, in combination with additional assumptions, the modified problem is easier to estimate.

Note also that the expression describing the optimal brand choice, $j_{ht} = \arg\max_{j \in \{1,2,\ldots,J\}} \{\beta_h \, a_{jxt} - \alpha_h \, p_{jxt} + \xi_{jxt} + \varepsilon_{hjxt}\}$, is a "standard" multinomial logit model with the caveat that prices are endogenous explanatory variables because they depend on the unobserved

attributes in ξ_{jxt}. We describe below how to deal with this endogeneity problem. With household level data, dealing with the endogeneity of prices is much simpler than with aggregate data on market shares. More specifically, we do not need to use Monte Carlo simulation techniques, or an iterative algorithm to compute the "average utilities" $\{\delta_{jxt}\}$.

To reduce the dimension of the state space, Hendel and Nevo (2006) introduce the following assumption. Let $\omega_h(\mathbf{p}_t)$ be the vector with the inclusive values for every possible size $\{\omega_h(x,\mathbf{p}_t) : x = 1,2,...,X\}$.

Assumption: The vector $\omega_h(\mathbf{p}_t)$ is a sufficient statistic of the information in $\mathbf{p}_t$ that is useful to predict $\omega_h(\mathbf{p}_{t+1})$:

$$\Pr(\omega_h(\mathbf{p}_{t+1}) \mid \mathbf{p}_t) = \Pr(\omega_h(\mathbf{p}_{t+1}) \mid \omega_h(\mathbf{p}_t))$$

In words, the vector $\omega_h(\mathbf{p}_t)$ contains all the relevant information in $\mathbf{p}_t$ needed to obtain the probability distribution of $\omega_h(\mathbf{p}_{t+1})$ conditional on $\mathbf{p}_t$. Instead of all the prices and attributes, we only need a single index for each size. Two vectors of prices that yield the same (vector of) current inclusive values imply the same distribution of future inclusive values. This assumption is violated if individual prices have predictive power above and beyond the predictive power of $\omega_h(\mathbf{p}_t)$.

The inclusive values can be estimated outside the dynamic demand model. Therefore, the assumption can be tested and somewhat relaxed by including additional statistics of prices in the state space. Note that $\omega_h(\mathbf{p}_t)$ is consumer specific: different consumers value a given set of products differently and therefore this assumption does not further restrict the distribution of heterogeneity.

Given this assumption, the integrated value function is $\bar{V}_h(i_{ht}, \omega_{ht})$, which includes only $X+1$ variables, instead of $3*J*X+1$ state variables.

7.4 Estimation

7.4.1 Estimation of brand choice

Let j_{ht} represent the brand choice of household h at period t. Under the assumption that there is product differentiation in purchasing but not in consumption or in the cost of inventory holding, a household brand choice is a static decision problem. Given $x_{ht} = x$, with $x > 0$, the optimal brand choice is:

$$j_{ht} = \arg \max_{j \in \{1,2,...,J\}} \{\beta_h\, a_{jxt} - \alpha_h\, p_{jxt} + \xi_{jxt} + \varepsilon_{hjxt}\}$$

The estimation of demand models of differentiated products, either static or dynamic, should deal with two important issues. The first is the endogeneity of prices. The model implies that p_{jxt} depends on observed and unobserved products attributes, and therefore p_{jxt} and ξ_{jxt} are not independently distributed. The second issue is that the model should allow for rich heterogeneity in consumers' marginal utilities of product attributes, β_h and α_h. Using consumer-level data (instead of aggregate market share data) facilitates significantly the econometric solution of these issues.

Consumer-level scanner datasets contain rich information on household socioeconomic characteristics. Let z_h be a vector of observable socioeconomic characteristics that have a potential effect on demand, for instance, income, family size, age distribution

of children and adults, education, occupation, type of housing, etc. We assume that β_h and α_h depend on this vector of household characteristics:

$$\begin{aligned}\beta_h &= \beta_0 + (z_h - \bar{z})\sigma_\beta \\ \alpha_h &= \alpha_0 + (z_h - \bar{z})\sigma_\alpha\end{aligned}$$

β_0 and α_0 are scalar parameters that represent the marginal utility of advertising and income, respectively, for the average household in the sample. $\bar{z}$ is the vector of household attributes of the average household in the sample. σ_β and σ_α are $K \times 1$ vectors of parameters that represent the effect of household attributes on marginal utilities. Therefore, the utility of purchasing can be written as:

$$\begin{aligned}&\left[\beta_0 + (z_h - \bar{z})\sigma_\beta\right] a_{jxt} - \left[\alpha_0 + (z_h - \bar{z})\sigma_\alpha\right] p_{jxt} + \xi_{jxt} + \varepsilon_{hjxt} \\ =\ &\left[\beta_0\, a_{jxt} - \alpha_0\, p_{jxt} + \xi_{jxt}\right] + (z_h - \bar{z})\left[a_{jxt}\ \sigma_\beta - p_{jxt}\ \sigma_\alpha\right] + \varepsilon_{hjxt} \\ =\ &\delta_{jxt} + (z_h - \bar{z})\ \sigma_{jxt} + \varepsilon_{hjxt}\end{aligned}$$

where $\delta_{jxt} \equiv \beta_0\, a_{jxt} - \alpha_0\, p_{jxt} + \xi_{jxt}$, and $\sigma_{jxt} \equiv a_{jxt}\ \sigma_\beta - p_{jxt}\ \sigma_\alpha$. δ_{jxt} is a scalar that represents the utility of product (j,x,t) for the average household in the sample. σ_{jxt} is a vector and each element in this vector represents the effect of a household attribute on the utility of product (j,x,t).

In fact, it is possible to allow also for interactions between the observable household attributes and the unobservable product attributes, such that we have a term $\lambda_h \xi_{jxt}$ where $\lambda_h = 1 + (z_h - \bar{z})\sigma_\lambda$. With this more general specification, we still have that $\delta_{jxt} \equiv \beta_0\ a_{jxt} - \alpha_0\ p_{jxt} + \xi_{jxt}$, but now $\sigma_{jxt} \equiv a_{jxt}\ \sigma_\beta - p_{jxt}\ \sigma_\alpha + \xi_{jxt}\sigma_\lambda$.

Dummy-Variables Maximum Likelihood + IV estimator

Given this representation of the brand choice model, the probability that a household with attributes z_h purchases brand j at period t given that it buys x units of the product is:

$$P_{hjxt} = \frac{\exp\left\{\delta_{jxt} + (z_h - \bar{z})\ \sigma_{jxt}\right\}}{\sum_{k=1}^{J} \exp\left\{\delta_{kxt} + (z_h - \bar{z})\ \sigma_{kxt}\right\}}$$

Given a sample with a large number of households, we can estimate δ_{jxt} and σ_{jxt} for every (j,x,t) in a multinomial logit model with probabilities $\{P_{hjxt}\}$. For instance, we can estimate these "incidental parameters" δ_{jxt} and σ_{jxt} separately for every value of (x,t). For $(t=1,x=1)$ we select the subsample of households who purchase $x=1$ unit of the product at week $t=1$. Using this subsample, we estimate the vector of $J(K+1)$ parameters $\{\delta_{j11}, \sigma_{j11} : j = 1,2,...,J\}$ by maximizing the multinomial log-likelihood function:

$$\sum_{h=1}^{H} 1\{x_{h1} = 1\} \sum_{j=1}^{J} 1\{j_{h1} = j\}\ \ln P_{hj11}$$

We can proceed in the same way to estimate all the parameters $\{\delta_{jxt}, \sigma_{jxt}\}$.

This estimator is consistent as H goes to infinity for fixed T, X, and J. For a given (finite) sample, there are some requirements on the number of observations in order

to be able to estimate the incidental parameters. For every value of (x,t), the number of incidental parameters to estimate is $J(K+1)$, and the number of observations is equal to the number of households who purchase x units at week t, that is, $H(x,t) = \sum_{h=1}^{H} 1\{x_{ht} = x\}$. We need that $H(x,t) > J(K+1)$. For instance, with $J = 25$ products and $K = 4$ household attributes, we need $H(x,t) > 125$, for every week t and every size x. We may need a very large number of households H in the sample in order to satisfy these conditions. An assumption that may eliminate this problem is that the utility from brand choice is proportional to quantity: $x(\beta_h\ a_{jt} - \alpha_h\ p_{jt} + \xi_{jt} + \varepsilon_{hjt})$. Under this assumption, we have that for every week t, the number of incidental parameters to estimate is $J(K+1)$, but the number of observations is now equal to the number of households who purchase any quantity $x > 0$ at week t, that is, $H(t) = \sum_{h=1}^{H} 1\{x_{ht} > 0\}$. We need that $H(t) > J(K+1)$ which is a much weaker condition.

Given estimates of the incidental parameters, $\{\hat{\delta}_{jxt}, \hat{\sigma}_{jxt}\}$, we can now estimate the structural parameters β_0, α_0, σ_β, and σ_α using an IV (or GMM) method. For the estimation of β_0 and α_0, we have that:

$$\hat{\delta}_{jxt} = \beta_0\ a_{jxt} - \alpha_0\ p_{jxt} + \xi_{jxt} + e_{jxt}$$

where e_{jxt} represents the estimation error $(\hat{\delta}_{jxt} - \delta_{jxt})$. This is a linear regression where the regressor p_{jxt} is endogenous. We can estimate this equation by IV using the so-called "BLP instruments", that is, the characteristics other than price of products other than j, $\{a_{kxt} : k \neq j\}$. Of course, there are other approaches to deal with the endogeneity of prices in this equation. For instance, we could consider the following Error-Component structure in the endogenous part of the error term: $\xi_{jxt} = \xi_{jx}^{(1)} + \xi_{jxt}^{(2)}$ where $\xi_{jxt}^{(2)}$ is assumed not serially correlated. Then, we can control for $\xi_{jx}^{(1)}$ using product-size dummies, and use lagged values of prices and other product attributes to deal with the endogeneity of prices that comes from the correlation with the transitory shock $\xi_{jxt}^{(2)}$.

For the estimation of σ_β, and σ_α, we have the system of equations:

$$\hat{\sigma}_{jxt} \quad = \quad a_{jxt}\sigma_\beta - p_{jxt}\ \sigma_\alpha + \xi_{jxt}\sigma_\lambda + e_{jxt}$$

where e_{jxt} represents the estimation error $(\hat{\sigma}_{jxt} - \sigma_{jxt})$. We have one equation for each household attribute. We can estimate each of these equations using the same IV procedure as for the estimation of β_0 and α_0.

Once we have estimated $(\beta_0,\ \alpha_0,\ \sigma_\beta,\ \sigma_\alpha)$, we can also obtain estimates of ξ_{jxt} as residuals from the estimated equation. We can also get consistent estimates of the marginal utilities β_h and α_h as:

$$\hat{\beta}_h \quad = \quad \hat{\beta}_0 + (z_h - \bar{z})\hat{\sigma}_\beta$$

$$\hat{\alpha}_h \quad = \quad \hat{\alpha}_0 + (z_h - \bar{z})\hat{\sigma}_\alpha$$

Finally, we can get estimates of the inclusive values:

$$\hat{\omega}_{hxt} \quad = \quad \ln\left(\sum_{j=1}^{J} \exp\left\{\hat{\beta}_h\ a_{jxt} - \hat{\alpha}_h\ p_{jxt} + \hat{\xi}_{jxt}\right\}\right)$$

Control function approach

The previous approach, though simple, has a certain limitation: we need to have, for every week in the sample, a large enough number of households making positive purchases. However, this requirement is not needed for identification of the parameters, only for the implementation of the simple two-step dummy variables approach to deal with the endogeneity of prices.

When our sample does not satisfy that requirement, there is another simple method that we can use. This method is a control function approach that is in the spirit of the methods proposed by Rivers and Vuong (1988), **blundell_powell_2004** (**blundell_powell_2004**), and in the specific context of demand of differentiated products, **petrin_train_2010** (**petrin_train_2010**).

If firms choose prices to maximize profits, we expect that prices depend on the own product characteristics and also on the characteristics of competing products: $p_{jxt} = f_{jxt}(a_t, \xi_t)$, where $a_t = \{a_{jxt} : \text{for any } j,x\}$, and $\xi_t = \{\xi_{jxt} : \text{for any } j,x\}$. Define the conditional mean function:

$$g^p_{jx}(a_t) \equiv \mathbb{E}(p_{jxt} \mid a_t) = \mathbb{E}(f_{jxt}(a_t, \xi_t) \mid a_t)$$

Then, we can write the regression equation:

$$p_{jxt} = g^p_{jx}(a_t) + e_{jxt}$$

where the error term e_{jxt} is by construction mean independent of a_t.

The first step of the control function method consists in the estimation of the conditional mean functions g^p_{jx} for every brand and size (j,x). Though we have a relatively large number of weeks in our dataset (more than 100 weeks in most scanner datasets), the number of variables in the vector a_t is $J * X$, which is a significantly large number. Therefore, we need to impose some restrictions on how the exogenous product characteristics in a_t affect prices. For instance, we may assume that,

$$g^p_{jx}(a_t) = g^p_{jx}\left(a_{jxt},\ \bar{a}_{j(-x)t},\ \bar{a}_{(-j)xt},\ \bar{a}_{(-jx)t} \right)$$

where $\bar{a}_{j(-x)t}$ is the sample mean of variable a at period t for all the products of brand j but with different size than x; $\bar{a}_{(-j)xt}$ is the sample mean for all the products with size x but with brand different than j; and $\bar{a}_{(-jx)t}$ is the sample mean for all the products with size different than x and brand different than j. Of course, we can consider more flexible specifications but still with a number of regressors much smaller than $J * X$.

The second step of the method is based on a decomposition of the error term ξ_{jxt} in two components: an endogenous component that is a deterministic function of the error terms in the first step, $e_t \equiv \{e_{jxt} : \text{for any } j \text{ and } x\}$, and an "exogenous" component that is independent of the price p_{jxt} once we have controlled for e_{jxt}. Define the conditional mean function:

$$g^{\xi}_{jx}(e_t) \equiv \mathbb{E}(\xi_{jxt} \mid e_t)$$

Then, we can write ξ_{jxt} as the sum of two components, $\xi_{jxt} = g^{\xi}_{jx}(e_t) + v_{jxt}$. By construction, the error term v_{jxt} is mean independent of e_t. Additionally, v_{jxt} is mean independent of all the product prices because prices depend only on the exogenous product characteristics a_t (that by assumption are independent of ξ_{jxt}) and on the "residuals" e_t (that

by construction are mean independent of v_{jxt}). Therefore, we can write the utility of product (j,x) as:

$$\beta_h \, a_{jxt} - \alpha_h \, p_{jxt} + g^{\xi}_{jx}(e_t) + (v_{jxt} + \varepsilon_{hjxt})$$

The term $g^{\xi}_{jx}(e_t)$ is the control function.

Under the assumption that $(v_{jxt} + \varepsilon_{hjxt})$ is iid extreme value type 1 distributed, we have that the brand choice probabilities conditional on $x_{ht} = x$ are:

$$P_{hjxt} = \frac{\exp\left\{\beta_0 a_{jxt} - \alpha_0 p_{jxt} + a_{jxt}(z_h - \bar{z})\sigma_\beta - p_{jxt}(z_h - \bar{z})\sigma_\alpha + g^{\xi}_{jx}(e_t)\right\}}{\sum_{k=1}^{J} \exp\left\{\beta_0 a_{kxt} - \alpha_0 p_{kxt} + a_{kxt}(z_h - \bar{z})\sigma_\beta - p_{kxt}(z_h - \bar{z})\sigma_\alpha + g^{\xi}_{kx}(e_t)\right\}}$$

where the control functions $\{g^{\xi}_{jx}(e_t)\}$ consist of a brand dummy and a polynomial in the residual variables $\{e_{jxt} : j = 1,2,...,J\}$. Then, we can estimate $(\beta_0, \alpha_0, \sigma_\beta, \sigma_\alpha)$ and the parameters of the control function by using Maximum Likelihood in this multinomial logit model. The log-likelihood function is:

$$\ell(\theta) = \sum_{h=1}^{H} \sum_{t=1}^{T} \sum_{x=1}^{X} \sum_{j=1}^{J} 1\{x_{ht} = x \, , \; j_{ht} = j\} \; \ln P_{hjxt}$$

As in the previous method, once we have estimated these parameters, we can construct consistent estimates of the inclusive values ω_{hxt}.

7.4.2 Estimation of quantity choice

As mentioned above, the lack of data on household inventories is a challenging econometric problem because this is a key state variable in a dynamic demand model for the demand of a storable good. Also, this is not a "standard" unobservable variable, as it follows a stochastic process that is endogenous. That is, not only do inventories affect purchasing decisions, but purchasing decisions also affect the evolution of inventories.

The approach used by **erdem_keane_2003** (**erdem_keane_2003**) to deal with this problem is to assume that household inventories is a (deterministic) function of the "number of weeks (duration) since last purchase", T_{ht}, and the quantity purchased in the last purchase, $x^{\ell ast}_{ht}$:

$$i_{ht} = f_h(x^{\ell ast}_{ht}, T_{ht})$$

In general, this assumption holds under two conditions: (1) consumption is deterministic; and (2) when a new purchase is made, the existing inventory at the beginning of the week is consumed or scrapped. For instance, suppose that these conditions hold and that the level of consumption is constant $c_{ht} = c_h$. Then,

$$i_{ht+1} = \max\left\{ 0 \; ; \; x^{\ell ast}_{ht} - c_h \, T_{ht} \right\}$$

The constant consumption can be replaced by a consumption rate that depends on the level of inventories. For instance, $c_{ht} = \lambda_h i_{ht}$. Then:

$$i_{ht+1} = \max\left\{ 0 \; ; \; (1 - \lambda_h)^{T_{ht}} \, x^{\ell ast}_{ht} \right\}$$

Using this approach, the state variable i_{ht} should be replaced by the state variables $(x_{ht}^{\ell ast}, T_{ht})$, but the rest of the features of the model remain the same. The parameters c_h or λ_h can be estimated together with the rest of the parameters of the structural model. Also, we may not need to solve for the optimal consumption decision.

There is no doubt that using observable variables to measure inventories is very useful for the estimation of the model and for identification. It also provides a more intuitive interpretation of the identification of the model.

The individual level data provide the probability of purchase conditional on current prices, and past purchases of the consumer (amounts purchased and duration from previous purchases): $\Pr(x_{ht}|x_{ht}^{\ell ast}, T_{ht}, \mathbf{p}_t)$. Suppose that we observe that this probability is not a function of past behavior $(x_{ht}^{\ell ast}, T_{ht})$. We would then conclude that dynamics are not relevant, and that consumers are purchasing for immediate consumption and not for inventory. On the other hand, if we observe that the purchase probability is a function of past behavior, and we assume that preferences are stationary, then we conclude that there is dynamic behavior.

Regarding the identification of storage costs, consider the following example. Suppose we observe two consumers who face the same price process and purchase the same amount over a relatively long period. However, one of them purchases more frequently than the other. This variation leads us to conclude that this consumer has higher storage costs. Therefore, the storage costs are identified from the average duration between purchases.

Hendel and Nevo use a different approach, though the identification of their model is based on the same intuition.

7.5 Empirical Results

To Be Completed

7.6 Dynamic Demand of Differentiated Durable Products

- **gowrisankaran_rysman_2009** (**gowrisankaran_rysman_2009**). TBW

8. Dynamic Games: Model and Metho

8.1 Introduction

The following chapter deals with methods and applications of empirical dynamic games of oligopoly competition. More generally, some of the methods that we will describe can be applied other applied fields such as political economy (for instance, competition between political parties), or international economics (for instance, ratification of international treaties), among others.

Dynamic games are powerful tools for the analysis of phenomena characterized by **dynamic strategic interactions** between multiple agents. By *dynamic strategic interactions* we mean that:

> **(a)** players' current decisions affect their own and other players' payoffs in the future (that is, *multi-agent dynamics*);
>
> **(b)** players' decisions are forward-looking in the sense that they take into account the implications on their own and on their rivals' future behavior and how this behavior affects future payoffs (that is, *strategic behavior*).

Typical sources of dynamic strategic interactions are decisions that are partially irreversible (costly to reverse) or that involve sunk costs. Some examples in the context of firm oligopoly competition: **(1)** entry-exit in markets; **(2)** introduction of a new product; timing of the release of a new movie; **(3)** repositioning of product characteristics; **(4)** investment in capacity, or equipment, or R&D, or quality, or goodwill, or advertising; **(5)** pricing of a durable good; pricing when demand is characterized by consumer switching costs; **(6)** production when there is learning-by-doing.

Taking into account dynamic strategic interactions may change substantially our interpretation of some economic phenomena or the implications of some public policies. We have already discussed some examples from recent applications in IO: **(1)** Short-run and long-run responses to changes in industry regulations (**ryan_2006** ,**ryan_2006**); **(2)** Product repositioning in differentiated product markets (Sweeting ,2007); **(3)** Dynamic aspects of network competition (**aguirregabiria_ho_2008** ,**aguirregabiria_ho_2008**).

Road Map: 1. Structure of empirical dynamic games; 2. Identification; 3. Estimation; 4. Dealing with unobserved heterogeneity; 5. Empirical Applications; 5.1. Dynamic effects of industry regulation (**ryan_2006** ,**ryan_2006**); 5.2. Product repositioning in differentiated product markets (Sweeting ,2007); 5.3. Dynamic aspects of network competition (**aguirregabiria_ho_2008** ,**aguirregabiria_ho_2008**).

8.2 Dynamic version of Bresnahan-Reiss model

The following section is based on Bresnahan and Reiss (Bresnahan and Reiss):
Complete information; homogeneous firms

8.2.1 Motivation

Suppose that we have panel data of M markets over T periods of time.

$$\text{Data} = \{ n_{mt}, X_{mt} : m = 1,2,...,M; t = 1,2,...,T \}$$

In these data, we observe how the number of firms grows or declines in the market. Suppose that we do not know the gross changes in the number of firms, that is, we do not observe the number of new entrants, en_{mt}, and number of exits, ex_{mt}. We only observe the net change $n_{mt} - n_{mt-1} = en_{mt} - ex_{mt}$.

To explain the observed variation, across markets and over time, in the number of firms, we could estimate the BR static model that we have considered so far. The only difference is that now we have multiple realizations of the game both because the game is played at different locations and because it is played at different periods of time.

However, the static BR model imposes a strong and unrealistic restriction on this type of panel data. According to the static model, the number of firms in the previous period, n_{mt-1}, does not play a role in the determination of the current number of firms n_{mt}. This is because the model considers that the profit of an active firm is the same regardless of whether it was active in the previous period or not. That is, the model assumes that either there are no entry costs, or that entry costs are paid every period in which the firm is active such that both new entrants and incumbents should pay these costs. Of course, this assumption is very unrealistic for most industries.

Bresnahan and Reiss (1994) propose and estimate a dynamic extension of their static model of entry. This dynamic model distinguishes between incumbents and potential entrants and takes into account the existence of sunk entry costs. The model is simple but interesting, and useful because of its simplicity. We could call it a "semi-structural" model. It is structural in the sense that it is fully consistent with dynamic game of entry-exit in an oligopoly industry. But it is only "semi" in the sense that it does not model explicitly how the future expected value function of an incumbent firm depends on the sunk-cost. Ignoring this relationship has clear computational advantages in the estimation of the model. However it has also limitations in terms of the type of counterfactuals and empirical questions that can be studied using this model.

8.2.2 Model

Let n_t be the number of active firms in the market at period t. n_t belongs to the set $\{0,1,...,N\}$ where N is a large but finite number. Let $V(n_t, X_t) - \varepsilon_t$ be the value function

of an active firm in a market with exogenous characteristics (X_t, ε_t) and number of firms n_t. The additive error term ε_t can be interpreted as an iid shock in the fixed cost of being active in the market. The function $V(n,X)$ is strictly decreasing in n.

This value function does not include the cost of entry. Let EC be the entry cost that a new entrant should pay to be active in the market at period t. Let SV be the scrapping value of a firm that decides to exit from the market. For the moment, we consider that EC and SV are constant parameters but we will discuss later how this assumption can be relaxed.

An important and obvious condition is that $SV \leq EC$. That is, firms cannot make profits by constantly entering and exiting in a market. It is an obvious arbitrage condition. The parameter $EC - SV$ is called the sunk entry cost, that is, it is the part of the entry cost that is sunk and cannot be recovered upon exit. This can include administrative costs, costs of market research, and in general any investment in capital that is firm specific and therefore will not have market value when the firm exits the market.

The values or payoffs of incumbents and potential entrants are: Incumbent that decides to stay: $V(n_t, X_t) - \varepsilon_t$; Incumbent that exits: SV; New entrant: $V(n_t, X_t) - \varepsilon_t - EC$; Potential entrant stays out: 0.

Below, we describe the entry-exit equilibrium conditions that determine the equilibrium number of firms n_t as a function of (X_t, ε_t).

Regime 1: Exit. Suppose that $n_{t-1} > 0$ and $V(n_{t-1}, X_t) - \varepsilon_t < SV$. That is, at the beginning of period t, the values of the exogenous variables X_t and ε_t are realized, and the incumbent firms from the previous period find out that the value of being active in the market is smaller than the scrapping value of the firm. Therefore, these firms want to exit.

It should be clear that under this regime there is no entry. Since $SV \leq EC$, we have that $V(n_{t-1}, X_t) - \varepsilon_t < EC$ and therefore $V(n_{t-1}+1, X_t) - \varepsilon_t < EC$. The value for a new entrant is smaller than the entry cost and therefore there is no entry.

Therefore, incumbent firms will start exiting the market up to the point where either: (a) there are no more firms in the market, that is, $n_t = 0$; or (b) there are still firms in the market and the value of an active firm is greater or equal to the scrapping value. The equilibrium number of firms in this regime is given by the conditions:

$$\begin{cases} n_t = 0 & \text{if} \quad V(1, X_t) - \varepsilon_t < SV \\ \text{OR} & \\ n_t = n > 0 & \text{if} \quad \{V(n, X_t) - \varepsilon_t \geq SV\} \quad \text{AND} \quad \{V(n+1, X_t) - \varepsilon_t < SV\} \end{cases}$$

The condition $\{V(n_t, X_t) - \varepsilon_t \geq SV\}$ says that an active firm in the market does not want to exit. Condition $\{V(n_t + 1, X_t) - \varepsilon_t < SV\}$ establishes that if there were any number of firms in the market greater than n_t, firms would prefer to exit.

Summarizing, **Regime 1 [Exit]** is described by the following condition on exogenous variables $\{n_{t-1} > 0\}$ and $\{\varepsilon_t > V(n_{t-1}, X_t) - SV\}$, and this condition implies that:

$$n_t < n_{t-1}$$

and n_t is determined by

$$\begin{cases} n_t = 0 & \text{if} \quad V(1, X_t) - \varepsilon_t < SV \\ \text{OR} & \\ n_t = n > 0 & \text{if} \quad V(n+1, X_t) - SV < \varepsilon_t \leq V(n, X_t) - SV \end{cases}$$

Regime 2: Entry. Suppose that $n_{t-1} < N$ and $V(n_{t-1}+1, X_t) - \varepsilon_t \geq EC$. At the beginning of period t, potential entrants realize that the value of being active in the market is greater than the entry cost. Therefore, potential entrants want to enter in the market.

It should be clear that under this regime there is no exit. Since $SV \leq EC$ and $V(n_{t-1}+1, X_t) < V(n_{t-1}, X_t)$, we have that the condition $\{V(n_{t-1}+1, X_t) - \varepsilon_t \geq EC\}$ implies that $\{V(n_{t-1}, X_t) - \varepsilon_t > SV\}$. The value of an incumbent is greater than the scrapping value and therefore there is no exit.

Therefore, new firms will start entering the market up to the point where either: (a) there are no more potential entrants to enter in the market, that is, $n_t = N$; or (b) there are still potential entrants that may enter the market but the value of an active firm goes down to a level such that there are no more incentives for additional entry. The equilibrium number of firms in this regime is given by the conditions:

$$\begin{cases} n_t = N & \text{if} \quad V(N, X_t) - \varepsilon_t \geq EC \\ \text{OR} & \\ n_t = n < N & \text{if} \quad \{V(n, X_t) - \varepsilon_t \geq EC\} \quad \text{AND} \quad \{V(n+1, X_t) - \varepsilon_t < EC\} \end{cases}$$

Condition $\{V(n_t, X_t) - \varepsilon_t \geq EC\}$ says that the last firm that entered the market had an incentive to enter. Condition $\{V(n_t+1, X_t) - \varepsilon_t < EC\}$ establishes that the next firm entering the market would not get enough value to cover the entry cost.

Summarizing, **Regime 2 [Entry]** is described by the following condition on exogenous variables $\{n_{t-1} < N\}$ and $\{\varepsilon_t \leq V(n_{t-1}+1, X_t) - EC\}$, and this condition implies that:

$$n_t > n_{t-1}$$

and n_t is determined by

$$\begin{cases} n_t = N & \text{if} \quad V(N, X_t) - \varepsilon_t \geq EC \\ \text{OR} & \\ n_t = n < N & \text{if} \quad V(n+1, X_t) - EC < \varepsilon_t \leq V(n, X_t) - EC \end{cases}$$

Regime 3: Inaction. The third possible regime is given by the complementary conditions to those that define regimes 1 and 2. Under these conditions, incumbent firms do not want to exit and potential entrants do not want to enter.

$$\{n_t = n_{t-1}\} \text{ iff} \begin{cases} & \{n_{t-1} = 0\} \text{ AND } \{V(1, X_t) - \varepsilon_t < EC\} \\ \text{OR} & \\ & \{n_{t-1} = N\} \text{ AND } \{V(N, X_t) - \varepsilon_t \geq SV\} \\ \text{OR} & \\ & \{0 < n_{t-1} < N\} \text{ AND } \{V(n_{t-1}+1, X_t) - \varepsilon_t < EC\} \text{ AND } \{V(n_{t-1}, X_t) - \varepsilon_t \geq SV\} \end{cases}$$

Putting the three regimes together, we can obtain the probability distribution of the endogenous n_t conditional on (n_{t-1}, X_t). Assume that ε_t is i.i.d. and independent of X_t

with CDF F_ε. Then:

$$\Pr(n_t = n \mid n_{t-1},\, X_t) = \begin{cases} F_\varepsilon\left(\frac{V(n,X_t)-SV}{\sigma}\right) - F_\varepsilon\left(\frac{V(n+1,X_t)-SV}{\sigma}\right) & \text{if} \quad n < n_{t-1} \\ F_\varepsilon\left(\frac{V(n_{t-1},X_t)-SV}{\sigma}\right) - F_\varepsilon\left(\frac{V(n_{t-1}+1,X_t)-EC}{\sigma}\right) & \text{if} \quad n = n_{t-1} \\ F_\varepsilon\left(\frac{V(n,X_t)-EC}{\sigma}\right) - F_\varepsilon\left(\frac{V(n+1,X_t)-EC}{\sigma}\right) & \text{if} \quad n > n_{t-1} \end{cases}$$

It is interesting to compare this probability distribution of the number of firms with the one from the static BR model. In the static BR model:

$$\Pr(n_t = n \mid n_{t-1},\, X_t) = F_\varepsilon\left(\frac{V(n,X_t)}{\sigma}\right) - F_\varepsilon\left(\frac{V(n+1,X_t)}{\sigma}\right)$$

This is exactly the distribution that we get in the dynamic model when $EC = SV$. Note that under $EC = SV$, the sunk cost $EC - SV$ is zero and firms' entry-exit decisions are static.

When $EC > SV$ (positive sunk cost), the dynamic model delivers different predictions than the static model. There are two main differences. First, the number of firms is more persistent over time, that is, there is "structural state dependence" in the number of firms.

$$\Pr(n_t = n_{t-1} \mid n_{t-1},\, X_t) = \begin{cases} F_\varepsilon\left(\frac{V(n_{t-1},X_t)-SV}{\sigma}\right) - F_\varepsilon\left(\frac{V(n_{t-1}+1,X_t)-EC}{\sigma}\right) & \text{if} \quad EC > SV \\ F_\varepsilon\left(\frac{V(n_{t-1},X_t)}{\sigma}\right) - F_\varepsilon\left(\frac{V(n_{t-1}+1,X_t)}{\sigma}\right) & \text{if} \quad EC = SV \end{cases}$$

In the static model, all the persistence in the number of firms is because this variable is indivisible - it is an integer. However, in the dynamic model, sunk entry costs introduce more persistence. A purely transitory shock (in X_t or in ε_t) that increases the number of firms at some period t will have a persistent effect for several periods in the future.

Second, the number of firms responds asymmetrically to positive and negative shocks. Given $EC > SV$, it is possible to show that the upward response is less elastic than the downward response.

8.2.3 Identification

It is interesting to explore the identification of the model. With this model and data, we cannot identify nonparametrically the distribution of ε_t. So we make a parametric assumption on this distribution. For instance, we assume that ε_t has a $N(0,\sigma^2)$ distribution.

Define $P_{entry}(n_{t-1}, X_t)$ and $P_{exit}(n_{t-1}, X_t)$ as the probabilities of positive (entry) and negative (exit) changes in the number of firms, respectively. That is, $P_{entry}(n_{t-1}, X_t) \equiv \Pr(n_t > n_{t-1} \mid n_{t-1}, X_t)$ and $P_{exit}(n_{t-1}, X_t) \equiv \Pr(n_t < n_{t-1} \mid n_{t-1}, X_t)$. These probability functions are nonparametrically identified from our panel data on $\{n_t, X_t\}$.

The model predicts the following structure for the probabilities of entry and exit:

$$\begin{aligned} P_{entry}(n_{t-1}, X_t) &= \Pr(V(n_{t-1}+1, X_t) - \varepsilon_t > EC \mid X_t) = \\ &= \Phi\left(\frac{V(n_{t-1}+1, X_t) - EC}{\sigma}\right) \end{aligned}$$

and:

$$\begin{aligned} P_{exit}(n_{t-1}, X_t) &= \Pr\left(V(n_{t-1}, X_t) - \varepsilon_t < SV \mid X_t\right) \\ &= 1 - \Phi\left(\frac{V(n_{t-1}, X_t) - SV}{\sigma}\right) \end{aligned}$$

Using these expressions, it is simple to obtain that, for any (n_{t-1}, X_t):

$$\frac{EC - SV}{\sigma} = \Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right) - \Phi^{-1}\left(P_{entry}(n_{t-1} - 1, X_t)\right)$$

where Φ^{-1} is the inverse function of the CDF of ε_t.

Therefore, even with a nonparametric specification of the value function $V(n,X)$, we can identify the sunk cost up to scale. Note that this expression provides a clear intuition about the source of identification of this parameter. The magnitude of this parameter is identified by "a distance" between the probability of entry of potential entrants and the probability of staying for incumbents $(1 - P_{exit})$. In a model without sunk costs, both probabilities should be the same. In a model with sunk costs, the probability of staying in the market should be greater than the probability of entry.

Since we do not know the value of σ, the value of the parameter $\frac{EC-SV}{\sigma}$ is not meaningful from an economic point of view. However, based on the identification of $\frac{EC-SV}{\sigma}$ and the identification up to scale of the value function $V(n,X)$ (that we show below), it is possible to get an economically meaningful estimate of the importance of the sunk cost. Suppose that $V(n,X)/\sigma$ is identified. Then, we can identify the ratio:

$$\frac{EC - SV}{V(n,X)} = \frac{\frac{EC-SV}{\sigma}}{\frac{V(n,X)}{\sigma}}$$

For instance, we have $\frac{EC-SV}{V(1,X)}$ which is the percentage of the sunk cost over the value of a monopoly in a market with characteristics X.

Following the same argument as for the identification of the constant parameter $\frac{EC-SV}{\sigma}$, we can show the identification of a sunk cost that depends nonparametrically on the state variables (n_{t-1}, X_t). That is, we can identify a sunk cost function $\frac{EC(n_{t-1},X_t)-SV(n_{t-1},X_t)}{\sigma}$. This has economic interest. In particular, the dependence of the sunk cost with respect to the number of incumbents n_{t-1} is evidence of endogenous sunk costs (see Sutton's book titled *"Sunk Costs and Market Structure,"* MIT Press, 1991). Therefore, we can test nonparametrically for the existence of endogenous sunk costs by testing the dependence of the estimated function $\frac{EC(n_{t-1},X_t)-SV(n_{t-1},X_t)}{\sigma}$ with respect to n_{t-1}.

We can also use the probabilities of entry and exit to identify the value function $V(n,X)$. The model implies that:

$$\begin{aligned} \Phi^{-1}\left(P_{entry}(n_{t-1} - 1, X_t)\right) &= \frac{V(n_{t-1},X_t)-EC}{\sigma} \\ \Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right) &= \frac{V(n_{t-1},X_t)-SV}{\sigma} \end{aligned}$$

The left-hand-side of these equations is identified from the data. From these expressions, it should be clear that we cannot identify EC/σ separately from a constant term in the value function (a fixed cost), and we cannot identify SV/σ separately from a constant term in the value function.

Let $-FC$ be the constant term or fixed cost in the value function. More formally, define the parameter FC as the expected value:

$$FC \equiv -\mathbb{E}\left(V(n_{t-1}, X_t)\right)$$

Also define the function $V^*(n_{t-1}, X_t)$ as the deviation of the value function with respect to its mean:

$$\begin{aligned} V^*(n_{t-1}, X_t) &\equiv V(n_{t-1}, X_t) - -\mathbb{E}\left(V(n_{t-1}, X_t)\right) \\ &= V(n_{t-1}, X_t) + FC \end{aligned}$$

Also, define $EC^* \equiv EC + FC$, and $SV^* \equiv SV + FC$ such that, by definition, $V(n_{t-1}, X_t) - EC = V^*(n_{t-1}, X_t) - EC^*$, and $V(n_{t-1}, X_t) - SV = V^*(n_{t-1}, X_t) - SV^*$.

Then, $\frac{EC^*}{\sigma}$, $\frac{SV^*}{\sigma}$, and $V^*(n_{t-1}, X_t)/\sigma$ are identified nonparametrically from the following expressions:

$$\begin{aligned} \frac{EC^*}{\sigma} &= E\left(\Phi^{-1}\left(P_{entry}(n_{t-1} - 1, X_t)\right)\right) \\ \frac{SV^*}{\sigma} &= \mathbb{E}\left(\Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right)\right) \end{aligned}$$

And

$$\begin{aligned} \frac{V^*(n_{t-1}, X_t)}{\sigma} &= \Phi^{-1}\left(P_{entry}(n_{t-1} + 1, X_t)\right) - \mathbb{E}\left(\Phi^{-1}\left(P_{entry}(n_{t-1} + 1, X_t)\right)\right) \\ &\text{and} \\ \frac{V^*(n_{t-1}, X_t)}{\sigma} &= \Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right) - \mathbb{E}\left(\Phi^{-1}\left(1 - P_{exit}(n_{t-1}, X_t)\right)\right) \end{aligned}$$

In fact, we can see that the function $V^*(.,.)$ is over identified: it can be identified either from the probability of entry or from the probability of exit. This provides over-identification restrictions that can be used to test the restrictions or assumptions of the model.

Again, one of the main limitations of this model is the assumption of homogeneous firms. In fact, as an implication of that assumption, the model predicts that there should not be simultaneous entry and exit. This prediction is clearly rejected in many panel datasets on industry dynamics.

8.2.4 Estimation of the model

Given a parametric assumption about the distribution of ε_t, and a parametric specification of the value function $V(n, X)$, we can estimate the model by conditional maximum likelihood. For instance, suppose that ε_t is i.i.d. across markets and over time with a distribution $N(0, \sigma^2)$, and the value function is linear in parameters:

$$V(n_t, X_t) = g(n_t, X_t)'\beta - FC$$

where $g(.,.)$ is a vector of known functions, and β is a vector of unknown parameters.

Let θ be the vectors of parameters to estimate:

$$\theta = \{ \beta/\sigma, EC^*/\sigma, SV/\sigma \}$$

Then, we can estimate θ using the conditional maximum likelihood estimator:

$$\hat{\theta} = \arg\max_{\theta} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{n=0}^{N} 1\{n_{mt} = n\} \log \Pr(n \mid n_{mt-1},\ X_{mt}; \theta)$$

where:

$$\Pr(n_t = n \mid n_{t-1},\ X_t) = \begin{cases} \Phi\left(g(n,X_t)'\frac{\beta}{\sigma} - \frac{SV}{\sigma} \right) - \Phi\left(g(n+1,X_t)'\frac{\beta}{\sigma} - \frac{SV}{\sigma} \right) & \text{if} \quad n < n_{t-1} \\ \Phi\left(g(n,X_t)'\frac{\beta}{\sigma} - \frac{SV}{\sigma} \right) - \Phi\left(g(n+1,X_t)'\frac{\beta}{\sigma} - \frac{EC}{\sigma} \right) & \text{if} \quad n = n_{t-1} \\ \Phi\left(g(n,X_t)'\frac{\beta}{\sigma} - \frac{EC}{\sigma} \right) - \Phi\left(g(n+1,X_t)'\frac{\beta}{\sigma} - \frac{EC}{\sigma} \right) & \text{if} \quad n > n_{t-1} \end{cases}$$

Based on the previous identification results, we can also construct a simple least squares estimator of θ. Let $\hat{P}_{mt}^{entry}$ and $\hat{P}_{mt}^{exit}$ be nonparametric Kernel estimates of $P_{entry}(n_{mt-1}+1, X_{mt})$ and $P_{exit}(n_{mt-1}, X_{mt})$, respectively. The model implies that:

$$\begin{aligned} \Phi^{-1}\left(\hat{P}_{mt}^{entry}\right) &= \left(-\frac{EC^*}{\sigma}\right) + g(n_{mt-1},X_{mt})'\frac{\beta}{\sigma} + e_{mt}^{entry} \\ \Phi^{-1}\left(1-\hat{P}_{mt}^{exit}\right) &= \left(-\frac{SV^*}{\sigma}\right) + g(n_{mt-1},X_{mt})'\frac{\beta}{\sigma} + e_{mt}^{exit} \end{aligned}$$

where e_{mt}^{entry} and e_{mt}^{exit} are error terms that come from the estimation error in $\hat{P}_{mt}^{entry}$ and $\hat{P}_{mt}^{exit}$. We can put together these regression equations in a single regression as:

$$Y_{dmt} = D_{dmt}\left(-\frac{EC^*}{\sigma}\right) + (1-D_{dmt})\left(-\frac{SV^*}{\sigma}\right) + g(n_{mt-1},X_{mt})'\frac{\beta}{\sigma} + e_{mt}$$

where $Y_{dmt} \equiv D_{dmt}\ \Phi^{-1}\left(\hat{P}_{mt}^{entry}\right) + (1-D_{dmt})\Phi^{-1}\left(1-\hat{P}_{mt}^{exit}\right)$; the subindex d represents the "regime", $d \in \{entry, exit\}$, and D_{dmt} is a dummy variable that is equal to one when $d = entry$ and it is equal to zero when $d = exit$.

OLS estimation of this linear regression equation provides a consistent estimator of θ. This estimator is not efficient but we can easily obtain an asymptotically efficient estimator by making one Newton-Raphson iteration in the maximization of the likelihood function.

8.2.5 Structural model and counterfactual experiments

This dynamic model is fully consistent with a dynamic game of entry-exit. However, the value function $V(n,X)$ is not a primitive or a structural function. It implicitly depends

on the one-period profit function, on the entry cost EC, on the scrapping value SV, and on the equilibrium of the model (that is, on equilibrium firms' strategies).

The model and the empirical approach that we have described above does not make explicit the relationship between the primitives of the model and the value function, or how this value function depends on the equilibrium transition probability of the number of firms, $P^*(n_{t+1}|n_t, X_t)$. This "semi-structural" approach has clear advantages in terms of computational and conceptual simplicity. However, it also has its limitations. We discuss here its advantages and limitations.

Similar approaches have been proposed and applied for the estimation of dynamic models of occupational choice by **geweke_keane_2001** (**geweke_keane_2001**) and **hoffmann_2009** (**hoffmann_2009**). This type of approach is different to other methods that have been proposed and applied to the estimation of dynamic structural models and that also try to reduce the computational cost in estimation, such as Hotz and Miller (1993 and 1994) and Aguirregabiria and Mira (2002 and 2007).

To understand the advantages and limitations of Bresnahan and Reiss' "semi-structural" model of industry dynamics, it is useful to relate the value function $V(n_t, X_t)$ with the actual primitives of the model. Let $\pi(n_t, X_t, \varepsilon_t)$ be the profit function of an incumbent firm that stays in the market. Therefore:

$$V(n_t, X_t) = \mathbb{E}\left(\sum_{j=0}^{\infty} \delta^j \left[(1 - Exit_{t+j}) \, \pi(n_{t+1}, X_{t+j}, \varepsilon_{t+j}) + Exit_{t+j} SV \right] \mid n_t, X_t \right)$$

where δ is the discount factor, and $Exit_{t+j}$ is a binary variable that indicates if the firm exits from the market at period $t+j$ (that is, $Exit_{t+j} = 1$) or stays in the market (that is, $Exit_{t+j} = 0$). The expectation is taken over all future paths of the state variables $\{n_{t+1}, X_{t+j}, \varepsilon_{t+j}\}$. In particular, this expectation depends on the stochastic process that follows the number of firms in equilibrium and that is governed by the transition probability $\Pr(n_{t+1}|n_t, X_t)$.

The transition probability $\Pr(n_{t+1}|n_t, X_t)$ is determined in equilibrium and it depends on all the structural parameters of the model. More specifically, this transition probability can be obtained as the solution of a fixed point problem. Solving this fixed point problem is computationally demanding. The "semi-structural" approach avoids this computational cost by ignoring the relationship between the value function $V(n_t, X_t)$ and the structural parameters of the model. This can provide huge computational advantages, especially when the dimension of the state space of (n_t, X_t) is large and/or when the dynamic game may have multiple equilibria.

These significant computational gains come with a cost. The range of predictions and counterfactual experiments that we can make using the estimated "semi-structural" model is very limited. In particular, we cannot make predictions about how the equilibrium transition $\Pr(n_{t+1}|n_t, X_t)$ (or the equilibrium steady-state distribution of n_t) changes when we perturb one the parameters in θ.

There are two types of problems in this model associated with implementing the predictions of counterfactual experiments. First, the parameters β are not structural such that we cannot change one of these parameters and assume that the rest will stay constant. In other words, we do not know what that type of experiment means.

Second, though EC^* and SV^* are structural parameters, the parameters β in the value function should depend on EC^* and SV^*, but we do not know the form of that

relationship. We cannot assume that EC^* or SV^* and β remains constant. In other words, that type of experiment does not have a clear interpretation or economic interest.

For instance, suppose that we want to predict how a 20% increase in the entry cost would affect the transition dynamics and the steady state distribution of the number of firms. If $\lambda_0 = EC^*/\sigma$ is our estimate of the value of the parameter in the sample, then its counterfactual value is $\lambda_1 = 1.2\lambda_0$. However, we also know that the value function should change. In particular, the value of an incumbent firm increases when the entry costs increases.

The "semi-structural" model ignores that the value function V will change as the result of the change in the entry cost. Therefore, it predicts that entry will decline, and that the exit/stay behavior of incumbent firms will not be affected because V and SV have not changed.

There are two errors in the prediction of the "semi-structural" model. First, it overestimates the decline in the amount of entry because it does not take into account that being an incumbent in the market now has more value. And second, it ignores that, for the same reason, exit of incumbent firms will also decline.

Putting these two errors together, we have that this counterfactual experiment using the "semi-structural" model can lead to a serious under-estimate of the number of firms in the counterfactual scenario.

Later in the chapter we will study other methods for the estimation of structural models of industry dynamics that avoid the computational cost of solving for the equilibrium of the game but that do not have the important limitations, in terms of counterfactual experiments, of the semi-structural model here.

Nevertheless, it is difficult to overemphasize the computational advantages of Bresnahan-Reiss' empirical model of industry dynamics. It is a useful model to obtain a first cut of the data, and to answer empirical questions that do not require the implementation of counterfactual experiments. For instance, we can test for endogenous sunk costs, or measure the magnitude of sunk costs relative to the value of an incumbent firm.

8.3 The structure of dynamic games of oligopoly competition

8.3.1 Basic Framework and Assumptions

Time is discrete and indexed by t. The game is played by N firms that we index by i. Following the standard structure in the Ericson and Pakes (1995) framework, firms compete in two different dimensions: a static dimension and a dynamic dimension. We denote the dynamic dimension as the "investment decision". Let a_{it} be the variable that represents the investment decision of firm i at period t. This investment decision can be an entry/exit decision, a choice of capacity, investment in equipment, R&D, product quality, other product characteristics, etc. Every period, given their capital stocks that can affect demand and/or production costs, firms compete in prices or quantities in a static Cournot or Bertand model. Let p_{it} be the static decision variables (for instance, price) of firm i at period t.

For simplicity and concreteness, we start by presenting a simple dynamic game of market entry-exit where every period incumbent firms compete à la Bertrand. In this entry-exit model, the dynamic investment decision a_{it} is a binary indicator of the event "firm i is active in the market at period t". The action is taken to maximize the expected

and discounted flow of profits in the market, $\mathbb{E}_t\left(\sum_{r=0}^{\infty}\delta^r\,\Pi_{it+r}\right)$ where $\delta\in(0,1)$ is the discount factor, and Π_{it} is firm i's profit at period t. The profits of firm i at time t are given by

$$\Pi_{it} = VP_{it} - FC_{it} - EC_{it}$$

where VP_{it} represents variable profits, FC_{it} is the fixed cost of operating, and EC_{it} is a one time entry cost. We now describe these different components of the profit function.

(a) Variable Profit Function. The variable profit VP_{it} is an "indirect" variable profit function that comes from the equilibrium of a static Bertrand game with differentiated product. Consider the simplest version of this type of model. Suppose that all firms have the same marginal cost c, and product differentiation is symmetric. Consumer utility of buying product i is $u_{it} = \nu - \alpha p_{it} + \varepsilon_{it}$, where ν and α are parameters, and ε_{it} is a consumer-specific i.i.d. extreme value type 1 random variable. Under these conditions, the equilibrium variable profit of an active firm depends only on the number of firms active in the market.

$$VP_{it} = (p_{it} - c)q_{it}$$

where p_{it} and q_{it} represent the price and the quantity sold by firm i at period t, respectively. According to this model, the quantity is:

$$q_{it} = H_t\,\frac{a_{it}\,\exp\{\nu-\alpha\,p_{it}\}}{1+\sum_{j=1}^{N}a_{jt}\,\exp\{\nu-\alpha\,p_{jt}\}} = H_t\,s_{it}$$

where H_t is the number of consumers in the market (market size) and s_{it} is the market share of firm i. Under the Nash-Bertrand assumption, the first order conditions for profit maximization are:

$$q_{it} + (p_{it}-c)\;(-\alpha)\;q_{it}\;(1-s_{it}) = 0$$

or

$$p_{it} = c + \frac{1}{\alpha\,(1-s_{it})}$$

Since all firms are identical, we consider a symmetric equilibrium, $p_t^* = p_{it}^*$, for every firm i. Therefore, $s_{it} = a_{it}\,s_t^*$, and:

$$s_t^* = \frac{\exp\{\nu-\alpha\,p_t^*\}}{1+n_t\,\exp\{\nu-\alpha\,p_t^*\}}$$

where $n_t \equiv \sum_{j=1}^{N}a_{jt}$ is the number of active firms at period t. Then, it is simple to show that the equilibrium price p_t^* is implicitly defined as the solution to the following fixed point problem:

$$p_t^* = \left(c+\frac{1}{\alpha}\right) + \frac{1}{\alpha}\left(\frac{\exp\{\nu-\alpha\,p_t^*\}}{1+(n_t-1)\,\exp\{\nu-\alpha\,p_t^*\}}\right)$$

It is simple to show that an equilibrium always exists. The equilibrium price depends on the number of firms active in the market, but in this model it does not depend on market size: $p_t^* = p^*(n_t)$. Similarly, the equilibrium market share s_t^* is a function of the number

of active firms: $s_t^* = s^*(n_t)$. Therefore, the indirect or equilibrium variable profit of an active firm is:

$$\begin{aligned} VP_{it} &= a_{it}\ H_t\ (p^*(n_t) - c)\ s^*(n_t) \\ &= a_{it}\ H_t\ \theta^{VP}(n_t) \end{aligned}$$

where $\theta^{VP}$is a function that represents variable profits per capita.

For most of the analysis below, we will consider that the researcher does not have access to information on prices and quantities. Therefore, we will treat $\{\theta^{VP}(1), \theta^{VP}(2), ..., \theta^{VP}(N)\}$ as parameters to estimate from the structural dynamic game.

Of course, we can extend the previous approach to incorporate richer forms of product differentiation. In fact, product differentiation can be endogenous. Suppose that the quality parameter v in the utility function can take A possible values: $v(1) < v(2) < ... < v(A)$. And suppose that the investment decision a_{it} combines an entry/exit decision with a "quality" choice decision. That is, $a_{it} \in \{0, 1, ..., A\}$ where $a_{it} = 0$ represents firm i not being active in the market, and $a_{it} = a > 0$ implies that firm i is active in the market with a product of quality a. It is straightforward to show that, in this model, the equilibrium variable profit of an active firm is:

$$VP_{it} = \sum_{a=1}^{A} 1\{a_{it} = a\}\ H_t\ \theta^{VP}(a, n_t^{(1)}, n_t^{(2)}, ..., n_t^{(A)})$$

where $\theta^{VP}$is the variable profit per capita that now depends on the firm's own quality, and on the number of competitors at each possible level of quality.

(b) Fixed Cost. The fixed cost is paid every period that the firm is active in the market, and it has the following structure:

$$FC_{it} = a_{it}\ \left(\theta_i^{FC} + \varepsilon_{it}\right)$$

θ_i^{FC} is a parameter that represents the mean value of the fixed operating cost of firm i. ε_{it} is a zero-mean shock that is private information to firm i. There are two main reasons why we incorporate these private information shocks in the model. First, as shown in **doraszelski_satterthwaite_2007** (**doraszelski_satterthwaite_2007**) it is a way to guarantee that the dynamic game has at least one equilibrium in pure strategies. And second, they are convenient econometric errors. If private information shocks are independent over time and over players, and unobserved to the researcher, they can 'explain' players heterogeneous behavior without generating endogeneity problems.

We will see later that the assumption of the private information shocks being the only unobservables for the researcher can be too restrictive. We will study how to incorporate richer forms of unobserved heterogeneity.

For the model with endogenous quality choice, we can generalize the structure of fixed costs:

$$FC_{it} = \sum_{a=1}^{A} 1\{a_{it} = a\}\ \left(\theta_i^{FC}(a) + \varepsilon_{it}(a)\right)$$

where now the mean value of the fixed cost, $\theta_i^{FC}(a)$, and the private information shock, $\varepsilon_{it}(a)$, depend on the level quality.

(c) Entry Cost and Repositioning costs. The entry cost is paid only if the firm was not active in the market at the previous period:

$$EC_{it} = a_{it}\ (1 - x_{it})\ \theta_i^{EC}$$

where x_{it} is a binary indicator that is equal to 1 if firm i was active in the market in period $t-1$, that is, $x_{it} \equiv a_{i,t-1}$, and θ_i^{EC} is a parameter that represents the entry cost of firm i. For the model with endogenous quality, we can also generalize this entry cost to also incorporate costs of adjusting the level of quality, or repositioning product characteristics. For instance,

$$\begin{aligned} EC_{it} &= 1\{x_{it} = 0\} \left(\textstyle\sum_{a=1}^{A} 1\{a_{it} = a\}\ \theta_i^{EC}(a)\right) \\ &+ 1\{x_{it} > 0\} \left(\theta_i^{AC(+)} 1\{a_{it} > x_{it}\} + \theta_i^{AC(-)} 1\{a_{it} < x_{it}\}\right) \end{aligned}$$

Now, $x_{it} = a_{i,t-1}$ again represents the firm's quality at previous period, $\theta_i^{EC}(a)$ is the cost of entry with quality a, and $\theta_i^{AC(+)}$ and $\theta_i^{AC(-)}$ represent the costs of increasing and reducing quality, respectively, once the firm is active.

The payoff relevant state variables of this model are: (1) market size H_t; (2) the incumbent status (or quality levels) of firms at previous period $\{x_{it} : i = 1, 2, ..., N\}$; and (3) the private information shocks $\{\varepsilon_{it} : i = 1, 2, ..., N\}$. The specification of the model is completed with the transition rules of these state variables. Market size follows an exogenous Markov process with transition probability function $F_H(H_{t+1}|H_t)$. The transition of the incumbent status is trivial: $x_{it+1} = a_{it}$. Finally, the private information shock ε_{it} is i.i.d. over time and independent across firms with CDF G_i.

Note that in this example, we consider that firms' dynamic decisions are made at the beginning of period t and they are effective during the same period. An alternative timing that has been considered in some applications is that there is a one-period time-to-build. That is, the decision is made at period t, and entry costs are paid at period t, but the firm is not active in the market until period $t+1$. This is in fact the timing of decisions in Ericson and Pakes (1995). All the results below can be generalized in a straightforward way to that case, and we will see empirical applications with that timing assumption.

8.3.2 Markov Perfect Equilibrium

Most of the recent literature in IO studying industry dynamics focuses on studying a Markov Perfect Equilibrium (MPE), as defined by **maskin_tirole_1988** (**maskin_tirole_1988**). The key assumption in this solution concept is that players' strategies are functions of only payoff-relevant state variables. We use the vector $\mathbf{x}_t$ to represent all the common knowledge state variables at period t, that is, $\mathbf{x}_t \equiv (H_t, x_{1t}, x_{2t}, ..., x_{Nt})$. In this model, the payoff-relevant state variables for firm i are $(\mathbf{x}_t, \varepsilon_{it})$.

Note that if private information shocks are serially correlated, the history of previous decisions contains useful information to predict the value of a player's private information, and it should be part of the set of payoff relevant state variables. Therefore, the assumption that private information is independently distributed over time has implications for the set of payoff-relevant state variables.

Let $\alpha = \{\alpha_i(\mathbf{x}_t, \varepsilon_{it}) : i \in \{1,2,...,N\}\}$ be a set of strategy functions, one for each firm. A MPE is a set of strategy functions α^* such that every firm is maximizing its value given the strategies of the other players. For given strategies of the other firms, the decision problem of a firm is a single-agent dynamic programming (DP) problem. Let $V_i^{\alpha}(\mathbf{x}_t, \varepsilon_{it})$ be the value function of this DP problem. This value function is the unique solution to the Bellman equation:

$$V_i^{\alpha}(\mathbf{x}_t, \varepsilon_{it}) = \max_{a_{it}} \left\{ \Pi_i^{\alpha}(a_{it}, \mathbf{x}_t) - \varepsilon_{it}(a_{it}) + \delta \int V_i^{\alpha}(\mathbf{x}_{t+1}, \varepsilon_{it+1}) \, dG_i(\varepsilon_{it+1}) \, F_i^{\alpha}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \right\} \tag{8.1}$$

where $\Pi_i^{\alpha}(a_{it}, \mathbf{x}_t)$ and $F_i^{\alpha}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t)$ are the expected one-period profit and the expected transition of the state variables, respectively, for firm i given the strategies of the other firms. For the simple entry/exit game, the expected one-period profit $\Pi_i^{\alpha}(a_{it}, \mathbf{x}_t)$ is:

$$\Pi_i^{\alpha}(a_{it}, \mathbf{x}_t) = a_{it} \left[H_t \sum_{n=0}^{N-1} \Pr\left(\sum_{j \neq i} \alpha_j(\mathbf{x}_t, \varepsilon_{jt}) = n \mid \mathbf{x}_t \right) \theta^{VP}(n+1) - \theta_i^{FC} - (1 - x_{it}) \theta_i^{EC} \right]$$

And the expected transition of the state variables is:

$$F_i^{\alpha}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \quad = \quad 1\{x_{it+1} = a_{it}\} \left[\prod_{j \neq i} \Pr\left(x_{j,t+1} = \alpha_j(\mathbf{x}_t, \varepsilon_{jt}) \mid \mathbf{x}_t \right) \right] F_H(H_{t+1} \mid H_t)$$

A player's best response function gives her optimal strategy if the other players behave, now and in the future, according to their respective strategies. In this model, the best response function of player i is:

$$\alpha_i^*(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it}} \{ v_i^{\alpha}(a_{it}, \mathbf{x}_t) - \varepsilon_{it}(a_{it}) \}$$

where $v_i^{\alpha}(a_{it}, \mathbf{x}_t)$ is the conditional choice value function that represents the value of firm i if: (1) the firm chooses alternative a_{it} today and then behaves optimally forever in the future; and (2) the other firms behave according to their strategies in α. By definition,

$$v_i^{\alpha}(a_{it}, \mathbf{x}_t) \equiv \Pi_i^{\alpha}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\alpha}(\mathbf{x}_{t+1}, \varepsilon_{it+1}) \, dG_i(\varepsilon_{it+1}) \, F_i^{\alpha}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t)$$

A Markov perfect equilibrium (MPE) in this game is a set of strategy functions α^* such that for any player i and for any $(\mathbf{x}_t, \varepsilon_{it})$ we have that:

$$\alpha_i^*(\mathbf{x}_t, \varepsilon_{it}) = \arg\max_{a_{it}} \left\{ v_i^{\alpha^*}(a_{it}, \mathbf{x}_t) - \varepsilon_{it}(a_{it}) \right\}$$

8.3.3 Conditional Choice Probabilities

Given a strategy function $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$, we can define the corresponding *Conditional Choice Probability (CCP)* function as :

$$\begin{aligned} P_i(a|\mathbf{x}) \quad &\equiv \quad \Pr\left(\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a \mid \mathbf{x}_t = \mathbf{x}\right) \\ &= \quad \int 1\{\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a\} \, dG_i(\varepsilon_{it}) \end{aligned}$$

Since choice probabilities are integrated over the continuous variables in ε_{it}, they are lower dimensional objects than the strategies α. For instance, when both a_{it} and $\mathbf{x}_t$ are discrete, CCPs can be described as vectors in a finite dimensional Euclidean space. In our entry-exit model, $P_i(1|\mathbf{x}_t)$ is the probability that firm i is active in the market given the state $\mathbf{x}_t$. Under standard regularity conditions, it is possible to show that there is a one-to-one relationship between strategy functions $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ and CCP functions $P_i(a|\mathbf{x}_t)$. From now on, we use CCPs to represent players' strategies, and use the terms 'strategy' and 'CCP' as interchangeable. We also use $\Pi_i^{\mathbf{P}}$ and $F_i^{\mathbf{P}}$ instead of Π_i^{α} and F_i^{α} to represent the expected profit function and the transition probability function, respectively.

Based on the concept of CCP, we can represent the equilibrium mapping and a MPE in a way that is particularly useful for the econometric analysis. This representation has two main features:

(1) a MPE is a vector of CCPs;

(2) a player's best response is an optimal response not only to the other players' strategies but also to her own strategy in the future.

A MPE is a vector of CCPs, $\mathbf{P} \equiv \{P_i(a|\mathbf{x}) : \text{for any } (i,a,x)\}$, such that for every firm and any state $\mathbf{x}$ the following equilibrium condition is satisfied:

$$P_i(a|\mathbf{x}) = \Pr\left(a = \arg\max_{a_i} \left\{ v_i^{\mathbf{P}}(a_i, \mathbf{x}) - \varepsilon_i(a_i) \right\} \mid \mathbf{x} \right)$$

The right hand side of this equation is a best response probability function. $v_i^{\mathbf{P}}(a_i, \mathbf{x})$ is a conditional choice probability function, but it has a slightly different definition than before. Now, $v_i^{\mathbf{P}}(a_i, \mathbf{x})$ represents the value of firm i if: (1) the firm chooses alternative a_i today; and (2) **all the firms**, including firm i, behave according to their respective CCPs in $\mathbf{P}$. The Representation Lemma in **aguirregabiria_ mira_2007** (**aguirregabiria_ mira_2007**) shows that every MPE in this dynamic game can be represented using this mapping. In fact, this result is a particular application of the so called "one-period deviation principle".

The form of this equilibrium mapping depends on the distribution of ε_i. For instance, in the entry/exit model, if ε_i is $N(0, \sigma_\varepsilon^2)$:

$$P_i(1|\mathbf{x}) = \Phi\left(\frac{v_i^{\mathbf{P}}(1,\mathbf{x}) - v_i^{\mathbf{P}}(0,\mathbf{x})}{\sigma_\varepsilon} \right)$$

In the model with endogenous quality choice, if $\varepsilon_i(a)$'s are extreme value type 1 distributed:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{ \frac{v_i^{\mathbf{P}}(a,\mathbf{x})}{\sigma_\varepsilon} \right\}}{\sum_{a'=0}^{A} \exp\left\{ \frac{v_i^{\mathbf{P}}(a',\mathbf{x})}{\sigma_\varepsilon} \right\}}$$

8.3.4 Computing $v_i^{\mathbf{P}}$ for arbitrary $\mathbf{P}$

Now, we describe how to obtain the conditional choice value functions $v_i^{\mathbf{P}}$. Since $v_i^{\mathbf{P}}$ is not based on the optimal behavior of firm i in the future, but just in an arbitrary behavior described by $P_i(.|.)$, calculating $v_i^{\mathbf{P}}$ does not require solving a DP problem, and it only implies a valuation exercise.

By definition:

$$v_i^{\mathbf{P}}(a_i,\mathbf{x}) = \Pi_i^{\mathbf{P}}(a_i,\mathbf{x}) + \delta \sum_{\mathbf{x}'} V_i^{\mathbf{P}}(\mathbf{x}')\, F_i^{\mathbf{P}}(\mathbf{x}'|a_i,\mathbf{x})$$

$\Pi_i^{\mathbf{P}}(a_i,\mathbf{x})$ is the expected current profit. In the entry/exit example:

$$\begin{aligned}\Pi_i^{\mathbf{P}}(a_i,\mathbf{x}) &= a_i \left[H \sum_{n=0}^{N-1} \Pr(n_{-i} = n \mid \mathbf{x}, \mathbf{P})\, \theta^{VP}(n+1) - \theta_i^{FC} - (1-x_i)\theta_i^{EC} \right] \\ &= a_i \left[\mathbf{z}_i^{\mathbf{P}}(\mathbf{x})\ \theta_i \right]\end{aligned}$$

where θ_i is the vector of parameters:

$$\theta_i = \left(\theta^{VP}(1),\, \theta^{VP}(2),\, ...,\theta^{VP}(N),\, \theta_i^{FC},\, \theta_i^{EC} \right)'$$

and $\mathbf{z}_i^{\mathbf{P}}(\mathbf{x})$ is the vector that depends only on the state $\mathbf{x}$ and on the CCPs at state $\mathbf{x}$, but not on structural parameters:

$$\mathbf{z}_i^{\mathbf{P}}(\mathbf{x}) = (H \Pr(n_{-i} = 1|\mathbf{x},\mathbf{P}),\, ...,\, H \Pr(n_{-i} = N-1|\mathbf{x},\mathbf{P}),\, -1,\, -(1-x_i))$$

For the dynamic game with endogenous quality choice, we can also represent the expected current profit $\Pi_i^{\mathbf{P}}(a_i,\mathbf{x})$ as:

$$\Pi_i^{\mathbf{P}}(a_i,\mathbf{x}) = \mathbf{z}_i^{\mathbf{P}}(a_i,\mathbf{x})\ \theta_i$$

The value function $V_i^{\mathbf{P}}$represents the value of firm i if all the firms, including firm i, behave according to their CCPs in $\mathbf{P}$. We can obtain $V_i^{\mathbf{P}}$ as the unique solution of the recursive expression:

$$V_i^{\mathbf{P}}(\mathbf{x}) = \sum_{a_i=0}^{A} P_i(a_i|\mathbf{x}) \left[\mathbf{z}_i^{\mathbf{P}}(a_i,\mathbf{x})\theta_i + \delta \sum_{\mathbf{x}'} V_i^{\mathbf{P}}(\mathbf{x}')\, F^{\mathbf{P}}(\mathbf{x}'|a_i,\mathbf{x}) \right]$$

When the space $\mathscr{X}$ is discrete and finite, we can obtain $V_i^{\mathbf{P}}$ as the solution of a system of linear equations of dimension $|\mathscr{X}|$. In vector form:

$$\begin{aligned}\mathbf{V}_i^{\mathbf{P}} &= \left[\sum_{a_i=0}^{A} P_i(a_i) * \mathbf{z}_i^{\mathbf{P}}(a_i) \right] \theta_i + \delta \left[\sum_{a_i=0}^{A} P_i(a_i) * \mathbf{F}_i^{\mathbf{P}}(a_i) \right] \mathbf{V}_i^{\mathbf{P}} \\ &= \bar{\mathbf{z}}_i^{\mathbf{P}}\ \theta_i + \delta\ \bar{\mathbf{F}}^{\mathbf{P}} \mathbf{V}_i^{\mathbf{P}}\end{aligned}$$

where $\bar{\mathbf{z}}_i^{\mathbf{P}} = \sum_{a_i=0}^{A} P_i(a_i) * \mathbf{z}_i^{\mathbf{P}}(a_i)$, and $\bar{\mathbf{F}}^{\mathbf{P}} = \sum_{a_i=0}^{A} P_i(a_i) * \mathbf{F}_i^{\mathbf{P}}(a_i)$. Then, solving for $\mathbf{V}_i^{\mathbf{P}}$, we have:

$$\begin{aligned}\mathbf{V}_i^{\mathbf{P}} &= \left(\mathbf{I} - \delta\ \bar{\mathbf{F}}^{\mathbf{P}} \right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}\ \theta_i \\ &= \mathbf{W}_i^{\mathbf{P}}\ \theta_i\end{aligned}$$

where $\mathbf{W}_i^{\mathbf{P}} = \left(\mathbf{I} - \delta\, \bar{\mathbf{F}}^{\mathbf{P}}\right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}$ is a matrix that only depends on CCPs and transition probabilities but not on θ.

Solving these expressions into the formula for the conditional choice value function, we have that:

$$v_i^{\mathbf{P}}(a_i, \mathbf{x}) = \tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x})\ \theta_i$$

where:

$$\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x}) = z_i^{\mathbf{P}}(a_i, \mathbf{x}) + \delta \sum_{\mathbf{x}'} F_i^{\mathbf{P}}(\mathbf{x}'|a_i, \mathbf{x})\ \mathbf{W}_i^{\mathbf{P}}$$

Finally, the equilibrium or best response mapping in the space of CCPs becomes:

$$P_i(a|\mathbf{x}) = \Pr\left(a = \arg\max_{a_i} \left\{ \tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x})\ \theta_i - \varepsilon_i(a_i) \right\} \mid \mathbf{x}\right)$$

For the entry/exit model with $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$:

$$P_i(1|\mathbf{x}) = \Phi\left(\left[\tilde{z}_i^{\mathbf{P}}(1, \mathbf{x}) - \tilde{z}_i^{\mathbf{P}}(0, \mathbf{x})\right] \frac{\theta_i}{\sigma_\varepsilon}\right)$$

In the model with endogenous quality choice with $\varepsilon_i(a)$'s extreme value type 1 distributed:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{\tilde{z}_i^{\mathbf{P}}(a, \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}}(a', \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}$$

Identification

First, let's summarize the structure of the dynamic game of oligopoly competition.

Let θ be the vector of structural parameters of the model, where $\theta = \{\theta_i : i = 1, 2, ..., N\}$ and θ_i includes the vector of parameters in the variable profit, fixed cost, and entry cost of firm i: for instance, in the entry-exit example, $\theta_i = (\theta^{VP}(1),\ \theta^{VP}(2),\ ...,\theta^{VP}(N),\ \theta_i^{FC},\ \theta_i^{EC})'$. Let $\mathbf{P}(\theta) = \{P_i(a|\mathbf{x}, \theta) : \text{for any } (i, a, \mathbf{x})\}$ be a MPE of the model associated with θ. $\mathbf{P}(\theta)$ is a solution to the following equilibrium mapping: for any $(i, a_i, \mathbf{x})$:

$$P_i(a_i|\mathbf{x}, \theta) = \frac{\exp\left\{\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}}(a', \mathbf{x}) \dfrac{\theta_i}{\sigma_\varepsilon}\right\}}$$

where the vector of values $\tilde{z}_i^{\mathbf{P}}(a, \mathbf{x})$ are

$$\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x}) = z_i^{\mathbf{P}}(a_i, \mathbf{x}) + \delta \sum_{\mathbf{x}'} F_i^{\mathbf{P}}(\mathbf{x}'|a_i, \mathbf{x})\ \mathbf{W}_i^{\mathbf{P}}$$

and $\mathbf{W}_i^{\mathbf{P}} = \mathbf{W}_i^{\mathbf{P}} = \left(\mathbf{I} - \delta\, \bar{\mathbf{F}}^{\mathbf{P}}\right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}$, and $z_i^{\mathbf{P}}(a_i, \mathbf{x})$ is a vector with the different components of the current expected profit. For instance, in the entry-exit example:

$$\mathbf{z}_i^{\mathbf{P}}(0, \mathbf{x}) = (0,\ 0,\ 0,\ ...0)$$

$$\mathbf{z}_i^{\mathbf{P}}(1, \mathbf{x}) = \left(H \Pr\left(n_{-i} = 1|\mathbf{x}, \mathbf{P}\right),\ ...,\ H \Pr\left(n_{-i} = N - 1|\mathbf{x}, \mathbf{P}\right),\ -1,\ -(1 - x_i)\right)$$

That is, $\tilde{z}_i^{\mathbf{P}}(a_i, \mathbf{x})$ represents the expected present value of the different components of the current profit of firm i if it chooses alternative a_i today, and then all the firms, including firm i, behave in the future according to their CCPs in the vector $\mathbf{P}$.

In general, we will use the function $\Psi_i(a_i, \mathbf{x}; \mathbf{P}, \theta)$ to represent the best response or equilibrium function that in our example is $\frac{\exp\left\{\tilde{z}_i^{\mathbf{P}}(a_i,\mathbf{x})\frac{\theta_i}{\sigma_\varepsilon}\right\}}{\Sigma_{a'=0}^{A}\exp\left\{\tilde{z}_i^{\mathbf{P}}(a',\mathbf{x})\frac{\theta_i}{\sigma_\varepsilon}\right\}}$. Then, we can represent in a compact form a MPE as:

$$\mathbf{P} = \Psi(\mathbf{P}, \theta)$$

where $\Psi(\mathbf{P}, \theta) = \{\Psi_i(a_i, \mathbf{x}; \mathbf{P}, \theta) : \text{for any } (i, a, \mathbf{x})\}$.

Our first goal is to use data on firms' investment decisions $\{a_{it}\}$ and state variables $\{x_{it}\}$ to estimate the parameters θ.

Our second goal is to use the estimated model to perform counterfactual analysis/experiments that will help us understand competition in this industry and to evaluate the effects of public policies or/and changes in structural parameters.

Data

In most applications of dynamic games in empirical IO, the researcher observes a random sample of M markets, indexed by m, over T periods of time, where the observed variables consists of players' actions and state variables. In the standard application in IO, the values of N and T are small, but M is large. Two aspects of the data deserve some comments. For the moment, we consider that the industry and the data are such that: (a) each firm is observed making decisions in every of the M markets; and (b) the researcher knows all the payoff relevant market characteristics that are common knowledge to the firms. We describe condition (a) as a data set with *global players*. For instance, this is the case in a retail industry characterized by competition between large retail chains which are potential entrants in any of the local markets that constitute the industry. With this type of data we can allow for rich firm heterogeneity that is fixed across markets and time by estimating firm-specific structural parameters, θ_i. This 'fixed-effect' approach to deal with firm heterogeneity is not feasible in data sets where most of the competitors can be characterized as *local players*, that is, firms specialized in operating in a few markets. Condition (b) rules out the existence of unobserved market heterogeneity. Though it is a convenient assumption, it is also unrealistic for most applications in empirical IO. Later we present estimation methods that relax conditions (a) and (b) and deal with unobserved market and firm heterogeneity.

Suppose that we have a random sample of M local markets, indexed by m, over T periods of time, where we observe:

$$Data = \{a_{mt},\ \mathbf{x}_{mt} : m = 1, 2, ..., M;\ t = 1, 2, ..., T\}$$

We want to use these data to estimate the model parameters in the population that has generated this data: $\theta^0 = \{\theta_i^0 : i \in I\}$.

Identification

A significant part of this literature has considered the following identification assumptions.

Assumption (ID 1): Single equilibrium in the data. Every observation in the sample comes from the same Markov Perfect Equilibrium, that is, for any observation (m,t), $\mathbf{P}^0_{mt} = \mathbf{P}^0$.

Assumption (ID 2): No unobserved common-knowledge variables. The only unobservables for the econometrician are the private information shocks ε_{imt} and the structural parameters θ.

Comments on these assumptions: .... The assumption of no unobserved common knowledge variables (for instance, no unobserved market heterogeneity) is particularly strong.

It is possible to relax these assumptions. We will see later identification and estimation when we relax assumption ID 2. The following is a standard regularity condition.

Assumption (ID 3): For some benchmark choice alternative, say $a_i = 0$, define $Z_{imt} \equiv \tilde{z}_i^{\mathbf{P}^0}(a_{imt}, \mathbf{x}_{mt}) - \tilde{z}_i^{\mathbf{P}^0}(0, \mathbf{x}_{mt})$. Then, $\mathbb{E}(Z'_{imt} Z_{imt})$ is a non-singular matrix.

Under assumptions ID-1 to ID-3, the proof of identification is straightforward. First, under assumptions ID-1 and ID-2, the equilibrium that has generated the data, $\mathbf{P}^0$, can be estimated consistently and nonparametrically from the data. For any $(i, a_i, \mathbf{x})$:

$$P_i^0(a_i|\mathbf{x}) = \Pr(a_{imt} = a_i \mid \mathbf{x}_{mt} = \mathbf{x})$$

For instance, we can estimate consistently $P_i^0(a_i|\mathbf{x})$ using the following simple kernel estimator:

$$P_i^0(a_i|\mathbf{x}) = \frac{\sum_{m,t} 1\{a_{imt} = a_i\} \, K\left(\frac{\mathbf{x}_{mt} - \mathbf{x}}{b_n}\right)}{\sum_{m,t} K\left(\frac{\mathbf{x}_{mt} - \mathbf{x}}{b_n}\right)}$$

Second, given that $\mathbf{P}^0$ is identified, we can identify also the expected present values $\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x})$ at the "true" equilibrium in the population. Third, we know that $\mathbf{P}^0$ is an equilibrium associated to θ^0. Therefore, the following equilibrium conditions should hold: for any $(i, a_i, \mathbf{x})$,

$$P_i^0(a_i|\mathbf{x}) = \frac{\exp\left\{\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x}) \frac{\theta_i^0}{\sigma_\varepsilon^0}\right\}}{\sum_{a'=0}^{A} \exp\left\{\tilde{z}_i^{\mathbf{P}^0}(a', \mathbf{x}) \frac{\theta_i^0}{\sigma_\varepsilon^0}\right\}}$$

It is straightforward to show that under Assumption ID-3, these equilibrium conditions identify $\frac{\theta_i^0}{\sigma_\varepsilon^0}$. For instance, in this logit example, we have that for $(i, a_i, \mathbf{x})$,

$$\ln\left(\frac{P_i^0(a_i|\mathbf{x})}{P_i^0(0|\mathbf{x})}\right) = \left[\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x}) - \tilde{z}_i^{\mathbf{P}^0}(0, \mathbf{x})\right] \frac{\theta_i^0}{\sigma_\varepsilon^0}$$

Define $Y_{imt} \equiv \ln\left(\frac{P_i^0(a_{imt}|\mathbf{x}_{mt})}{P_i^0(0|\mathbf{x}_{mt})}\right)$ and $Z_{imt} \equiv \tilde{z}_i^{\mathbf{P}^0}(a_{imt}, \mathbf{x}_{mt}) - \tilde{z}_i^{\mathbf{P}^0}(0, \mathbf{x}_{mt})$. Then,

$$Y_{imt} = Z_{imt} \, \frac{\theta_i^0}{\sigma_\varepsilon^0}$$

And we can also write this system as, $\mathbb{E}(Z'_{imt}Y_{imt}) = \mathbb{E}(Z'_{imt}Z_{imt})\ \dfrac{\theta_i^0}{\sigma_\varepsilon^0}$. Under assumption ID-3:

$$\frac{\theta_i^0}{\sigma_\varepsilon^0} = \mathbb{E}(Z'_{imt}Z_{imt})^{-1}\mathbb{E}(Z'_{imt}Y_{imt})$$

and $\dfrac{\theta_i^0}{\sigma_\varepsilon^0}$ is identified.

Note that under the single-equilibrium-in-the-data assumption, the multiplicity of equilibria in the model does not play any role in the identification of the structural parameters. The single-equilibrium-in-the-data assumption is sufficient for identification but it is not necessary. Sweeting (2009), **aguirregabiria_ mira_2009** (**aguirregabiria_ mira_2009**), and **paula_tang_2010** (**paula_tang_2010**) present conditions for the point-identification of games of incomplete information when there are multiple equilibria in the data.

Estimation

The use of an 'extended' or 'pseudo' likelihood (or alternatively GMM criterion) function plays an important role in the different estimation methods. For arbitrary values of the vector of structural parameters θ and firms' strategies $\mathbf{P}$, we define the following likelihood function of observed players' actions $\{a_{imt}\}$ conditional on observed state variables $\{\mathbf{x}_{mt}\}$:

$$Q(\theta,\mathbf{P}) = \sum_{i,m,t}\sum_{a_i=0}^{A} 1\{a_{imt} = a_i\}\ \ln\Psi_i(a_i,\mathbf{x}_{mt};\mathbf{P},\theta)$$

We call $Q(\theta,\mathbf{P})$ a 'Pseudo' Likelihood function because players' CCPs in $\mathbf{P}$ are arbitrary and do not represent the equilibrium probabilities associated with θ implied by the model.

An important implication of using arbitrary CCPs, instead of equilibrium CCPs, is that likelihood Q is a function and not a correspondence. To compute this pseudo likelihood, a useful construct is the representation of equilibrium in terms of CCPs, which we presented above.

We could also consider a Pseudo GMM Criterion function:

$$Q(\theta,\mathbf{P}) = -r(\theta,\mathbf{P})'\ \Omega\ r(\theta,\mathbf{P})$$

where Ω is the weighting matrix and $r(\theta,\mathbf{P})$ is the vector of moment conditions:

$$r(\theta,\mathbf{P}) = \frac{1}{MT}\sum_{m,t}\left[h(x_{mt})\otimes\begin{pmatrix} 1\{a_{imt} = a_i\} - \Psi_i(a_i,\mathbf{x}_{mt};\mathbf{P},\theta) \\ \dots \\ \text{for any } (i,a_i)\end{pmatrix}\right]$$

and $h(x_{mt})$ is a vector of functions of x_{mt} (instruments).

Full Maximum Likelihood

The dynamic game imposes the restriction that the strategies in $\mathbf{P}$ should be in equilibrium. The ML estimator is defined as the pair $(\hat{\theta}_{MLE},\hat{\mathbf{P}}_{MLE})$ that maximizes the pseudo

likelihood subject to the constraint that the strategies in $\hat{\mathbf{P}}_{MLE}$ are equilibrium strategies associated with $\hat{\theta}_{MLE}$. That is,

$$\begin{aligned}(\hat{\theta}_{MLE}, \hat{\mathbf{P}}_{MLE}) &= \arg\max_{(\theta,\mathbf{P})} Q(\theta, \mathbf{P}) \\ &\text{s.t. } P_i(a_i|\mathbf{x}) = \Psi_i(a_i, \mathbf{x}; \mathbf{P}, \theta) \text{ for any } (i, a_i, \mathbf{x})\end{aligned}$$

This is a constrained ML estimator that satisfies the standard regularity conditions for consistency, asymptotic normality and efficiency of ML estimation.

The numerical solution of the constrained optimization problem that defines these estimators requires one to search over an extremely large dimensional space. In the empirical applications of dynamic oligopoly games, the vector of probabilities $\mathbf{P}$ includes thousands or millions of elements. Searching for an optimum in that kind of space is computationally demanding. **su_judd_2008** (**su_judd_2008**) have proposed to use a MPEC algorithm, which is a general purpose algorithm for the numerical solution of constrained optimization problems. However, even using the most sophisticated algorithm such as MPEC, the optimization with respect to $(\mathbf{P}, \theta)$ can be extremely demanding when $\mathbf{P}$ has a high dimension.

Two-step methods

Let $\mathbf{P}^0$ be the vector with the population values of the probabilities $P_i^0(a_i|\mathbf{x}) \equiv \Pr(a_{imt} = a_i|\mathbf{x}_{mt} = \mathbf{x})$. Under the assumptions of "no unobserved common knowledge variables" and "single equilibrium in the data", the CCPs in $\mathbf{P}^0$ represent firms' strategies in the only equilibrium that is played in the data. These probabilities can be estimated consistently using standard nonparametric methods. Let $\hat{\mathbf{P}}^0$ be a consistent nonparametric estimator of $\mathbf{P}^0$. Given $\hat{\mathbf{P}}^0$, we can construct a consistent estimator of $\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x})$. Then, the two-step estimator of θ^0 is defined as:

$$\hat{\theta}_{2S} = \arg\max_{\theta} Q(\theta, \hat{\mathbf{P}}^0)$$

After the computation of the expected present values $\tilde{z}_i^{\mathbf{P}^0}(a_i, \mathbf{x})$, this second step of the procedure is computationally very simple. It consists only in the estimation of a standard discrete choice model, for instance, a binary probit/logit in our entry-exit example, or a conditional logit in our example with quality choice. Under standard regularity conditions, this two-step estimator is root-M consistent and asymptotically normal.

This idea was originally exploited, for estimation of single agent problems, by Hotz and Miller (1993) and Hotz et al. (1994). It was expanded to the estimation of dynamic games by **aguirregabiria_ mira_2007** (**aguirregabiria_ mira_2007**), Bajari, Benkard, and Levin (2007), Pakes, Ostrovsky, and Berry (2007), and Pesendorfer and Schmidt-Dengler (2008).

The main advantage of these two-step estimators is their computational simplicity. The first step is a simple nonparametric regression, and the second step is the estimation of a standard discrete choice model with a criterion function that in most applications is globally concave (for instance, such as the likelihood of a standard probit model in our entry-exit example). The main computational burden comes from the calculation of the present values $W_i^{\hat{\mathbf{P}}}(\mathbf{x})$. Though the computation of these present values may be

subject to a curse of dimensionality, the cost of obtaining a two-step estimator is several orders of magnitude smaller than solving (just once) for an equilibrium of the dynamic game. In most applications, this makes the difference between being able to estimate the model or not.

However, these two-step estimators have some important limitations. A first limitation is the restrictions imposed by the assumption of no unobserved common knowledge variables. Ignoring persistent unobservables, if present, can generate important biases in the estimation of structural parameters. We deal with this issue later.

A second problem is finite sample bias. The finite sample bias of the two-step estimator of θ^0 depends very importantly on the properties of the first-step estimator of $\mathbf{P}^0$. In particular, it depends on the rate of convergence and on the variance and bias of $\hat{\mathbf{P}}^0$. It is well-known that there is a **curse of dimensionality in the nonparametric estimation** of a regression function such as $\mathbf{P}^0$. The rate of convergence of the estimator (and its asymptotic variance) declines (increase) with the number of explanatory variables in the regression. The initial nonparametric estimator can be very imprecise in the samples available in actual applications, and this can generate serious finite sample biases in the two-step estimator of structural parameters.

In dynamic games with heterogeneous players, the number of observable state variables is proportional to the number of players, and therefore the so called *curse of dimensionality in nonparametric estimation* (and the associated bias of the two-step estimator) can be particularly serious. For instance, in our dynamic game of product quality choice, the vector of state variables contains the qualities of the N firms.

The source of this bias is well understood in two-step methods: $\hat{\mathbf{P}}$ enters nonlinearly in the sample moment conditions that define the estimator, and the expected value of a nonlinear function of $\hat{\mathbf{P}}$ is not equal to that function evaluated at the expected value of $\hat{\mathbf{P}}$. The larger the variance or the bias of $\hat{\mathbf{P}}$, the larger the bias of the two-step estimator of θ_0. To see this, note that the PML or GMM estimators in the second step are based on moment conditions at the true $\mathbf{P}^0$:

$$\mathbb{E}\left(h(x_{mt}) \left[1\{a_{imt} = a_i\} - \Psi_i(a_i, \mathbf{x}; \mathbf{P^0}, \theta) \right] \right) = 0$$

The same moment conditions evaluated at $\hat{\mathbf{P}}^0$ do not hold because of the estimation error:

$$\mathbb{E}\left(h(x_{mt}) \left[1\{a_{imt} = a_i\} - \Psi_i(a_i, \mathbf{x}; \hat{\mathbf{P}}^{\mathbf{0}}, \theta_0) \right] \right) \neq 0$$

This generates a finite sample bias. The best response function $\Psi_i(a_i, \mathbf{x}; \hat{\mathbf{P}}^{\mathbf{0}}, \theta_0)$ is a nonlinear function of the random vector $\hat{\mathbf{P}}_0$, and the expected value of a nonlinear function is not equal to the function evaluated at the expected value. The larger the finite sample bias or the variance of $\hat{\mathbf{P}}^{\mathbf{0}}$, the larger the bias of the two-step estimation of θ_0.

Recursive K-step estimators

To deal with finite sample bias, **aguirregabiria_ mira_2002** (**aguirregabiria_ mira_2002**,**aguirregabiria_** consider a recursive K-step extension. Given the two-step estimator $\hat{\theta}_{2S}$ and the initial nonparametric estimator of CCPs, $\hat{\mathbf{P}}^0$, we can construct a new estimator of CCPs, $\hat{\mathbf{P}}^1$, such that, for any $(i, a_i, \mathbf{x})$:

$$\hat{P}_i^1(a_i|\mathbf{x}) = \Psi_i(a_i, \mathbf{x}; \hat{\mathbf{P}}^{\mathbf{0}}, \hat{\theta}_{2S})$$

or in our example:

$$\hat{P}_i^1(a_i|\mathbf{x}) = \frac{\exp\left\{\tilde{z}_i^{\hat{\mathbf{P}}^0}(a_i,\mathbf{x})\ \hat{\theta}_{i,2S}\right\}}{\sum_{a'=0}^{A}\exp\left\{\tilde{z}_i^{\hat{\mathbf{P}}^0}(a',\mathbf{x})\ \hat{\theta}_{i,2S}\right\}}$$

This new estimator of CCPs exploits the parametric structure of the model, and the structure of best response functions. It seems intuitive that this new estimator of CCPs has better statistical properties than the initial nonparametric estimator, that is, smaller asymptotic variance, and smaller finite sample bias and variance. As we explain below, this intuition is correct as long as the equilibrium that generated the data is (Lyapunov) stable.

Under this condition, it seems natural to obtain a new two-step estimator by replacing $\hat{\mathbf{P}}^0$ with $\hat{\mathbf{P}}^1$ as the estimator of CCPs. Then, we can obtain the new estimator:

$$\hat{\theta} = \arg\max_{\theta} Q(\theta, \hat{\mathbf{P}}^1)$$

The same argument can be applied recursively to generate a sequence of $K-step$ estimators. Given an initial consistent nonparametric estimator $\hat{\mathbf{P}}^0$, the sequence of estimators $\{\hat{\theta}^K, \hat{\mathbf{P}}^K : K \geq 1\}$ is defined as:

$$\hat{\theta}^K = \arg\max_{\theta} Q(\theta, \hat{\mathbf{P}}^{K-1})$$

and

$$\hat{\mathbf{P}}^K = \Psi(\hat{\mathbf{P}}^{K-1}, \hat{\theta}^K)$$

Monte Carlo experiments in **aguirregabiria_ mira_2002** (**aguirregabiria_ mira_2002**,**aguirregabiria** and **kasahara_shimotsu_2008a** (**kasahara_shimotsu_2008a**,**kasahara_shimotsu_2009**) show that iterating in the NPL mapping can reduce significantly the finite sample bias of the two-step estimator. The Monte Carlo experiments in Pesendorfer and Schmidt-Dengler (2008) present a different, more mixed, picture. While for some of their experiments NPL iteration reduces the bias, in other experiments the bias remains constant or even increases. A closer look at the Monte Carlo experiments in Pesendorfer and Schmidt-Dengler shows that the NPL iterations provide poor results in those cases where the equilibrium that generates the data is not (Lyapunov) stable. As we explain below, this is not a coincidence. It turns out that the computational and statistical properties of the sequence of K-step estimators depend critically on the stability of the NPL mapping around the equilibrium in the data.

Convergence properties of recursive K-step estimators

To study the properties of these K-step estimators, it is convenient to represent the sequence $\{\hat{\mathbf{P}}^K : K \geq 1\}$ as the result of iterating in a fixed point mapping. For arbitrary $\mathbf{P}$, define the mapping:

$$\varphi(\mathbf{P}) \equiv \Psi(\mathbf{P},\ \hat{\theta}(\mathbf{P}))$$

where $\hat{\theta}(\mathbf{P}) \equiv \arg\max_{\theta}\ Q(\theta, \mathbf{P})$. The mapping $\varphi(\mathbf{P})$ is called the Nested Pseudo Likelihood (NPL) mapping.

The sequence of estimators $\{\hat{\mathbf{P}}^K : K \geq 1\}$ can be obtained by successive iterations in the mapping φ starting with the nonparametric estimator $\hat{\mathbf{P}}^0$, that is, for $K \geq 1$, $\hat{\mathbf{P}}^K = \varphi(\hat{\mathbf{P}}^{K-1})$.

Lyapunov stability. Let $\mathbf{P}^*$ be a fixed point of the NPL mapping such that $\mathbf{P}^* = \varphi(\mathbf{P}^*)$. We say that the mapping φ is Lyapunov-stable around the fixed point $\mathbf{P}^*$ if there is a neighborhood of $\mathbf{P}^*$, $\mathscr{N}$, such that successive iterations in the mapping φ starting at $\mathbf{P} \in \mathscr{N}$ converge to $\mathbf{P}^*$. A necessary and sufficient condition for Lyapunov stability is that the *spectral radius* of the Jacobian matrix $\partial\varphi(\mathbf{P}^*)/\partial\mathbf{P}'$ is smaller than one. The neighboring set $\mathscr{N}$ is denoted the dominion of attraction of the fixed point $\mathbf{P}^*$. The spectral radius of a matrix is the maximum absolute eigenvalue. If the mapping φ is twice continuously differentiable, then the spectral radius is a continuous function of $\mathbf{P}$. Therefore, if φ is Lyapunov stable at $\mathbf{P}^*$, for any $\mathbf{P}$ in the dominion of attraction of $\mathbf{P}^*$ we have that the spectral radius of $\partial\varphi(\mathbf{P})/\partial\mathbf{P}'$ is also smaller than one. Similarly, if $\mathbf{P}^*$ is an equilibrium of the mapping $\Psi(.,\theta)$, we say that this mapping is Lyapunov stable around $\mathbf{P}^*$ if and only if the *spectral radius* of the Jacobian matrix $\partial\Psi(\mathbf{P}^*,\theta)/\partial\mathbf{P}'$ is smaller than one.

There is a relationship between the stability of the NPL mapping and of the equilibrium mapping $\Psi(.,\theta^0)$ around $\mathbf{P}^0$ (that is, the equilibrium that generates the data). The Jacobian matrices of the NPL and equilibrium mapping are related by the following expression (see **kasahara_shimotsu_2009** ,**kasahara_shimotsu_2009**):

$$\frac{\partial\varphi(\mathbf{P}^0)}{\partial\mathbf{P}'} = M(\mathbf{P}^0)\,\frac{\partial\Psi(\mathbf{P}^0,\theta^0)}{\partial\mathbf{P}'}$$

where $M(\mathbf{P}^0)$ is an idempotent projection matrix $I - \Psi_\theta(\Psi_\theta'\ diag\{\mathbf{P}^0\}^{-1}\ \Psi_\theta)^{-1}\Psi_\theta'$ $diag\{\mathbf{P}^0\}^{-1}$, where $\Psi_\theta \equiv \partial\Psi(\mathbf{P}^0,\theta^0)/\partial\theta'$. In single-agent dynamic programming models, the Jacobian matrix $\partial\Psi(\mathbf{P}^0,\theta^0)/\partial\mathbf{P}'$ is zero (that is, zero Jacobian matrix property, **aguirregabiria_ mira_2002** ,**aguirregabiria_ mira_2002**). Therefore, for that class of models $\partial\varphi(\mathbf{P}^0)/\partial\mathbf{P}' = 0$ and the NPL mapping is Lyapunov stable around $\mathbf{P}^0$. In dynamic games, $\partial\Psi(\mathbf{P}^0,\theta^0)/\partial\mathbf{P}'$ is not zero. However, given that $M(\mathbf{P}^0)$ is an idempotent matrix, it is possible to show that the spectral radius of $\partial\varphi(\mathbf{P}^0)/\partial\mathbf{P}'$ is not larger than the spectral radius of $\partial\Psi(\mathbf{P}^0,\theta^0)/\partial\mathbf{P}'$. Therefore, Lyapunov stability of $\mathbf{P}^0$ in the equilibrium mapping implies stability of the NPL mapping.

Convergence of NPL iterations. Suppose that the true equilibrium in the population, $\mathbf{P}^0$, is Lyapunov stable with respect to the NPL mapping. This implies that with probability approaching one, as M goes to infinity, the (sample) NPL mapping is stable around a consistent nonparametric estimator of $\mathbf{P}^0$. Therefore, the sequence of K-step estimators converges to a limit $\hat{\mathbf{P}}^0_{\text{lim}}$ that is a fixed point of the NPL mapping, that is, $\hat{\mathbf{P}}^0_{\text{lim}} = \varphi(\hat{\mathbf{P}}^0_{\text{lim}})$. It is possible to show that this limit $\hat{\mathbf{P}}^0_{\text{lim}}$ is a consistent estimator of $\mathbf{P}^0$ (see **kasahara_shimotsu_2009** ,**kasahara_shimotsu_2009**). Therefore, under Lyapunov stability of the NPL mapping, if we start with a consistent estimator of $\mathbf{P}^0$ and iterate in the NPL mapping, we converge to a consistent estimator that is an equilibrium of the model. It is possible to show that this estimator is asymptotically more efficient than the two-step estimator (**aguirregabiria_ mira_2007** ,**aguirregabiria_ mira_2007**).

Pesendorfer and Schmidt-Dengler (2010) present an example where the sequence of K-step estimators converges to a limit estimator that is not consistent. As implied by the results presented above, the equilibrium that generates the data in their example is not Lyapunov stable. The concept of Lyapunov stability of the best response mapping at an equilibrium means that if we marginally perturb players' strategies, and then allow players to best respond to the new strategies, then we will converge to the original

equilibrium. This seems like a plausible equilibrium selection criterion. Ultimately, whether an unstable equilibrium is interesting depends on the application and the researcher's taste. Nevertheless, at the end of this section we present simple modified versions of the NPL method that can deal with data generated from an equilibrium that is not stable.

Reduction of finite sample bias

kasahara_shimotsu_2008a (**kasahara_shimotsu_2008a**,**kasahara_shimotsu_2009**) derive a second order approximation to the bias of the K-step estimators. They show that the key component in this bias is the distance between the first step and the second step estimators of $\mathbf{P}^0$, that is, $\left\|\varphi\left(\hat{\mathbf{P}}^0\right)-\hat{\mathbf{P}}^0\right\|$. An estimator that reduces this distance is an estimator with lower finite sample bias. Therefore, based on our discussion in point (b) above, the sequence of K-step estimators are decreasing in their finite sample bias if and only if the NPL mapping is Lyapunov stable around $\mathbf{P}^0$.

The Monte Carlo experiments in Pesendorfer and Schmidt-Dengler (2008) illustrate this point. They implement experiments using different DGPs: in some of them the data is generated from a stable equilibrium, and in others the data come from a non-stable equilibrium. It is simple to verify (see **aguirregabiria_ mira_2010** ,**aguirregabiria_ mira_2010**) that the experiments where NPL iterations do not reduce the finite sample bias are those where the equilibrium that generates the data is not (Lyapunov) stable.

Modified NPL algorithms

Note that Lyapunov stability can be tested after obtaining the first NPL iteration. Once we have obtained the two-step estimator, we can calculate the Jacobian matrix $\partial\varphi(\hat{\mathbf{P}}^0)/\partial\mathbf{P}'$ and its eigenvalues, and then check whether Lyapunov stability holds at $\hat{\mathbf{P}}^0$.

If the applied researcher considers that her data may have been generated by an equilibrium that is not stable, then it will be worthwhile to compute this Jacobian matrix and its eigenvalues. If Lyapunov stability holds at $\hat{\mathbf{P}}^0$, then we know that NPL iterations reduce the bias of the estimator and converge to a consistent estimator.

When the condition does not hold, then the solution to this problem is not simple. Though the researcher might choose to use the two-step estimator, the non-stability of the equilibrium has also important negative implications on the properties of this simple estimator. Non-stability of the NPL mapping at $\mathbf{P}^0$ implies that the asymptotic variance of the two-step estimator of $\mathbf{P}^0$ is larger than the asymptotic variance of the nonparametric reduced form estimator. To see this, note that the two-step estimator of CCPs is $\hat{\mathbf{P}}^1 = \varphi(\hat{\mathbf{P}}^0)$, and applying the delta method we have that $Var(\hat{\mathbf{P}}^1) = [\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}'] \, Var(\hat{\mathbf{P}}^{\mathbf{0}}) \, [\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}']'$. If the spectral radius of $\partial\varphi\left(\mathbf{P}^0\right)/\partial\mathbf{P}'$ is greater than 1, then $Var(\hat{\mathbf{P}}^1) > Var(\hat{\mathbf{P}}^0)$. This is a puzzling result because the estimator $\hat{\mathbf{P}}^0$ is nonparametric while the estimator $\hat{\mathbf{P}}^1$ exploits most of the structure of the model. Therefore, the non-stability of the equilibrium that generates the data is an issue for this general class of two-step or sequential estimators.

In this context, **kasahara_shimotsu_2009** (**kasahara_shimotsu_2009**) propose alternative recursive estimators based on fixed-point mappings other than the NPL that, by construction, are stable. Iterating in these alternative mappings is significantly more

costly than iterating in the NPL mapping, but these iterations guarantee reduction of the finite sample bias and convergence to a consistent estimator.

aguirregabiria_ mira_2010 (**aguirregabiria_ mira_2010**) propose two modified versions of the NPL algorithm that are simple to implement and that always converge to a consistent estimator with better properties than two-step estimators. A *first modified-NPL-algorithm* applies to dynamic games. The first NPL iteration is standard but in every successive iteration best response mappings are used to update guesses of each player's own future behavior without updating beliefs about the strategies of the other players. This algorithm always converges to a consistent estimator even if the equilibrium generating the data is not stable and it reduces monotonically the asymptotic variance and the finite sample bias of the two-step estimator.

The *second modified-NPL-algorithm* applies to static games and it consists in the application of the standard NPL algorithm both to the best response mapping and to the inverse of this mapping. If the equilibrium that generates the data is unstable in the best response mapping, it should be stable in the inverse mapping. Therefore, the NPL applied to the inverse mapping should converge to the consistent estimator and should have a larger value of the pseudo likelihood than the estimator that we converge to when applying the NPL algorithm to the best response mapping. Aguirregabiria and Mira illustrate the performance of these estimators using the examples in Pesendorfer and Schmidt-Dengler (2008, 2010).

Estimation using Moment Inequalities

Bajari, Benkard, and Levin (2007) proposed a two-step estimator in the spirit of the ones described before but with two important differences:

> **(a)** they use moment inequalities (instead of moment equalities);
> **(b)** they do not calculate exactly the present value $\mathbf{W}_i^{\mathbf{P}}(\mathbf{x}_t)$ but they approximate them using Monte Carlo simulation.

(a) and (b) are two different ideas than can be applied separately. Both of these two ideas have different merits and therefore we will discuss them separately.

Estimation using Moment Inequalities. Remember that $V_i^{\mathbf{P}}(\mathbf{x}_t)$ is the value of player i at state $\mathbf{x}_t$ when all the players behave according to their strategies in $\mathbf{P}$. In a model where the one-period payoff function is multiplicatively separable in the structural parameters, we have that

$$V_i^{\mathbf{P}}(\mathbf{x}_t) \;=\; W_i^{\mathbf{P}}(\mathbf{x}_t)\;\theta_i$$

and the matrix of present values $\mathbf{W}_i^{\mathbf{P}} \equiv \{W_i^{\mathbf{P}}(\mathbf{x}_t) : \mathbf{x}_t \in X\}$ can be obtained exactly as:

$$\mathbf{W}_i^{\mathbf{P}} \equiv \left(\mathbf{I} - \beta\;\mathbf{F}_i^{\mathbf{P}}\right)^{-1} \bar{\mathbf{z}}_i^{\mathbf{P}}$$

For notational simplicity, we will use $W_{it}^{\mathbf{P}}$ to represent $W_i^{\mathbf{P}}(\mathbf{x}_t)$.

Let's split the vector of choice probabilities $\mathbf{P}$ into the sub-vectors $\mathbf{P}_i$ and $\mathbf{P}_{-i}$,

$$\mathbf{P} \equiv (\mathbf{P}_i\;,\;\mathbf{P}_{-i})$$

where $\mathbf{P}_i$ are the probabilities associated to player i and $\mathbf{P}_{-i}$ contains the probabilities of players other than i. $\mathbf{P}^0$ is an equilibrium associated to θ^0. Therefore, $\mathbf{P}_i^0$ is firm i's best response to $\mathbf{P}_{-i}^0$, and for any $\mathbf{P}_i \neq \mathbf{P}_i^0$ the following inequality should hold:

$$W_{it}^{\left(\mathbf{P}_i^0, \mathbf{P}_{-i}^0\right)} \theta_i^0 \geq W_{it}^{\left(\mathbf{P}_i, \mathbf{P}_{-i}^0\right)} \theta_i^0$$

We can define an estimator of θ^0 based on these (moment) inequalities. There are infinite alternative policies $\mathbf{P}_i$, and therefore there are infinite moment inequalities. For estimation, we should select a finite set of alternative policies. This is a very important decision for this class of estimators (more below). Let H be a **(finite) set of alternative policies** for each player. Define the following criterion function:

$$R\left(\theta, \mathbf{P}^0\right) \equiv \sum_{i,m,t} \sum_{\mathbf{P} \in H} \left(\min \left\{ 0 \, ; \, \left[W_{imt}^{\left(\mathbf{P}_i^0, \mathbf{P}_{-i}^0\right)} - W_{imt}^{\left(\mathbf{P}_i, \mathbf{P}_{-i}^0\right)} \right] \theta_i \right\} \right)^2$$

This criterion function penalizes departures from the inequalities. Then, given an initial NP estimator of $\mathbf{P}^0$, say $\hat{\mathbf{P}}^0$, we can define the following estimator of θ^0 based on moment inequalities (MI):

$$\hat{\theta} = \arg\min_{\theta} R\left(\theta, \hat{\mathbf{P}}^0\right)$$

There are several relevant comments to make on this MI estimator: **(1)** Computational properties (relative to two-step ME estimators); **(2)** Point identification / Set identification; **(3)** How to choose the set of alternative policies?; **(4)** Statistical properties; **(5)** Continuous decision variables.

Computational Properties. **The two-step MI estimator is more computationally costly than a two-step ME estimator.** There at least three factors that contribute to this larger cost.

> **(a)** In both types of estimators, the main cost comes from calculating the present values $\mathbf{W}_i^{\mathbf{P}}$. In a 2-step ME estimator this evaluation is done once. In the MI estimator this is done as many times as there are alternative policies in the set H;
>
> **(b)** The ME criterion function $Q\left(\theta, \hat{\mathbf{P}}\right)$ is typically globally concave in θ, but $R\left(\theta, \hat{\mathbf{P}}\right)$ is not;
>
> **(c)** Set estimation versus point estimation. The MI estimator needs an algorithm for set optimization.

MI Estimator: Point / Set identification. This estimator is based on exactly the same assumptions as the 2-step moment equalities (ME) estimator. We have seen that θ^0 **is point identified** by the moment equalities of the ME estimators (for instance, by the pseudo likelihood equations). Therefore, if the set H of alternative policies is large enough, then θ^0 should be point identified as the unique minimizer of $R\left(\theta, \mathbf{P}^0\right)$. However, it is very costly to consider a set H with many alternative policies. For the type of H sets which are considered in practice, minimizing $R\left(\theta, \mathbf{P}^0\right)$ does not uniquely identify θ^0. Therefore, θ^0 **is set identified.**

How to choose the set of alternative policies? The choice of the alternative policies in the set H plays a key role in the statistical properties (for instance, precision, bias) of this estimator. However, there is no clear rule on how to select these policies.

Statistical properties of MI estimator (relative to ME). The MI estimator is not more 'robust' than the ME estimator. Both estimators are based on exactly the same model and assumptions. Set identification. **Asymptotically, the MI estimator is less efficient than the ME estimator.** The efficient 2-step Moment Equalities (ME) estimator has lower asymptotic variance than the MI estimator, even as the set H becomes very large.

Continuous decision variables. BBL show that, when combined with simulation techniques to approximate the values $\{W_{it}^{\mathbf{P}}\}$, the MI approach can be easily applied to the **estimation of dynamic games with continuous decision variables.** In fact, the BBL estimator of a model with continuous decision variable is basically the same as with a discrete decision variable**.** The ME estimator of models with continuous decision variable may be more complicated.

A different approach to construct inequalities in dynamic games. In a MPE a player's equilibrium strategy is her best response not only within the class of Markov strategies but also within the class of non Markov strategies: for instance, strategies that vary over time. Maskin and Tirole: if all the other players use Markov strategies, a player does not have any gain from using non Markov strategies.

Suppose that to construct the inequalities $W_{it}^{(\mathbf{P}_i^0,\mathbf{P}_{-i}^0)}\ \theta_i^0 \geq W_{it}^{(\mathbf{P}_i,\mathbf{P}_{-i}^0)}\ \theta_i^0$ we use alternative strategies which are non-Markov.

In a MPE a player equilibrium strategy is her best response not only within the class of Markov strategies but also within the class of non Markov strategies: for instance, strategies that vary over time. Maskin and Tirole: if all the other players use Markov strategies, a player does not have any gain from using non Markov strategies.

More specifically, suppose that the alternative strategy of player i is

$$\mathbf{P}_i = \{P_{it}(\mathbf{x}_t) : t = 1,2,3,....;\ \mathbf{x}_t \in X\}$$

with the following features.

(a) Two-periods deviation: $P_{it} \neq P_i^0$, $P_{it+1} \neq P_i^0$, but $P_{it+s} = P_i^0$ for any $s \geq 2$;

(b) P_{it+1} is constructed in such a way that it compensates the effects of the perturbation P_{it} on the distribution of $\mathbf{x}_{t+2}$ conditional on $\mathbf{x}_t$, that is,

$$F_x^{\mathbf{P}^0(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t) = F_x^{\mathbf{P}^0(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t)$$

Given this type of alternative policies, we have that the value differences

$$W_{it}^{(\mathbf{P}_i^0,\mathbf{P}_{-i}^0)}\ \theta_i^0 \geq W_{it}^{(\mathbf{P}_i,\mathbf{P}_{-i}^0)}\ \theta_i^0$$

only depend on differences between expected payoffs at periods t and $t+1$. We do not have to use simulation, invert huge matrices, etc, and we can consider thousands (or even millions) of alternative policies.

Dealing with Unobserved Heterogeneity

So far, we have maintained the assumption that the only unobservables for the researcher are the private information shocks that are i.i.d. over firms, markets, and time. In most applications in IO, this assumption is not realistic and it can be easily rejected by the data. Markets and firms are heterogenous in terms of characteristics that are payoff-relevant for firms but unobserved to the researcher. Not accounting for this heterogeneity may generate significant biases in parameter estimates and in our understanding of competition in the industry.

For instance, in the empirical applications in Aguirregabiria and Mira (2007) and Collard-Wexler (2006), the estimation of a model without unobserved market heterogeneity implies estimates of strategic interaction between firms (that is, competition effects) that are close to zero or even have the opposite sign to the one expected under competition. In both applications, including unobserved heterogeneity in the models results in estimates that show significant and strong competition effects.

Aguirregabiria and Mira (2007) and Arcidiacono and Miller (2008) have proposed methods for the estimation of dynamic games that allow for persistent unobserved heterogeneity in players or markets. Here we concentrate on the case of permanent unobserved market heterogeneity in the profit function.

$$\Pi_{imt} \;=\; z_i^{\mathbf{P}}(a_i, \mathbf{x}_{mt})\; \theta_{ii}^{EC} - \sigma_{\xi i}\; \xi_m - \varepsilon_{imt}$$

$\sigma_{\xi i}$ is a parameter, and ξ_m is a time-invariant 'random effect' that is common knowledge to the players but unobserved to the researcher.

The distribution of this random effect has the following properties: (A.1) it has a discrete and finite support $\{\xi^1, \xi^2, \ldots, \xi^L\}$, each value in the support of ξ represents a 'market type', and we index market types by $\ell \in \{1, 2, ..., L\}$; (A.2) it is i.i.d. over markets with probability mass function $\lambda_\ell \equiv \Pr(\xi_m = \xi^\ell)$; and (A.3) it does not enter into the transition probability of the observed state variables, that is, $\Pr(\mathbf{x}_{mt+1} \mid \mathbf{x}_{mt}, \mathbf{a}_{mt}, \xi_m) = F_x(\mathbf{x}_{mt+1} \mid \mathbf{x}_{mt}, \mathbf{a}_{mt})$. Without loss of generality, ξ_m has mean zero and unit variance because the mean and the variance of ξ_m are incorporated in the parameters θ_i^{FC} and $\sigma_{\xi i}$, respectively. Also, without loss of generality, the researcher knows the points of support $\{\xi^\ell : \ell = 1, 2, \ldots, L\}$ though the probability mass function $\{\lambda_\ell\}$ is unknown.

Assumption (A.1) is common when dealing with permanent unobserved heterogeneity in dynamic structural models. The discrete support of the unobservable implies that the contribution of a market to the likelihood (or pseudo likelihood) function is a finite mixture of likelihoods under the different possible best responses that we would have for each possible market type. With continuous support we would have an infinite mixture of best responses and this could complicate significantly the computation of the likelihood. Nevertheless, as we illustrate before, using a pseudo likelihood approach and a convenient parametric specification of the distribution of ξ_m simplifies this computation such that we can consider many values in the support of this unobserved variable at a low computational cost. Assumption (A.2) is also standard when dealing with unobserved heterogeneity. Unobserved spatial correlation across markets does not generate inconsistency of the estimators that we present here because the likelihood equations that define the estimators are still valid moment conditions under spatial correlation. Incorporating spatial correlation in the model, if present in the data, would improve the

efficiency of the estimator but at a significant computational cost. Assumption (A.3) can be relaxed, and in fact the method by Arcidiacono-Miller deals with unobserved heterogeneity both in payoffs and transition probabilities.

Each market type ℓ has its own equilibrium mapping (with a different level of profits given ξ^ℓ) and its own equilibrium. Let $\mathbf{P}_\ell$ be a vector of strategies (CCPs) in market-type ℓ: $\mathbf{P}_\ell \equiv \{P_{i\ell}(\mathbf{x}_t) : i = 1,2,...,N; \mathbf{x}_t \in \mathscr{X}\}$. The introduction of unobserved market heterogeneity also implies that we can relax the assumption of only 'a single equilibrium in the data' to allow for different market types having different equilibria. It is straightforward to extend the description of an equilibrium mapping in CCPs to this model. A vector of CCPs $\mathbf{P}_\ell$ is a MPE for market type ℓ if and only if for every firm i and every state $\mathbf{x}_t$ we have that: $P_{i\ell}(\mathbf{x}_t) = \Phi\left(\tilde{\mathbf{z}}_i^{\mathbf{P}_\ell}(\mathbf{x}_t,\xi^\ell)\ \theta_i + \tilde{e}_i^{\mathbf{P}_\ell}(\mathbf{x}_t,\xi^\ell)\right)$, where now the vector of structural parameters θ_i is $\left\{\theta_{i,0}^{VP}, ..., \theta_{i,N-1}^{VP}, \theta_i^{FC}, \theta_i^{EC}, \sigma_{\xi i}\right\}$ which includes $\sigma_{\xi i}$, and the vector $\tilde{\mathbf{z}}_i^{\mathbf{P}_\ell}(\mathbf{x}_t,\xi^\ell)$ has a similar definition as before with the only difference being that it has one more component associated with $-\xi^\ell$. Since the points of support $\left\{\xi^\ell : \ell = 1,2,\ldots,L\right\}$ are known to the researcher, she can construct the equilibrium mapping for each market type.

Let λ be the vector of parameters in the probability mass function of ξ, that is, $\lambda \equiv \{\lambda_\ell : \ell = 1,2,...,L\}$, and let $\mathbf{P}$ be the set of CCPs for every market type, $\{\mathbf{P}_\ell : \ell = 1,2,...,L\}$. The (conditional) pseudo log likelihood function of this model is $Q(\theta,\lambda,\mathbf{P}) = \sum_{m=1}^M \log \Pr(\mathbf{a}_{m1},\ \mathbf{a}_{m2},\ ...,\ \mathbf{a}_{mT} \mid \mathbf{x}_{m1},\ \mathbf{x}_{m2},...,\ \mathbf{x}_{mT};\ \theta,\lambda,\mathbf{P})$. We can write this function as $\sum_{m=1}^M \log q_m(\theta,\lambda,\mathbf{P})$, where $q_m(\theta,\lambda,\mathbf{P})$ is the contribution of market m to the pseudo likelihood:

$$q_m(\theta,\lambda,\mathbf{P}) \quad = \quad \sum_{\ell=1}^{L} \lambda_{\ell|\mathbf{x}_{m1}} \left[\prod_{i,t} \Phi\left(\tilde{\mathbf{z}}_{im\ell t}^{\mathbf{P}_\ell}\ \theta_i + \tilde{e}_{im\ell t}^{\mathbf{P}_\ell}\right)^{a_{imt}} \Phi\left(-\tilde{\mathbf{z}}_{im\ell t}^{\mathbf{P}_\ell}\ \theta_i - \tilde{e}_{im\ell t}^{\mathbf{P}_\ell}\right)^{1-a_{imt}}\right]$$

where $\tilde{\mathbf{z}}_{im\ell t}^{\mathbf{P}_\ell} \equiv \tilde{\mathbf{z}}_i^{\mathbf{P}_\ell}(\mathbf{x}_{mt},\xi^\ell)$, $\tilde{e}_{im\ell t}^{\mathbf{P}_\ell} \equiv \tilde{e}_i^{\mathbf{P}_\ell}(\mathbf{x}_{mt},\xi^\ell)$, and $\lambda_{\ell|\mathbf{x}}$ is the conditional probability $\Pr(\xi_m = \xi^\ell | \mathbf{x}_{m1} = \mathbf{x})$. The conditional probability distribution $\lambda_{\ell|x}$ is different from the unconditional distribution λ_ℓ. In particular, ξ_m is not independent of the predetermined endogenous state variables that represent market structure. For instance, we expect a negative correlation between the indicators of incumbent status, s_{imt}, and the unobserved component of the fixed cost ξ_m, that is, markets where it is more costly to operate tend to have a smaller number of incumbent firms. This is the so called *initial conditions problem* (Heckman ,1981). In short panels (for T relatively small), not taking into account this dependence between ξ_m and $\mathbf{x}_{m1}$ can generate significant biases, similar to the biases associated with ignoring the existence of unobserved market heterogeneity. There are different ways to deal with the initial conditions problem in dynamic models (Heckman ,1981). One possible approach is to derive the joint distribution of $\mathbf{x}_{m1}$ and ξ_m implied by the equilibrium of the model. That is the approach proposed and applied in Aguirregabiria and Mira (2007) and Collard-Wexler (2006). Let $\mathbf{p}^{\mathbf{P}_\ell} \equiv \{p^{\mathbf{P}_\ell}(\mathbf{x}_t) : \mathbf{x}_t \in \mathscr{X}\}$ be the ergodic or steady-state distribution of $\mathbf{x}_t$ induced by the equilibrium $\mathbf{P}_\ell$ and the transition F_x. This stationary distribution can be simply obtained as the solution to the following system of linear equations: for every value $\mathbf{x}_t \in \mathscr{X}$, $p^{\mathbf{P}_\ell}(\mathbf{x}_t) = \sum_{\mathbf{x}_{t-1}\in\mathscr{X}} p^{\mathbf{P}_\ell}(\mathbf{x}_{t-1})\ F_x^{\mathbf{P}_\ell}(\mathbf{x}_t \mid \mathbf{x}_{t-1})$, or in vector form, $\mathbf{p}^{\mathbf{P}_\ell} = \mathbf{F}_x^{\mathbf{P}_\ell}\ \mathbf{p}^{\mathbf{P}_\ell}$ subject to $\mathbf{p}^{\mathbf{P}_\ell\prime}\mathbf{1} = 1$. Given the ergodic distributions for the L market types, we can apply Bayes'

rule to obtain:

$$\lambda_{\ell|\mathbf{x}_{m1}} = \frac{\lambda_\ell \; p^{\mathbf{P}_\ell}(\mathbf{x}_{m1})}{\sum_{\ell'=1}^{L} \lambda_{\ell'} \; p^{\mathbf{P}_{\ell'}}(\mathbf{x}_{m1})} \tag{8.2}$$

Note that given the CCPs $\{\mathbf{P}_\ell\}$, this conditional distribution does not depend on parameters in the vector θ, only on the distribution λ. Given this expression for the probabilities $\{\lambda_{\ell|\mathbf{x}_{m1}}\}$, we have that the pseudo likelihood in (**??**) only depends on the structural parameters θ and λ and the incidental parameters $\mathbf{P}$.

For the estimators that we discuss here, we maximize $Q(\theta,\lambda,\mathbf{P})$ with respect to (θ,λ) for given $\mathbf{P}$. Therefore, the ergodic distributions $p^{\mathbf{P}_\ell}$ are fixed during this optimization. This implies a significant reduction in the computational cost associated with the initial conditions problem. Nevertheless, in the literature of finite mixture models, it is well known that optimization of the likelihood function with respect to the mixture probabilities λ is a complicated task because the problem is plagued with many local maxima and minima. To deal with this problem, Aguirregabiria and Mira (2007) introduce an additional parametric assumption on the distribution of ξ_m that simplifies significantly the maximization of $Q(\theta,\lambda,\mathbf{P})$ for fixed $\mathbf{P}$. They assume that the probability distribution of unobserved market heterogeneity is such that the only unknown parameters for the researcher are the mean and the variance which are included in θ_i^{FC} and $\sigma_{\xi i}$, respectively. Therefore, they assume that the distribution of ξ_m (that is, the points of support and the probabilities λ_ℓ) are known to the researcher. For instance, we may assume that ξ_m has a discretized standard normal distribution with an arbitrary number of points of support L. Under this assumption, the pseudo likelihood function is maximized only with respect to θ for given $\mathbf{P}$. Avoiding optimization with respect to λ simplifies importantly the computation of the different estimators that we describe below.

NPL estimator. As defined above, the NPL mapping φ is the composition of the equilibrium mapping and the mapping that provides the maximand in θ to $Q(\theta,\mathbf{P})$ for given $\mathbf{P}$. That is, $\varphi(\mathbf{P}) \equiv \Psi(\hat{\theta}(\mathbf{P}),\mathbf{P})$ where $\hat{\theta}(\mathbf{P}) \equiv \arg\max_\theta Q(\theta,\mathbf{P})$. By definition, an NPL fixed point is a pair $(\hat{\theta},\hat{\mathbf{P}})$ that satisfies two conditions: (a) $\hat{\theta}$ maximizes $Q(\theta,\hat{\mathbf{P}})$; and (b) $\hat{\mathbf{P}}$ is an equilibrium associated to $\hat{\theta}$. The NPL estimator is defined as the NPL fixed point with the maximum value of the likelihood function. The NPL estimator is consistent under standard regularity conditions (Aguirregabiria and Mira ,2007, Proposition 2).

When the equilibrium that generates the data is Lyapunov stable, we can compute the NPL estimator using a procedure that iterates in the NPL mapping, as described in section 3.2 to obtain the sequence of K-step estimators (that is, NPL algorithm). The main difference is that now we have to calculate the steady-state distributions $\mathbf{p}(\mathbf{P}_\ell)$ to deal with the initial conditions problem. However, the pseudo likelihood approach also reduces significantly the cost of dealing with the initial conditions problem. This NPL algorithm proceeds as follows. We start with L arbitrary vectors of players' choice probabilities, one for each market type: $\{\hat{\mathbf{P}}_\ell^0 : \ell = 1,2,...,L\}$. Then, we perform the following steps. Step 1: For every market type we obtain the steady-state distributions and the probabilities $\{\lambda_{\ell|x_{m1}}\}$. Step 2: We obtain a pseudo maximum likelihood estimator of θ as $\hat{\theta}^1 = \arg\max_\theta Q(\theta,\hat{\mathbf{P}}^0\})$. Step 3: Update the vector of players' choice probabilities using the best response probability mapping. That is, for market type ℓ,

firm i and state $\mathbf{x}$, $\hat{P}^1_{i\ell}(\mathbf{x}) = \Phi(\tilde{\mathbf{z}}_i^{\hat{\mathbf{P}}^0_\ell}(\mathbf{x},\xi^\ell)\hat{\theta}^1_i + \tilde{e}_i^{\hat{\mathbf{P}}^0_\ell}(\mathbf{x},\xi^\ell))$. If, for every type ℓ, $||\hat{\mathbf{P}}^1_\ell - \hat{\mathbf{P}}^0_\ell||$ is smaller than a predetermined small constant, then stop the iterative procedure and keep $\hat{\theta}^1$ as a candidate estimator. Otherwise, repeat steps 1 to 4 using $\hat{\mathbf{P}}^1$ instead of $\hat{\mathbf{P}}^0$.

The NPL algorithm, upon convergence, finds an NPL fixed point. To guarantee consistency, the researcher needs to start the NPL algorithm from different CCP's in case there are multiple NPL fixed points. This situation is similar to using a gradient algorithm, designed to find a local root, in order to obtain an estimator which is defined as a global root. Of course, this global search aspect of the method makes it significantly more costly than the application of the NPL algorithm in models without unobserved heterogeneity. This is the additional computational cost that we have to pay for dealing with unobserved heterogeneity. Note, however, that this global search can be parallelized in a computer with multiple processors.

Arcidiacono and Miller (2008) extend this approach in several interesting and useful ways. First, they consider a more general form of unobserved heterogeneity that may enter both in the payoff function and in the transition of the state variables. Second, to deal with the complexity in the optimization of the likelihood function with respect to the distribution of the finite mixture, they combine the NPL method with an EM algorithm. Third, they show that for a class of dynamic decision models, that includes but it is not limited to optimal stopping problems, the computation of the inclusive values $\tilde{\mathbf{z}}^{\mathbf{P}_\ell}_{im\ell t}$ and $\tilde{e}^{\mathbf{P}_\ell}_{im\ell t}$ is simple and it is not subject to a 'curse of dimensionality', that is, the cost of computing these value for given $\mathbf{P}_\ell$ does not increase exponentially with the dimension of the state space. Together, these results provide a relatively simple approach to estimate dynamic games with unobserved heterogeneity of finite mixture type. Note that Lyapunov stability of each equilibrium type that generates the data is a necessary condition for the NPL and the Arcidiacono-Miller algorithms to converge to a consistent estimator.

K**kasahara_shimotsu_2008a (kasahara_shimotsu_2008a)**. The estimators of finite mixture models in Aguirregabiria and Mira (2007) and Arcidiacono and Miller (2008) consider that the researcher cannot obtain consistent nonparametric estimates of market-type CCPs $\{\mathbf{P}^0_\ell\}$. **kasahara_shimotsu_2008b** (**kasahara_shimotsu_2008b**), based on previous work by **hall_zhou_2003** (**hall_zhou_2003**), have derived sufficient conditions for the nonparametric identification of market-type CCPs $\{\mathbf{P}^0_\ell\}$ and the probability distribution of market types, $\{\lambda^0_\ell\}$. Given the nonparametric identification of market-type CCPs, it is possible to estimate structural parameters using a two-step approach similar to the one described above. However, this two-step estimator has three limitations that do not appear in two-step estimators without unobserved market heterogeneity. First, the conditions for nonparametric identification of $\mathbf{P}^0$ may not hold. Second, the nonparametric estimator in the first step is a complex estimator from a computational point of view. In particular, it requires the minimization of a sample criterion function with respect to the large dimensional object $\mathbf{P}$. This is in fact the type of computational problem that we wanted to avoid by using two-step methods instead of standard ML or GMM. Finally, the finite sample bias of the two-step estimator can be significantly more severe when $\mathbf{P}^0$ incorporates unobserved heterogeneity and we estimate it nonparametrically.

8.4 Reducing the State Space

Although two-step and sequential methods are computationally much cheaper than full solution-estimation methods, they are still impractical for applications where the dimension of the state space is large. The cost of computing exactly the matrix of present values $\mathbf{W}^{\mathbf{P}}_{z,i}$ increases cubically with the dimension of the state space. In the context of dynamic games, the dimension of the state space increases exponentially with the number of heterogeneous players. Therefore, the cost of computing the matrix of present values may become intractable even for a relatively small number of players.

A **simple approach to deal with this curse of dimensionality is to assume that** players are homogeneous and the equilibrium is symmetric. For instance, in our dynamic game of market entry-exit, when firms are heterogeneous, the dimension of the state space is $|H| * 2^N$, where $|H|$ is the number of values in the support of market size H_t. To reduce the dimensionality of the state space, we need to assume that: (a) only the number of competitors (and not their identities) affects the profit of a firm; (b) firms are homogeneous in their profit function; and (c) the selected equilibrium is symmetric.

Under these conditions, the payoff relevant state variables for a firm i are $\{H_t, s_{it}, n_{t-1}\}$ where s_{it} is its own incumbent status, and n_{t-1} is the total number of active firms at period $t-1$. The dimension of the state space is $|H| * 2 * (N+1)$ that increases only linearly with the number of players.[1] It is clear that the assumption of homogeneous firms and symmetric equilibrium can reduce substantially the dimension of the state space, and it can be useful in some empirical applications. Nevertheless, there are many applications where this assumption is too strong. For instance, in applications where firms produce differentiated products.

To deal with this issue, Hotz et al. (1994) proposed an estimator that uses Monte Carlo simulation techniques to approximate the values $\mathbf{W}^{\mathbf{P}}_{z,i}$. Bajari, Benkard, and Levin (2007) have extended this method to dynamic games and to models with continuous decision variables. This approach has proved useful in some applications. Nevertheless, it is important to be aware that in those applications with large state spaces, simulation error can be sizeable and it can induce biases in the estimation of the structural parameters. In those cases, it is worthwhile to reduce the dimension of the state space by making additional structural assumptions. That is the general idea in the inclusive-value approach that we have discussed in section 2 and that can be extended to the estimation of dynamic games. Different versions of this idea have been proposed and applied by Nevo and Rossi (2008), **maceriai_2007** (**maceriai_2007**), **rossi_2009** (**rossi_2009**), and **aguirregabiria_ho_2009** (**aguirregabiria_ho_2009**).

To present the main ideas, we consider here a dynamic game of quality competition in the spirit of Pakes and McGuire (1994), the differentiated product version of the Ericson-Pakes model. There are N firms in the market, that we index by i, and B brands or differentiated products, that we index by b. The set of brands sold by firm i is $\mathscr{B}_i \subset \{1,2,...,B\}$. Demand is given by a model similar to that of Section 2.1: consumers choose one of the B products offered in the market, or the outside good. The utility that consumer h obtains from purchasing product b at time t is $U_{hbt} = x_{bt} - \alpha$ $p_{bt} + u_{hbt}$, where x_{bt} is the quality of the product, p_{bt} is the price, α is a parameter, and u_{hbt} represents consumer specific taste for product b. These idiosyncratic errors

[1] This is a particular example of the 'exchangeability assumption' proposed by Pakes and McGuire (2001).

are identically and independently distributed over (h,b,t) with type I extreme value distribution. If the consumer decides not to purchase any of the goods, she chooses the outside option that has a mean utility normalized to zero. Therefore, the aggregate demand for product b is $q_{bt} = H_t \exp\{x_{bt} - \alpha\ p_{bt}\}\ [1 + \sum_{b'=1}^{B} \exp\{x_{b't} - \alpha p_{b't}\}]^{-1}$, where H_t represents market size at period t. The market structure of the industry at time t is characterized by the vector $\mathbf{x}_t = (H_t, x_{1t}, x_{2t}, ..., x_{Bt})$. Every period, firms take as given current market structure and decide simultaneously their current prices and their investment in quality improvement. The one-period profit of firm i can be written as

$$\Pi_{it} = \sum_{b \in \mathscr{B}_i} (p_{bt} - mc_b)\ q_{bt} - FC_b - (c_b + \varepsilon_{bt})\ a_{bt} \tag{8.3}$$

where $a_{bt} \in \{0,1\}$ is the binary variable that represents the decision to invest in quality improvement of product b; mc_b, FC_b, and c_b are structural parameters that represent marginal cost, fixed operating cost, and quality investment cost for product b, respectively; and ε_{bt} is an iid private information shock in the investment cost. Product quality evolves according to a transition probability $f_x(x_{bt+1}|a_{bt}, x_{bt})$. For instance, in the Pakes-McGuire model, $x_{bt+1} = x_{bt} - \zeta_t + a_{bt}\ v_{bt}$ where ζ_t and v_{bt} are two independent and non-negative random variables that are independently and identically distributed over (b,t).

In this model, price competition is static. The Nash-Bertrand equilibrium determines prices and quantities as functions of market structure $\mathbf{x}_t$, that is, $p_b^*(\mathbf{x}_t)$ and $q_b^*(\mathbf{x}_t)$. Firms' quality choices are the result of a dynamic game. The one-period profit function of firm i in this dynamic game is $\Pi_i(\mathbf{a}_{it}, \mathbf{x}_t) = \sum_{b \in \mathscr{B}_i} (p_i^*(\mathbf{x}_t) - mc_b)\ q_b^*(\mathbf{x}_t) - FC_b - (c_b + \varepsilon_{bt})\ a_{bt}$, where $\mathbf{a}_{it} \equiv \{a_{bt} : b \in \mathscr{B}_i\}$. This dynamic game of quality competition has the same structure as the game that we have described in Section 3.1 and it can be solved and estimated using the same methods. However, the dimension of the state space increases exponentially with the number of products and the solution and estimation of the model becomes impractical even when B is not too large.

Define the *cost adjusted inclusive value* of firm i at period t as $\omega_{it} \equiv \log[\sum_{b \in \mathscr{B}_i} exp\{x_{bt} - \alpha\ mc_b\}]$.This value is closely related to the inclusive value that we have discussed in Section 2. It can be interpreted as the net quality level, or a value added of sort, that the firm is able to produce in the market. Under the assumptions of the model, the variable profit of firm i in the Nash-Bertrand equilibrium can be written as a function of the vector of inclusive values $\omega_t \equiv (\omega_{1t}, \omega_{2t}, ..., \omega_{Nt}) \in \Omega$, that is, $\sum_{b \in \mathscr{B}_i} (p_i^*(\mathbf{x}_t) - mc_b)\ q_b^*(\mathbf{x}_t) = vp_i(\omega_t)$. Therefore, the one-period profit Π_{it} is a function $\tilde{\Pi}_i(\mathbf{a}_{it}, \omega_t)$. The following assumption is similar to Assumption A2 made in Section 2 and it establishes that given vector ω_t the rest of the information contained in the in $\mathbf{x}_t$ is redundant for the prediction of future values of ω.

Assumption: The transition probability of the vector of inclusive values ω_t from the point of view a firm (that is, conditional on a firm's choice) is such that $\Pr(\omega_{t+1} \mid \mathbf{a}_{it}, \mathbf{x}_t) = \Pr(\omega_{t+1} \mid \mathbf{a}_{it}, \omega_t)$.

Under these assumptions, ω_t is the vector of payoff relevant state variables in the dynamic game. The dimension of the space Ω increases exponentially with the number of firms but not with the number of brands. Therefore, the dimension of Ω can be much smaller than the dimension of the original state space of $\mathbf{x}_t$ in applications where the number of brands is large relative to the number of firms.

Of course, the assumption of sufficiency of ω_t in the prediction of next period ω_{t+1} is not trivial. In order to justify it we can put quite strong restrictions on the stochastic process of quality levels. Alternatively, it can be interpreted in terms of limited information, and/or bounded rationality. For instance, a possible way to justify this assumption is that firms face the same type of computational burdens that we do. Limiting the information that they use in their strategies reduces a firm's computational cost of calculating a best response.

Note that the dimension of the space of ω_t still increases exponentially with the number of firms. To deal with this curse of dimensionality, **aguirregabiria_ho_2009** (**aguirregabiria_ho_2009**) consider a stronger inclusive value / sufficiency assumption. Let vp_{it} be the variable profit of firm i at period t. Assumption: $\Pr(\omega_{it+1}, vp_{it+1} \mid \mathbf{a}_{it}, \mathbf{x}_t) = \Pr(\omega_{it+1}, vp_{it+1} \mid \mathbf{a}_{it}, \omega_{it}, vp_{it})$. Under this assumption, the vector of payoff relevant state variables in the decision problem of firm i is (ω_{it}, vp_{it}) and the dimension of the space of (ω_{it}, vp_{it}) does not increase with the number of firms.

8.5 Counterfactual experiments with multiple equilibria

One of the attractive features of structural models is that they can be used to predict the effects of new counterfactual policies. This is a challenging exercise in a model with multiple equilibria. Under the assumption that our data has been generated by a single equilibrium, we can use the data to identify which of the multiple equilibria is the one that we observe. However, even under that assumption, we still do not know which equilibrium will be selected when the values of the structural parameters are different to the ones that we have estimated from the data. For some models, a possible approach to deal with this issue is to calculate all of the equilibria in the counterfactual scenario and then draw conclusions that are robust to whatever equilibrium is selected. However, this approach is of limited applicability in dynamic games of oligopoly competition because the different equilibria typically provide contradictory predictions for the effects we want to measure.

Here we describe a simple homotopy method that has been proposed in **aguirregabiria_2009** (**aguirregabiria_2009**) and applied in **aguirregabiria_ho_2009** (**aguirregabiria_ho_2009**). Under the assumption that the equilibrium selection mechanism, which is unknown to the researcher, is a smooth function of the structural parameters, we show how to obtain a Taylor approximation to the counterfactual equilibrium. Despite the equilibrium selection function being unknown, a Taylor approximation of that function, evaluated at the estimated equilibrium, depends on objects that the researcher knows.

Let $\Psi(\theta, \mathbf{P})$ be the equilibrium mapping such that an equilibrium associated with θ can be represented as a fixed point $\mathbf{P} = \Psi(\theta, \mathbf{P})$. Suppose that there is an equilibrium selection mechanism in the population under study, but we do not know that mechanism. Let $\pi(\theta)$ be the selected equilibrium given θ. The approach here is quite agnostic with respect to this equilibrium selection mechanism: it only assumes that there is such a mechanism, and that it is a smooth function of θ. Since we do not know the mechanism, we do not know the form of the mapping $\pi(\theta)$ for every possible θ. However, we know that the equilibrium in the population, $\mathbf{P}^0$, and the vector of the structural parameters in the population, θ^0, belong to the graph of that mapping, that is, $\mathbf{P}^0 = \pi(\theta^0)$.

Let θ^* be the vector of parameters under the counterfactual experiment that we want

to analyze. We want to know the counterfactual equilibrium $\pi(\theta^*)$ and compare it to the factual equilibrium $\pi(\theta^0)$. Suppose that Ψ is twice continuously differentiable in θ and P. The following is the key assumption to implement the homotopy method that we describe here.

Assumption: The equilibrium selection mechanism is such that π is a continuous differentiable function within a convex subset of Θ that includes θ^0 and θ^*.

That is, the equilibrium selection mechanism does not "jump" between the possible equilibria when we move over the parameter space from θ^0 to θ^*. This seems a reasonable condition when the researcher is interested in evaluating the effects of a change in the structural parameters but "keeping constant" the same equilibrium type as the one that generates the data.

Under these conditions, we can make a Taylor approximation to $\pi(\theta^*)$ around θ^0 to obtain:

$$\pi(\theta^*) = \pi\left(\theta^0\right) + \frac{\partial \pi\left(\theta^0\right)}{\partial \theta'}\left(\theta^* - \theta^0\right) + O\left(\left\|\theta^* - \theta^0\right\|^2\right) \tag{8.4}$$

We know that $\pi\left(\theta^0\right) = \mathbf{P}^0$. Furthermore, by the implicit function theorem, $\partial \pi\left(\theta^0\right)/\partial \theta'$ $= \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \theta' + \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \mathbf{P}' \; \partial \pi\left(\theta^0\right)/\partial \theta'$. If $\mathbf{P}^0$ is not a singular equilibrium then $I - \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \mathbf{P}'$ is not a singular matrix and $\partial \pi\left(\theta^0\right)/\partial \theta' = (I - \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \mathbf{P}')^{-1} \; \partial \Psi(\theta^0, \mathbf{P}^0)/\partial \theta'$. Solving this expression into the Taylor approximation, we have the following approximation to the counterfactual equilibrium:

$$\hat{\mathbf{P}}^* = \hat{\mathbf{P}}^0 + \left(I - \frac{\partial \Psi(\hat{\theta}^0, \hat{\mathbf{P}}^0)}{\partial \mathbf{P}'}\right)^{-1} \frac{\partial \Psi(\hat{\theta}^0, \hat{\mathbf{P}}^0)}{\partial \theta'}\left(\theta^* - \hat{\theta}^0\right) \tag{8.5}$$

where $(\hat{\theta}^0, \hat{\mathbf{P}}^0)$ represents our consistent estimator of $(\theta^0, \mathbf{P}^0)$. It is clear that $\hat{\mathbf{P}}^*$ can be computed given the data and θ^*. Under our assumptions, $\hat{\mathbf{P}}^*$ is a consistent estimator of the linear approximation to $\pi(\theta^*)$.

As in any Taylor approximation, the order of magnitude of the error depends on the distance between the value of the structural parameters in the factual and counterfactual scenarios. Therefore, this approach can be inaccurate when the counterfactual experiment implies a large change in some of the parameters. For these cases, we can combine the Taylor approximation with iterations in the equilibrium mapping. Suppose that $\mathbf{P}^*$ is a (Lyapunov) stable equilibrium. And suppose that the Taylor approximation $\hat{\mathbf{P}}^*$ belongs to the dominion of attraction of $\mathbf{P}^*$. Then, by iterating in the equilibrium mapping $\Psi(\theta^*, .)$ starting at $\hat{\mathbf{P}}^*$ we will obtain the counterfactual equilibrium $\mathbf{P}^*$. Note that this approach is substantially different to iterating in the equilibrium mapping $\Psi(\theta^*, .)$ starting with the equilibrium in the data $\hat{\mathbf{P}}^0$. This approach will return the counterfactual equilibrium $\mathbf{P}^*$ *if and only if* $\hat{\mathbf{P}}^0$ belongs to the dominion of attraction of $\mathbf{P}^*$. This condition is stronger than the one establishing that the Taylor approximation $\hat{\mathbf{P}}^*$ belongs to the domination of attraction of $\mathbf{P}^*$.

9. Dynamic Games: Applications

9.1 Environmental Regulation in the Cement Industry

Ryan studies the effects in the US cement industry of the 1990 Amendments to Air Clean Act. Ryan's model presents a dynamic game of oligopoly competition where firms compete in quantities but they also make investment decisions in capacity and in market entry/exit, and they are heterogeneous in their different costs, that is,marginal costs, fixed costs, capacity investment costs, and sunk entry costs.

Below, we examine the following points of the paper. (a) Motivation and Empirical Questions; (b) The US Cement Industry; (c) The Regulation (Policy Change); (d) Empirical Strategy; (e) Data; (f) Model; (g) Estimation and Results.

9.1.1 Motivation and Empirical Questions

Most previous studies that measure the welfare effects of environmental regulation (ER) have ignored dynamic effects of these policies.

ER has potentially important effects on firms' entry and investment decisions, and, in turn, these can have important welfare effects.

This paper estimates a dynamic game of entry/exit and investment in the US Portland cement industry.

The estimated model is used to evaluate the welfare effects of the 1990 Amendments to the Clean Air Act (CAA).

9.1.2 The US Cement Industry

For the purpose of this paper, the most important features of the US cement industry are: (1) Indivisibilities in capacity investment, and economies of scale; (2) Highly polluting and energy intensive industry; and (3) Local competition, and highly concentrated local markets

Indivisibilities in capacity investment, and economies of scale. Portland cement is the binding material in concrete, which is a primary construction material. It is produced by first pulverizing limestone and then heating it at very high temperatures in a rotating kiln furnace. These kilns are the main piece of equipment. Plants can have one or more kilns (indivisibilities). Marginal cost increases rapidly when a kiln is close to full capacity.

Highly polluting and energy intensive industry. The industry generates a large amount of pollutants by-products. High energy requirements and pollution make the cement industry an important target of environmental policies.

Local competition, and highly concentrated local markets. Cement is a commodity difficult to store and transport, as it gradually absorbs water out of the air rendering it useless. This is the main reason why the industry is spatially segregated into regional markets. These regional markets are very concentrated.

9.1.3 The Regulation (Policy Change)

In 1990, the Amendments to the Clean Air Act (CAA) added new categories of regulated emissions. Also, cement plants were required to undergo an environmental certification process. It has been the most important new environmental regulation affecting this industry in the last three decades. This regulation may have increased sunk costs, fixed operating costs or even investment costs in this industry.

9.1.4 Empirical Strategy

Previous evaluations of these policies have ignored effects on entry/exit and on firms' investment. They have found that the regulation contributed to reduce marginal costs and therefore prices. Positive effects on consumer welfare and total welfare. Ignoring effects on entry/exit and on firms' investment could imply an overestimate of these positive effects.

Ryan specifies a model of the cement industry, where oligopolists make optimal decisions over entry, exit, production, and investment given the strategies of their competitors. He estimates the model for the cement industry using a 20 year panel and allowing the structural parameters to differ before and after the 1990 regulation. Changes in cost parameters are attributed to the new regulation. The MPEs before and after the regulation are computed and they are used for welfare comparisons.

Comments on this empirical approach and its potential limitations: (a) anticipation of the policy; (b) technological change; (c) learning about the new policy.

9.1.5 Data

Period: 1980 to 1999 (20 years); 27 regional markets. Index local markets by m, plants by i and years by t.

$$Data = \{S_{mt},\ W_{mt},\ P_{mt},\ n_{mt},\ q_{imt},\ i_{imt},\ s_{imt}\}$$

S_{mt} = Market size; W_{mt} = Input prices (electricity prices, coal prices, natural gas prices, and manufacturing wages); P_{mt} = Output price; n_{mt} = Number of cement plants; q_{imt} = Quantity produced by plant i; s_{imt} = Capacity of plant i (number and capacity of kilns); i_{imt} = Investment in capacity by plant i.

9.1.6 Model

Regional homogenous-goods market. Firms compete in quantities in a static equilibrium, but they are subject to capacity constraints. Capacity is the most important strategic variable. Firms invest in future capacity and this decision is partly irreversible (and therefore dynamic). Incumbent firms also make optimal decisions over whether to exit.

Inverse demand curve (iso-elastic):

$$\log P_{mt} = \alpha_{mt} + \frac{1}{\varepsilon}\log Q_{mt}$$

Production costs:

$$\begin{aligned} C(q_{imt}) &= (MCOST + \omega_{imt})\, q_{imt} \\ &\quad + CAPCOST * I\left\{\frac{q_{imt}}{s_{imt}} > BINDING\right\}\left(\frac{q_{imt}}{s_{imt}} - BINDING\right)^2 \end{aligned}$$

s_{imt} = installed capacity; q_{imt}/s_{imt} = degree of capacity utilization; ω_{imt} = private information shock; $MCOST$, $CAPCOST$ and $BINDING$ are parameters.

Investment costs

$$\begin{aligned} IC_{imt} &= I\{i_{imt} > 0\}\left(ADJPOS + INVMCPOS * i_{imt} + INVMCPOS2 * i_{imt}^2\right) \\ &\quad + I\{i_{imt} < 0\}\left(ADJNEG + INVMCNEG * i_{imt} + INVMCNEG2 * i_{imt}^2\right) \end{aligned}$$

Entry costs

$$EC_{imt} = I\{s_{imt} = 0 \text{ and } i_{imt} > 0\} \left(SUNK + \varepsilon_{imt}^{EC}\right)$$

In equilibrium, investment is a function:

$$i_{imt} = i(\alpha_{mt}, W_{mt}, s_{imt}, s_{-imt})$$

Similarly, entry and exit probabilities depend on $(\alpha_{mt}, W_{mt}, s_{imt}, s_{-imt}, \varepsilon_{imt}^{EC})$.

9.1.7 Estimation and Results

Estimation of demand curve. Includes local market region fixed effects (estimated with 19 observations per market). Instruments: local variation in input prices. The market specific demand shocks, A_{mt}, are estimated as residuals in this equation.

Estimation of variable production costs. From the Cournot equilibrium conditions. Firm specific cost shocks, ω_{imt}, are estimated as residuals in this equation.

Estimation of investment functions. Assumption:

$$i_{imt} = i(\alpha_{mt}, W_{mt}, s_{imt}, s_{-imt}) = i\left(\alpha_{mt}, W_{mt}, s_{imt}, \sum_{j \neq i} s_{jmt}\right)$$

9.2 Dynamic game of store location

Opening (or closing) a store is a forward-looking decision with significant non-recoverable entry costs, mainly owing to capital investments which are both firm and location-specific. The sunk cost of setting up new stores, and the dynamic strategic behavior associated with them, are potentially important forces behind the configuration of the spatial market structure that we observe in retail markets. We now present an extension of the previous model that incorporates these dynamic considerations.

Time is discrete and indexed by $t \in \{\ldots, 0, 1, 2, \ldots\}$. At the beginning of period t a firm's network of stores is represented by the vector $a_{it} \equiv \{a_{i\ell t} : \ell = 1, 2, \ldots, L\}$, where $a_{i\ell t}$ is the number of stores that firm i operates in location ℓ at period t. For simplicity, we maintain the assumption that a firm can have at most one store in a location, such that $a_{i\ell t} \in \{0, 1\}$. The market structure at period t is represented by the vector $a_t \equiv \{a_{it} : i = 1, 2, \ldots, N\}$ capturing the store network of all firms. Following the structure in the influential work on dynamic games of oligopoly competition by Ericson and Pakes (1995) and Pakes and McGuire (1994), at every period t the model has two stages, similar to the ones described in the static game above. In the second stage, taking the vector of firms' store networks a_t as given, retail chains compete in prices in exactly the same way as in the Bertrand model described in section 2.1.2. The equilibrium in this Bertrand game determines the indirect variable profit function, $VP_i^*(a_t; z_t)$, where z_t is a vector of exogenous state variables in demand and costs. Some components of z_t may be random variables, and their future values may not be known at the current period. In the first stage, every firm decides its network of stores in the next period, $a_{i,t+1}$, and pays at period t the entry and exit costs associated to opening and closing stores. The period profit of a firm is $\pi_i(a_{i,t+1}, a_t, z_t) = VP_i^*(a_t; z_t) - FC(a_{it}; z_t) - AC_i(a_{i,t+1}, a_{it})$,

where FC_i is the fixed cost of operating the network, and AC_i is the cost of adjusting the network from a_{it} to $a_{i,t+1}$, that is, costs of opening and closing stores. A firm chooses its new network $a_{i,t+1}$ to maximize the sum of its discounted expected future profits.

A Markov perfect equilibrium of this dynamic game is an N-tuple of strategy functions $\{\alpha_i^*(a_t, z_t) : i = 1, 2, \ldots, N\}$ such that every firm maximizes its expected intertemporal profit:

$$\alpha_i^*(a_t, z_t) = \arg\max_{a_{i,t+1}} \left[\pi_i(a_{i,t+1}, a_t, z_t) + \delta \, \mathbb{E}_t(V_i^{\alpha^*}(a_{i,t+1}, \alpha_{-i}^*(a_t, z_t), z_{t+1}) \right] \tag{9.1}$$

where $\delta \in (0,1)$ is the discount factor, and $V_i^{\alpha^*}(a_{it}, a_{-it}, z_t)$ is the value of firm i when firms' networks are equal to a_t, the value of exogenous state variables is z_t, and the other firms follow strategies α_{-i}^*.

9.2.1 Single-store firms

When the entry cost is partially sunk, firms' entry decisions depend on their incumbency status, and dynamic models become more relevant. The role of sunk entry costs in shaping market structure in an oligopoly industry was first empirically studied by Bresnahan and Reiss (1994). They estimate a two-period model using panel data on the number of dentists. Following recent developments in the econometrics of dynamic games of oligopoly competition, several studies have estimated dynamic games of market entry-exit in different retail industries.

Aguirregabiria and Mira (2007) estimate dynamic games of market entry and exit for five different retail industries: restaurants, bookstores, gas stations, shoe shops, and fish shops. They use annual data from a census of Chilean firms created for tax purposes by the Chilean Internal Revenue Service during the period 1994–99. The estimated models show significant differences in fixed costs, entry costs, and competition effects across the five industries, and these three parameters provide a precise description of the observed differences in market structure and entry-exit rates between the five industries. Fixed operating costs are a very important component of total profits of a store in the five industries, and they range between 59 percent (in restaurants) to 85 percent (in bookstores) of the variable profit of a monopolist in a median market. Sunk entry costs are also significant in the five industries, and they range between 31 percent (in shoe shops) and 58 percent (in gas stations) of a monopolist variable profit in a median market. The estimates of the parameter that measures the competition effect show that restaurants are the retailers with the smallest competition effects, which might be explained by a higher degree of horizontal product differentiation in this industry.

Suzuki (2013) examines the consequence of tight land use regulation on market structure of hotels through its impacts on entry costs and fixed costs. He estimates a dynamic game of entry-exit of mid-scale hotels in Texas that incorporates detailed measures of land use regulation into cost functions of hotels. The estimated model shows that imposing stringent regulation increases costs considerably and has substantial effects on market structure and hotel profits. Consumers also incur a substantial part of the costs of regulation in the form of higher prices.

Dunne et al. (2013) estimate a dynamic game of entry and exit in the retail industries of dentists and chiropractors in the US, and use the estimated model to evaluate the effects on market structure of subsidies for entry in small geographic markets, that is,

markets that were designated by the government as Health Professional Shortage Areas (HPSA). The authors compare the effects of this subsidy with those of a counterfactual subsidy on fixed costs, and they find that subsidies on entry costs are cheaper, or more effective for the same present value of the subsidy.

Yang (2020) extends the standard dynamic game of market entry-exit in a retail market by incorporating information spillovers from incumbent firms to potential entrants. In his model, a potential entrant does not know a market-specific component in the level of profitability of a market (for example, a component of demand or operating costs). Firms learn about this profitability only when they actually enter that market. In this context, observing incumbents staying in this market is a positive signal for potential entrants about the quality of this market. Potential entrants use these signals to update their beliefs about the profitability of the market (that is, Bayesian updating). These information spillovers from incumbents may contribute to explaining why we observe retail clusters in some geographic markets. Yang estimates his model using data from the fast food restaurant industry in Canada, which goes back to the initial conditions of this industry in Canada. He finds significant evidence supporting the hypothesis that learning from incumbents induces retailers to herd into markets where others have previously done well in, and to avoid markets where others have previously failed in.

9.2.2 Multi-store firms

A structural empirical analysis of economies of density, cannibalization, or spatial entry deterrence in retail chains requires the specification and estimation of models that incorporate dynamics, multi-store firms, and spatial competition. Some recent papers present contributions on this research topic.

Holmes (2011) studies the temporal and spatial pattern of store expansion by Walmart during the period 1971–2005. He proposes and estimates a dynamic model of entry and store location by a multi-store firm similar to the one that we have described in section 2.1.3 above. The model incorporates economies of density and cannibalization between Walmart stores, though it does not model explicitly competition from other retailers or chains (for example, Kmart or Target), and therefore it abstracts from dynamic strategic considerations such as spatial entry deterrence. The model also abstracts from price variation and assumes that Walmart sets constant prices across all stores and over time. However, Holmes takes into account three different types of stores and plants in the Walmart retail network: regular stores that sell only general merchandise; supercenters, that sell both general merchandise and food; and distribution centers, which are the warehouses in the network, that have also two different types: general and food. The distinction between these types of stores and warehouses is particularly important to explain the evolution of the Walmart retail network over time and space. In the model, every year Walmart decides the number and the geographic location of new regular stores, supercenters, and general and food distribution centers. Economies of density are channeled through the benefits of stores being close to distribution centers. The structural parameters of the model are estimated using the Moment Inequalities estimation method in Pakes et al. (2015). More specifically, moment inequalities are constructed by comparing the present value of profits from Walmart's actual expansion decision with the present value from counterfactual expansion decisions, which are slight deviations from the observed ones. Holmes finds that Walmart obtains large savings in

distribution costs by having a dense store network.

Igami and Yang (2016) study the trade-off between cannibalization and spatial pre-emption in the fast-food restaurant industry, for example, McDonalds, Burger King, and so on. Consider a chain store that has already opened its first store in a local market. Opening an additional store increases this chain's current and future variable profits by, first, attracting more consumers and, second, preventing its rivals' future entries (pre-emption). However, the magnitude of this increase could be marginal when the new store steals customers from its existing store (cannibalization). Whether opening a new store economically makes sense or not depends on the size of the entry cost. Igami and Yang estimate a dynamic structural model and find the quantitative importance of preemptive motives. However, they do not model explicitly spatial competition, or allow for multiple geographic locations within their broad definition of a geographic market.

Schiraldi, Smith, and Takahashi (2012) study store location and spatial competition between UK supermarket chains. They propose and estimate a dynamic game similar to the one in Aguirregabiria and Vicentini (2016) that we have described in section 2.1.3. A novel and interesting aspect of this application is that the authors incorporate the regulator's decision to approve or reject supermarkets' applications for opening a new store in a specific location. The estimation of the model exploits a very rich dataset from the U.K. supermarket industry on exact locations and dates of store openings/closings, applications for store opening, approval/rejection decisions by the regulator, as well as rich data of consumer choices and consumer locations. The estimated model is used to evaluate the welfare effects of factual and counterfactual decision rules by the regulator.

9.3 Product repositioning in differentiated product markets

(Sweeting ,2007) To Be Completed

9.4 Dynamic Game of Airlines Network Competition

9.4.1 Motivation and Empirical Questions

An airline network is a description of the city-pairs (or airport pairs) that the airline connects with non-stop flights. The first goal of this paper is to develop a **dynamic game of network competition between airlines**, a model that can be estimated using publicly available data.

The model endogenizes airlines' networks, and the dynamics of these networks. Prices and quantities for each airline-route are also endogenous in the model. It extends previous work by Hendricks, Piccione, and Tan (1995, 1999) on airline networks, and previous literature on structural models of the airline industry: Berry (1990, 1992), **berry_carnall_2006** (**berry_carnall_2006**), Ciliberto and Tamer (2009).

The second of the paper is to apply this model to study empirically the contribution of demand, cost, and strategic factors to explain why most companies in the US airline industry operate using **hub-and-spoke networks**. The model incorporates **different hypotheses** that have been suggested in the literature to explain **hub-and-spoke networks**. We estimate the model and use counterfactual experiments to obtain the contribution of demand, costs and strategic factors.

Hub-and-Spoke Networks

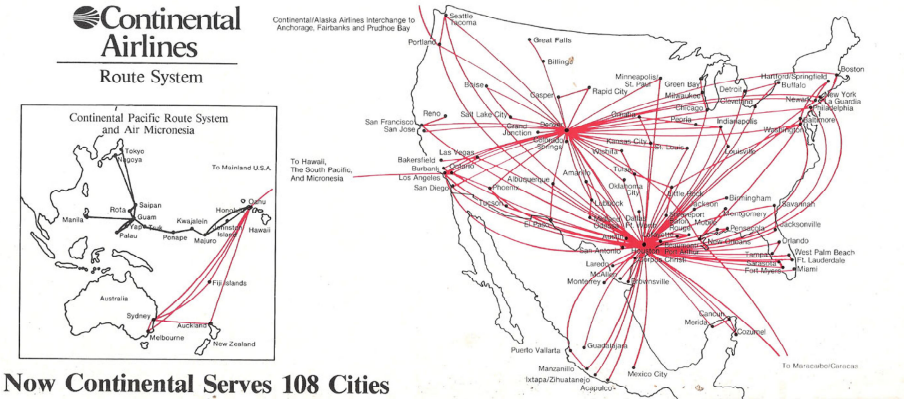


Figure 9.1: Hub & Spoke Networks: Continental route map in 1983

Hypotheses that have been suggested in the literature to explain airlines' adoption of hub-spoke networks:

- **Demand:** Travellers may be willing to pay for the services associated with an airline's scale of operation in an airport.

- **Costs:** Economies of scale at the plane level (marginal costs); Economies of scope at the airport level (fixed costs and entry costs); Contracts with airports (fixed costs and entry costs).

- **Strategic:** Entry deterrence (Hendricks, Piccione, and Tan ,1997).

The paper has several contributions to the literature on empirical dynamic games of oligopoly competition: (1) first application of dynamic network competition; (2) first paper to study empirically the strategic entry-deterrence aspect of hub-and-spoke networks; (3) first paper to apply the inclusive-values approach to a dynamic game; and (4) it proposes and implements a new method to make counterfactual experiments in dynamic games.

9.4.2 Model: Dynamic Game of Network Competition

N airlines and C cities, exogenously given. In our application, $N = 22$ and $C = 55$.

City-Pairs and Routes. Given the C cities, there are $M \equiv C(C-1)/2$ **non-directional city-pairs** (or markets). For each city-pair, an airline decides whether to operate non-stop flights. A **route** (or path) is a **directional round-trip between 2 cities**. A route may or may not have stops. A route-airline is a product, and there is a demand for each route-airline product. Airlines choose prices for each route they provide.

Networks. We index city-pairs by m, airlines by i, and time (quarters) by t. $x_{imt} \in \{0,1\}$ is a binary indicator for the event "airline i operates non-stop flights in city-pair m". $x_{it} \equiv \{x_{imt} : m = 1,2,...,M\}$ is the network of airline i at period t. The network x_{it} describes all the routes (products) that the airline provides, and whether they are non-stop or stop routes. The industry network is $x_t \equiv \{x_{it} : i = 1,2,...,N\}$.

Airlines' Decisions. An airline network x_{it} determines the set of routes (products) that the airline provides, that we denote by $L(x_{it})$. Every period, active airlines in a route compete in prices. Price competition determines variable profits for each airline. Every period (quarter), each airline decides also its network for next period. There is *time-to-build*. We represent this decision as $a_{it} \equiv \{a_{imt} : m = 1,2,...,M\}$, though $a_{imt} \equiv x_{imt+1}$.

Profit Function. The airline's total profit function is:

$$\begin{aligned} \Pi_{it} &= \sum_{r \in L(x_{it})} (p_{irt} - c_{irt}) q_{irt} \\ &- \sum_{m=1}^{M} a_{imt} \left(FC_{imt} + (1 - x_{imt}) \, EC_{imt} \right) \end{aligned}$$

$(p_{irt} - c_{irt})q_{irt}$ is the variable profit in route r. FC_{imt} and EC_{imt} are fixed cost and entry cost in city-pair m.

Network effects in demand and costs. An important feature of the model is that demand, variable costs, fixed costs, and entry costs depend on the scale of operation (number of connections) of the airline in the origin and destination airports of the city-pair. Let HUB_{imt} be the "hub size" of airline i in market m at period t as measured by the total number of connections to other cities that airline i has in the origin and destination cities of market m at the beginning of period t. This is the most important endogeneous state variable of this model. It is endogenous because, though HUB_{imt} does not depend on the entry-exit decision of the airline in market m, a_{imt-1}, it does depend on the airline's entry-exit decisions in any other market that has common cities with market m, $\{a_{im't-1}$ for $m' \neq m$ and markets m' and m have common cities$\}$.

This implies that markets are interconnected through these hub-size effects. Entry-exit in a market has implications of profits in other markets. An equilibrium of this model is an equilibrium for the whole airline industry and not only for a single city-pair.

Dynamic Game / Strategy Functions. Airlines maximize intertemporal profits, are forward-looking, and take into account the implications of their entry-exit decisions on future profits and on the expected future reaction of competitors. Airlines' strategies depend only on payoff-relevant state variables, that is, Markov perfect equilibrium assumption. An airline's payoff-relevant information at quarter t is $\{x_t, \mathbf{z}_t, \varepsilon_{it}\}$. Let $\sigma \equiv \{\sigma_i(x_t, \mathbf{z}_t, \varepsilon_{it}) : i = 1,2,...,N\}$ be a set of strategy functions, one for each airline. A MPE is a set of strategy functions such that each airline's strategy maximizes the value of the airline for each possible state and taking as given other airlines' strategies.

9.4.3 Data

Airline Origin and Destination Survey (DB1B) collected by the Office of Airline Information of the BTS. Period 2004-Q1 to 2004-Q4. $C = 55$ largest metropolitan areas. $N = 22$ airlines. City Pairs: $M = (55 * 54)/2 = 1,485$.

Airlines: Passengers and Markets

	Airline (Code)	# Passengers (in thousands)	# City-Pairs in 2004-Q4 (maximum = 1,485)
1.	Southwest (WN)	25,026	373
2.	American (AA)[3]	20,064	233
3.	United (UA)[4]	15,851	199
4.	Delta (DL)[5]	14,402	198
5.	Continental (CO)[6]	10,084	142
6.	Northwest (NW)[7]	9,517	183
7.	US Airways (US)	7,515	150
8.	America West (HP)[8]	6,745	113
9.	Alaska (AS)	3,886	32
10.	ATA (TZ)	2,608	33
11.	JetBlue (B6)	2,458	22

Airlines, their Hubs, and Hub-Spoke Ratios

Airline (Code)	1st largest hub	Hub-Spoke Ratio (%) One Hub	2nd largest hub	Hub-Spoke Ratio (%) Two Hubs
Southwest	Las Vegas (35)	9.3	Phoenix (33)	18.2
American	Dallas (52)	22.3	Chicago (46)	42.0
United	Chicago (50)	25.1	Denver (41)	45.7
Delta	Atlanta (53)	26.7	Cincinnati (42)	48.0
Continental	Houston (52)	36.6	New York (45)	68.3
Northwest	Minneapolis (47)	25.6	Detroit (43)	49.2
US Airways	Charlotte (35)	23.3	Philadelphia (33)	45.3
America West	Phoenix (40)	35.4	Las Vegas (28)	60.2
Alaska	Seattle (18)	56.2	Portland (10)	87.5
ATA	Chicago (16)	48.4	Indianapolis (6)	66.6
JetBlue	New York (13)	59.0	Long Beach (4)	77.3

Figure 9.2: Cumulative Hub-and-Spoke Ratios

Distribution of Markets by Number of Incumbents	
Markets with 0 airlines	35.44%
Markets with 1 airline	29.06%
Markets with 2 airlines	17.44%
Markets with 3 airlines	9.84%
Markets with 4 or more airlines	8.22%

Number of Monopoly Markets by Airline	
Southwest	157
Northwest	69
Delta	56
American	28
Continental	24
United	17

Entry and Exit	
	All Quarters
Distribution of Markets by Number of New Entrants	
Markets with 0 Entrants	84.66%
Markets with 1 Entrant	13.37%
Markets with 2 Entrants	1.69%
Markets with 3 Entrants	0.27%
Distribution of Markets by Number of Exits	
Markets with 0 Exits	86.51%
Markets with 1 Exit	11.82%
Markets with 2 Exits	1.35%
Markets with more 3 or 4 Exits	0.32%

9.4.4 Specification and Estimation of Demand

Demand. Let $d \in \{0,1\}$ be the indicator of "direct" or non-stop flight. Let q_{irdt} be the number of tickets sold by airline i for route r, type of flight d, at quarter t. For a given route r and quarter t, the quantities $\{q_{irdt}$: for every airline i and $d = 0,1\}$ come from a system of demand of differentiated product. More specifically, we consider a Nested Logit demand. For notational simplicity, we omit here the subindexes (r,t), but the demand system refers to a specific route and quarter.

Let H be the number of travelers in the route. Each traveler in the route demands only one trip (per quarter) and chooses which product to purchase. The indirect utility of a traveler who purchases product (i,d) is $U_{id} = b_{id} - p_{id} + v_{id}$, where p_{id} is the price of product (i,d), b_{id} is the "quality" or willingness to pay for the product of the average consumer in the market, and v_{id} is a consumer-specific component that captures consumer heterogeneity in preferences. Product quality b_{ird} depends on exogenous characteristics of the airline and the route, and on the endogenous "hub-size" of the airline in the origin and destination airports.

$$b_{id} \quad = \quad \alpha_1\, d + \alpha_2\, HUB_i + \alpha_3\, DIST + \xi_i^{(1)} + \xi^{(2)} + \xi_{id}^{(3)}$$

α_1 to α_3 are parameters. *DIST* is the flown distance between the origin and destination cities of the route. $\xi_i^{(1)}$ is an airline fixed-effect that captures between-airlines differences in quality which are constant over time and across markets. $\xi^{(2)}$ represents the interaction of (origin and destination) city dummies and time dummies. These terms account for demand shocks, such as seasonal effects, which vary across cities and over time. $\xi_{id}^{(3)}$ is a demand shock that is airline and route specific. The variable HUB_i represents the "hub size" airline i in the origin and destination airports of the route r.

In the Nested Logit, we have that $v_{id} = \sigma_1 \, v_i^{(1)} + \sigma_2 \, v_{id}^{(2)}$, where $v_i^{(1)}$ and $v_{id}^{(2)}$ are independent Type I extreme value random variables, and σ_1 and σ_2 are parameters that measure the dispersion of these variables, with $\sigma_1 \geq \sigma_2$. A property of the nested logit model is that the demand system can be represented using the following closed-form demand equations:

$$\ln(s_{id}) - \ln(s_0) \quad = \quad \frac{b_{id} - p_{id}}{\sigma_1} + \left(1 - \frac{\sigma_2}{\sigma_1}\right) \ln\left(s_{id}^*\right) \tag{9.2}$$

where s_0 is the share of the outside alternative in route r, that is, $s_{0r} \equiv 1 - \sum_{i=1}^{N}(s_{ir0} + s_{ir1})$, and s_{id}^* is the market share of product (i,d) within the products of airline i in this route, that is, $s_{id}^* \equiv s_{id} / (s_{i0} + s_{i1})$.

Therefore, we have the following demand regression equation:

$$\ln(s_{irdt}) - \ln(s_{0rdt}) \quad = \quad W_{irdt} \, \alpha + \left(\frac{-1}{\sigma_1}\right) p_{irdt} + \left(1 - \frac{\sigma_2}{\sigma_1}\right) \ln\left(s_{irdt}^*\right) + \xi_{irdt}^{(3)} \tag{9.3}$$

The regressors in vector W_{irdt} are: dummy for nonstop-flight, hub-size, distance, airline dummies, origin-city dummies $\times$ time dummies, and destination-city dummies $\times$ time dummies.

Issues: Is HUB_{irt} correlated with $\xi_{irdt}^{(3)}$? Are the BLP instruments (HUB size of competing airlines in route r at period t) valid in this equation, that is, are they correlated with $\xi_{irdt}^{(3)}$?

ASSUMPTION D1: Idiosyncratic demand shocks $\{\xi_{irdt}^{(3)}\}$ are not serially correlated over time.

ASSUMPTION D2: The idiosyncratic demand shock $\{\xi_{irdt}^{(3)}\}$ is private information of the corresponding airline. Furthermore, the demand shocks of two different airlines at two different routes are independently distributed.

Under assumption *D1*, the hub-size variable is not correlated with $\xi_{irdt}^{(3)}$ because HUB_{irt} is predetermined. Under assumption D2, HUB sizes of competing airlines in route r at period t are not correlated with $\xi_{irdt}^{(3)}$ and they are valid instruments for price p_{irdt}. Note that both assumptions D1 and D2 are testable. We can use the residuals of $\xi_{irdt}^{(3)}$ to test for no serial correlation (assumption D1) and no spatial correlation (assumption D2) in the residuals.

Table 7 presents estimates of the demand system.

Table 7
Demand Estimation(1)
Data: 85,497 observations. 2004-Q1 to 2004-Q4

	OLS		IV	
FARE (in \$100) $\left(-\frac{1}{\sigma_1}\right)$	-0.329	(0.085)	-1.366	(0.110)
ln(s*) $\left(1-\frac{\sigma_2}{\sigma_1}\right)$	0.488	(0.093)	0.634	(0.115)
NON-STOP DUMMY	1.217	(0.058)	2.080	(0.084)
HUBSIZE-ORIGIN (in million people)	0.032	(0.005)	0.027	(0.006)
HUBSIZE-DESTINATION (in million people)	0.041	(0.005)	0.036	(0.006)
DISTANCE	0.098	(0.011)	0.228	(0.017)
σ_1 **(in \$100)**	3.039	(0.785)	0.732	(0.059)
σ_2 **(in \$100)**	1.557	(0.460)	0.268	(0.034)
Test of Residuals Serial Correlation				
m1$\sim N(0,1)$ **(p-value)**	0.303	(0.762)	0.510	(0.610)

(1) All the estimations include airline dummies, origin-airport dummies $\times$ time dummies, and destination-airport dummies $\times$ time dummies. Stadard errors in parentheses.

The most important result is that the effect of hub-size on demand is statistically significant but very small: on average consumers are willing to pay approx. \$2 for an additional connection of the airline at the origin or destination airports ($\$2 \simeq \$100 * (0.027/1.366)$).

9.4.5 Specification and Estimation of Marginal Cost

Static Bertrand competition between airlines active in a route imply:

$$p_{irdt} - c_{irdt} = \frac{\sigma_1}{1-\bar{s}_{irt}}$$

where $\bar{s}_{irt} = (e_{ir0t} + e_{ir1t})^{\sigma_2/\sigma_1}[1 + \sum_{j=1}^{N}(e_{jr0t} + e_{jr1t})^{\sigma_2/\sigma_1}]^{-1}$, $e_{irdt} \equiv \exp\{(b_{irdt} - p_{irdt})/\sigma_2\}$. Then, given the estimated demand parameters we can obtain estimates of the marginal costs c_{irdt}.

We are interested in estimating the effect of "hub-size" on marginal costs. We estimated the following model for marginal costs:

$$c_{irdt} = W_{irdt}\,\delta + \omega_{irdt}$$

where the regressors in vector W_{irdt} are: dummy for nonstop-flight, hub-size, distance, airline dummies, origin-city dummies $\times$ time dummies, and destination-city dummies $\times$ time dummies.

Again, under the assumption that the error term ω_{irdt} is not serially correlated, hub-size is an exogenous regressor and we can estimate the equation for marginal costs using OLS.

Table 8
Marginal Cost Estimation(1)
Data: 85,497 observations. 2004-Q1 to 2004-Q4
Dep. Variable: Marginal Cost in $100

	Estimate (Std. Error)
NON-STOP DUMMY	0.006 (0.010)
HUBSIZE-ORIGIN (in million people)	-0.023 (0.009)
HUBSIZE-DESTINATION (in million people)	-0.016 (0.009)
DISTANCE	5.355 (0.015)
Test of Residuals Serial Correlation	
m1$\sim N(0,1)$ **(p-value)**	0.761 (0.446)

(1) All the estimations include airline dummies, origin-airport dummies $\times$ time dummies, and destination-airport dummies $\times$ time dummies.

Again, the most important result from this estimation is that the effect of hub-size on marginal cost is statistically significant but very small: on average an additional connection of the airline at the origin or destination airports implies a reduction in marginal cost between $1.6 and $2.3.

9.4.6 Simplifying assumptions for solution and estimation of dynamic game of network competition

The next step is the estimation of the effects if hub-size on fixed operating costs and sunk entry-costs. We consider the following structure in these costs.

$$FC_{imt} = \gamma_1^{FC} + \gamma_2^{FC} HUB_{imt} + \gamma_3^{FC} DIST_m + \gamma_{4i}^{FC} + \gamma_{5c}^{FC} + \varepsilon_{imt}^{FC}$$

$$EC_{imt} = \eta_1^{EC} + \eta_2^{EC} HUB_{imt} + \eta_3^{EC} DIST_m + \eta_{4i}^{EC} + \eta_{5c}^{EC}$$

where γ_{4i}^{FC} and η_{4i}^{EC} are airline fixed effects, and γ_{5c}^{FC} and η_{5c}^{EC} are city (origin and destination) fixed effects. ε_{imt}^{FC} is a private information shock. The parameters in these functions are estimated using data on airlines entry-exit decisions and the dynamic game.

However, this dynamic game has really a large dimension. Given the number of cities and airlines in our empirical analysis, the number of possible industry networks is $|X| = 2^{NM} \simeq 10^{10,000}$ (much larger than all the estimates of the number of atoms in the observable universe, around 10^{100}). We should make simplifying assumptions.

We consider **two types of simplifying assumptions** that reduce the dimension of the dynamic game and make its solution and estimation manageable.

1. An **airline's choice of network is decentralized** in terms of the separate decisions of local managers.

2. The state variables of the model can be aggregated in a vector of **inclusive-values** that belongs to a space with a much smaller dimension than the original state space.

(1) **Decentralizing the Airline's Choice of Network.** Each airline has M local managers, one for each city-pair. A local manager decides whether to operate or not non-stop flights in her local-market: that is, she chooses a_{imt}. The private information shock ε_{imt}^{FC} is private information of the manager (i,m).

IMPORTANT: A local manager is not only concerned about profits in her own route. She internalizes the effects of her own entry-exit decision in many other routes. This is very important to allow for entry deterrence effects of hub-and-spoke networks.

ASSUMPTION: Let R_{imt} be the sum of airline i's variable profits over all the routes that include city-pair m as a segment. *Local managers maximize the expected and discounted value of*

$$\Pi_{imt} \equiv R_{imt} - a_{imt}\left(FC_{imt} + (1 - x_{imt})EC_{imt}\right).$$

(2) Inclusive-Values. Decentralization of the decision simplifies the computation of players' best responses, but the state space of the decision problem of a local manager is still huge. Notice that the profit of a local manager depends only on the state variables:

$$\mathbf{x}_{imt}^{*} \equiv (x_{imt},\, R_{imt},\, HUB_{imt})$$

ASSUMPTION: The *vector* $\mathbf{x}_{imt}^{*}$ *follows a controlled first-order Markov Process:*

$$\Pr\left(\mathbf{x}_{im,t+1}^{*} \mid \mathbf{x}_{imt}^{*}, a_{imt}, \mathbf{x}_t, \mathbf{z}_t\right) = \Pr\left(\mathbf{x}_{im,t+1}^{*} \mid \mathbf{x}_{imt}^{*}, a_{imt}\right)$$

A MPE of this game can be describe as a vector of probability functions, one for each local-manager:

$$P_{im}(\mathbf{x}_{imt}^{*}) : i = 1,2,...,N;\ m = 1,2,...,M$$

$P_{im}(\mathbf{x}_{imt}^{*})$ is the probability that local-manager (i,m) decides to be active in city-pair m given the state $\mathbf{x}_{imt}^{*}$. An equilibrium exists. The model typically has multiple equilibria.

9.4.7 Estimation of dynamic game of network competition

We use the Nested Pseudo Likelihood (NPL) method.

Table 9
Estimation of Dynamic Game of Entry-Exit(1)
Data: 1,485 markets × 22 airlines × 3 quarters = 98,010 observations

	Estimate (Std. Error) (in thousand $)
Fixed Costs (quarterly):(2)	
$\gamma_1^{FC} + \gamma_2^{FC}$**mean hub-size** $+\gamma_3^{FC}$**mean distance (average fixed cost)**	119.15 (5.233)
γ_2^{FC} **(hub-size in # cities connected)**	-1.02 (0.185)
γ_3^{FC} **(distance, in thousand miles)**	4.04 (0.317)
Entry Costs:	
$\eta_1^{EC} + \eta_2^{EC}$**mean hub-size** $+\eta_2^{EC}$**mean distance (average entry cost)**	249.56 (6.504)
η_2^{EC} **(hub-size in # cities connected)**	-9.26 (0.140)
η_3^{EC} **(distance, in thousand miles)**	0.08 (0.068)
σ_ε	8.402 (1.385)
β	0.99 (not estimated)
Pseudo R-square	0.231

(1) All the estimations include airline dummies, and city dummies.
(2) Mean hub size = 25.7 million people. Mean distance (nonstop flights) = 1996 miles

• Goodness of fit:

Table 10
Comparison of Predicted and Actual Statistics of Market Structure
1,485 city-pairs (markets). Period 2004-Q1 to 2004-Q4

		Actual (Average All Quarters)	Predicted (Average All Quarters)
	Herfindahl Index (median)	5338	4955
Distribution of Markets by Number of Incumbents	Markets with 0 airlines	35.4%	29.3%
	" " 1 airline	29.1%	32.2%
	" " 2 airlines	17.4%	24.2%
	" " 3 airlines	9.8%	8.0%
	" " $\geq$4 airlines	8.2%	6.2%
Number (%) of Monopoly Markets for top 6 Airlines	Southwest	151 (43.4%)	149 (38.8%)
	Northwest	66 (18.9%)	81 (21.1%)
	Delta	57 (16.4%)	75 (19.5%)
	American	31 (8.9%)	28 (7.3%)
	Continental	27 (7.7%)	27 (7.0%)
	United	16 (4.6%)	24 (6.2%)
Distribution of Markets by Number of New Entrants	Markets with 0 Entrants	84.7%	81.9%
	" " 1 Entrant	13.4%	16.3%
	" " 2 Entrants	1.7%	1.6%
	" " $\geq$3 Entrants	0.3%	0.0%
Distribution of Markets by Number of Exits	Markets with 0 Exits	86.5%	82.9%
	" " 1 Exit	11.8%	14.6%
	" " 2 Exits	1.4%	1.4%
	" " $\geq$3 Exits	0.3%	0.0%

9.4.8 Counterfactual Experiments

To deal with multiple equilibria or equilibrium selection in the counterfactual experiment, we use the homotopy method that we saw in the previous chapter.

Table 11
Counterfactual Experiments
Hub-and-Spoke Ratios when Some Structural Parameters Become Zero

		Method 1: Taylor Approximation			
		Experiment 1	**Experiment 2**	**Experiment 3**	**Experiment 4**
		No hub-size effects	No hub-size effects	No hub-size effects	No complementarity
Carrier	Observed	in variable profits	in fixed costs	in entry costs	across markets
Southwest	18.2	17.3	15.6	8.9	16.0
American	42.0	39.1	36.5	17.6	29.8
United	45.7	42.5	39.3	17.8	32.0
Delta	48.0	43.7	34.0	18.7	25.0
Continental	68.3	62.1	58.0	27.3	43.0
Northwest	49.2	44.3	36.9	18.7	26.6
US Airways	45.3	41.7	39.0	18.1	34.4

		Method II: Policy Iterations Starting from Taylor Approx.			
		Experiment 1	**Experiment 2**	**Experiment 3**	**Experiment 4**
		No hub-size effects	No hub-size effects	No hub-size effects	No complementarity
Carrier	Observed	in variable profits	in fixed costs	in entry costs	across markets
Southwest	18.2	16.9	14.4	8.3	16.5
American	42.0	37.6	34.2	16.6	24.5
United	45.7	40.5	37.3	15.7	30.3
Delta	48.0	41.1	32.4	17.9	22.1
Continental	68.3	60.2	57.4	26.0	42.8
Northwest	49.2	40.8	35.0	17.2	23.2
US Airways	45.3	39.7	37.1	16.4	35.2

Experiment 1: Counterfactual model: $\alpha_2 = \alpha_3 = \delta_2 = \delta_3 = 0$
Experiment 2: Counterfactual model: $\gamma_2^{FC} = 0$
Experiment 3: Counterfactual model: $\eta_2^{EC} = 0$
Experiment 4: Counterfactual model: Variable profit of local manager in city-pair AB includes only variable profits from non-stop routes AB and BA.

Main results:

-Hub-size effects on **demand, variable costs and fixed operating costs** are significant but can **explain very little of the propensity to adopt hub-spoke networks**.

- **Hub-size effects on Sunk Entry Costs are large**. This is the most important factor to explain hub-spoke networks.

- **Strategic factors: hub-spoke network as a strategy to deter entry** is the second most important factor for some of the largest carriers (Northwest and Delta).

9.5 Dynamic strategic behavior in firms' innovation

9.5.1 Competition and Innovation: static analysis

Competition and Innovation. Long lasting debate on the effect of competition on innovation (for instance, Schumpeter, Arrow). Apparently, there are contradictory results between a good number of theory papers showing that "competition" has a negative effect on innovation (**dasgupta_stiglitz_1980** ,**dasgupta_stiglitz_1980**; **spence_1984** ,**spence_1984**), and a good number of reduced-form empirical papers showing a positive relationship between measures of competition and measures of innovation (**porter_1990** ,**porter_1990**; **geroski_1990** ,**gerosk_1990**; **blundell_griffith_1999** ,**blundell_griffith_1999**). Vives (2008) presents a systematic theoretical analysis of this problem that tries to explain the apparent disparities between existing theoretical and empirical results.

Competition and Innovation: Vives (2008) considers:

[1] Different sources of exogenous increase in competition. (i) reduction in entry cost; (ii) increase in market size; (iii) increase in degree of product substitutability.

[2] Different types of innovation. (i) process or cost-reduction innovation; (ii) product innovation / new products.

[3] Different models of competition and specifications. (i) Bertrand; (ii) Cournot

[4] Specification of demand. linear, CES, exponential, logit, nested logit.

Vives shows that [1] the form of increase in competition; and [2] the type of innovation are key to determine a positive or a negative relationship between competition and innovation. However, the results are very robust to: [3] the form of competition (Bertrand or Cournot) and [4] the specification of the demand system.

Model. Static model with symmetric firms, endogenous entry. Profit of firm i: $\pi_j = \left[p_j - c(z_j)\right]\ s\ d(p_j, p_{-j}, n; \alpha) - z_j - F$, $s =$ market size; $n =$ number of firm; $d(p_j, p_{-j}, n; \alpha) =$ demand per-consumer; $\alpha =$ degree of substitutability; $c(z_j) =$ marginal cost (constant); $z_i =$ expenditure in cost reduction; $c' < 0$ and $c'' > 0$; $F =$ entry cost.

Equilibrium. Nash equilibrium for simultaneous choice of (p_j, z_j). Symmetric equilibrium. There is endogenous entry. Marginal condition w.r.t cost-reduction R&D (z) is: $-c'(z)\ s\ d(p,n;\alpha) - 1 = 0$. Since $c'' > 0$, this implies $z = g(s\ d(p,n;\alpha))$, where g is an increasing function. The incentive to invest in cost reduction increases with output per firm, $q \equiv s\ d(p,n;\alpha)$.

Any exogenous change in competition (say in α, s, or F) has three effects on output per firm and therefore on investment in cost-reduction R&D.

$$\frac{dz}{d\alpha} = g'(q)\left[\frac{\partial\left[s\ d(p,n;\alpha)\right]}{\partial\alpha} + \frac{\partial\left[s\ d(p,n;\alpha)\right]}{\partial p}\frac{\partial p}{\partial\alpha} + \frac{\partial\left[s\ d(p,n;\alpha)\right]}{\partial n}\frac{\partial n}{\partial\alpha}\right]$$

$\frac{\partial [s\, d(p,n;\alpha)]}{\partial \alpha}$ is the **direct demand effect,** $\frac{\partial [s\, d(p,n;\alpha)]}{\partial p}\frac{\partial p}{\partial \alpha}$ is the **price pressure effect,** $\frac{\partial [s\, d(p,n;\alpha)]}{\partial n}\frac{\partial n}{\partial \alpha}$ is the **number of entrants effect.** The effects of different changes in competition on cost-reduction R&D can be explained in terms of these three effects.

Summary of comparative statics. (i) Increase in market size. - Increases per-firm expenditures in cost-reduction; - Effect on product innovation (# varieties) can be either positive or negative. **(ii) Reduction in cost of market entry.** - Reduces per-firm expenditures in cost-reduction; - Increases number of firms and varieties. **(iii) Increase in degree of product substitution.** - Increases per-firm expenditures in cost-reduction; - # varieties may increase or decline.

Some limitations in this analysis. The previous analysis is static, without uncertainty, with symmetric and single product firms. Therefore, the following factors that relate competition and innovation are absent from the analysis. (1) **Preemptive motives.** (2) **Cannibalization of own products**. (3) **Increasing uncertainty** in returns to R&D due competition (asymmetric info)**.** To study these factors, we need dynamic games with uncertainty, and asymmetric multi-product firms.

9.5.2 Creative destruction: incentives to innovate of incumbents and new entrants

Innovation and creative destruction (Igami ,2017). Innovation, the creation of new products and technologies, necessarily implies the "destruction" of existing products, technologies, and firms. In other words, the survival of existing products / technologies / firms is at the cost of preempting the birth of new ones. The speed (and the effectiveness) of the innovation process in an industry depends crucially on the dynamic strategic interactions between "old" and "new" products/technologies. Igami (2017) studies these interactions in the context of the Hard-Disk-Drive (HDD) industry during 1981-1998.

HDD: Different generations of products

HDD: Different generations of products

Adoption new tech: Incumbents vs. New Entrants

Adoption new tech: Incumbents vs. New Entrants. Igami focuses on the transition from 5.25 to 3.5 inch products. He consider three main factors that contribute to the relative propensity to innovate of incumbents and potential entrants. C**annibalization.** For incumbents, the introduction of a new product reduces the demand for their pre-existing products. **Preemption.** Early adoption by incumbents can deter entry and competition from potential new entrants. **Differences in entry/innovation costs.** It can play either way. Incumbents have knowledge capital and **economies of scope**, but they also have **organizational inertia**.

Data. **Market shares: New/Old products**

Average Prices: New/Old products

Average Quality: New/Old products

Market Structure: New/Old products

Model. Market structure at period t is described by four type of firms according to the products they produce: $s_t = \{N_t^{old}, N_t^{both}, N_t^{new}, N_t^{pe}\}$

• Initially, $N_0^{both} = N_0^{new} = 0$. Timing within a period t:

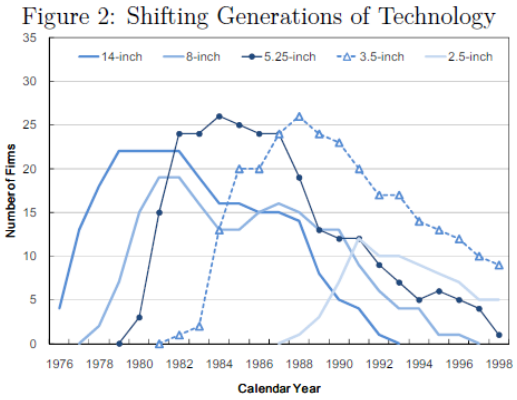


Figure 9.3: Hard drives: Different generations of products

1 Incumbents compete (à la Cournot) $\rightarrow$ Period profits $\pi_t(s_{it}, s_{-it})$

2. The N_t^{old} firms draw private info shocks and simultaneously choose $a_{it}^{old} \in \{exit, stay, innovate\}$

3. The N_t^{both} observe a_t^{old}, draw private info shocks, and simultaneously choose $a_{it}^{both} \in \{exit, stay\}$

4. The N_t^{new} observe a_t^{old}, a_t^{both},draw private info shocks, and simultaneously choose $a_{it}^{new} \in \{exit, stay\}$

5. The N_t^{pe} observe a_t^{old}, a_t^{both}, a_t^{new}, draw private info shocks, and simultaneously choose $a_{it}^{pe} \in \{entry, noentry\}$.

Given these choices, next period market structure is obtained, s_{t+1}, and demand and cost variables evolve exogenously. Why imposing this order of move? This Assumption, together with: - Finite horizon T; Homogeneous firms (up to the i.i.d. private info

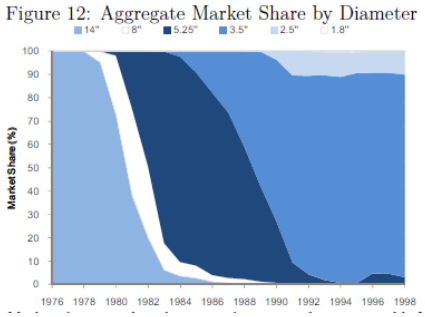


Figure 9.4: Hard drives: Different generations of products

shocks) within each type, implies that there is a **unique Markov Perfect equilibrium**. This is very convenient for estimation (Igami uses a standard/Rust Nested Fixed Point Algorithm for estimation) and especially for counterfactuals.

Demand. Simple logit model of demand. A product is defined as a pair {technology, quality}, where technology $\in \{old, new\}$ and quality represents different storage sizes. There is no differentiation across firms (perhaps true, but assumption comes from data limitations).

Estimation:

$$\ln\left(\frac{s_j}{s_k}\right) = \alpha_1\left[p_j - p_k\right] + \alpha_2\left[1_j^{new} - 1_k^{new}\right] + \alpha_3\left[x_j - x_k\right] + \xi_j - \xi_k$$

Data: multiple periods and regions. IVs: Hausman-Nevo. Prices in other regions.

Estimates of Demand

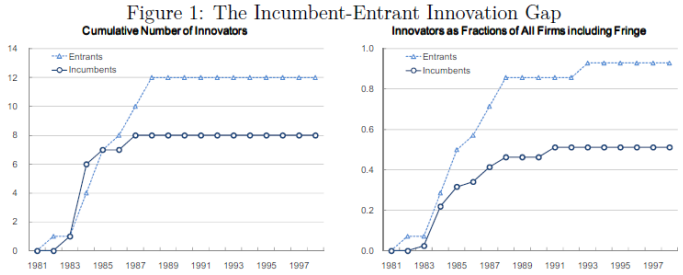


Figure 9.5: Adoption Propensity: Incumbents vs. New Entrants

Evolution of unobserved Quality (epsi)

Evolution of Marginal Costs

Evolution of Period Profits [keeping market structure]

Estimates of Dynamic Parameters

Estimates of Dynamic Parameters

Different estimates depending on the order of move within a period. Cost for innovation is smaller for incumbents than for new entrants ($\kappa^{inc} < \kappa^{pe}$). Organizational inertia does not seem an important factor. Magnitude of entry costs are comparable to the annual R&D budget of specialized HDD manufacturers, for instance, Seagate Tech: between $\$0.6B - \$1.6B$.

Estimated Model: Goodness of fit

Counterfactual: Removing Cannibalization

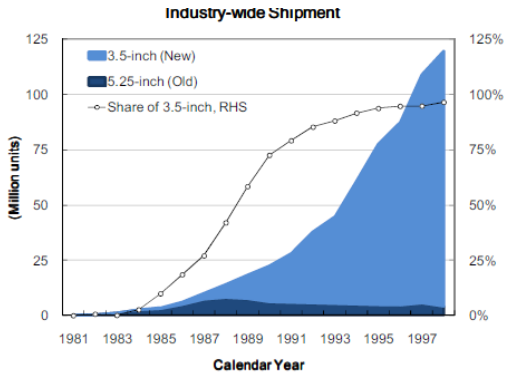


Figure 9.6: Market shares New/Old products

Counterfactual: Removing Preemption

9.5.3 Competition and innovation in the CPU industry: Intel and AMD

Studies competition between Intel and AMD in the PC microprocessor industry. Incorporates durability of the product as a potentially important factor. Two forces drive innovation: - competition between firms for the technological frontier; - since PCs have little physical depreciation, firms have the incentive to innovate to generate a technological depreciation of consumers' installed PCs that encourages them to upgrade [most of the demand during the period $> 89\%$ was upgrading]. Duopolists face both forces, whereas a monopolist faces only the latter (but in a stronger way).

The PC microprocessor industry. Very important to the economy: - Computer equipment manufacturing industry generated 25% of U.S. productivity growth from

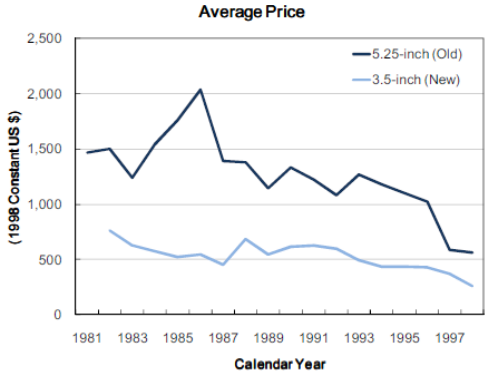


Figure 9.7: Average Prices: New/Old products

1960 to 2007. - Innovations in microprocessors are directly measured via improved performance on benchmark tasks. Most important: CPU speed. Interesting also from the point of view of antitrust: - In 2004: several antitrust lawsuits claiming Intel's anticompetitive practices, for instance, rewarding PC manufacturers that exclusively use Intel microprocessors. - Intel foreclosures AMD to access some consumers.- Intel settled these claims in 2009 with a $1.25 billion payment to AMD.

Market is essentially a duopoly, with AMD and Intel selling 95% CPUs. Firms have high R&D intensities, R&D/Revenue (1993-2004): - AMD 20% ; and Intel 11%. Innovation is rapid: new products are released nearly every quarter. CPU performance (speed) doubles every 7 quarters, that is, Moore's law. AMD and Intel extensively cross-license each other's technologies, that is, positive spillovers.

As microprocessors are durable, replacement drives are important part of demand.

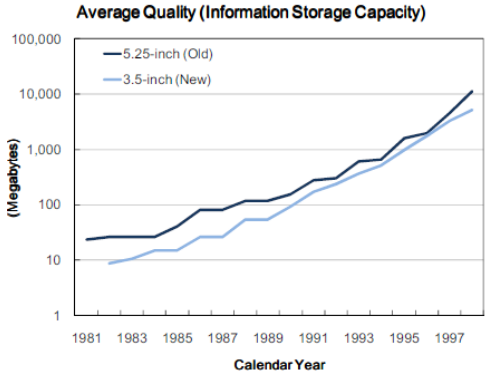


Figure 9.8: Average Quality: New/Old products

The importance of replacement is partly exogenous (new consumers arriving to the marker), and partly endogenous: speed of improvements in frontier microprocessors that encourages consumers to upgrade. In 2004, 82% of PC purchases were replacements. After an upgrade boom, prices and sales fall as replacement demand drops. Firms must continue to innovate to rebuild replacement demand.

Data. Proprietary data from a market research firm specializing in the microprocessor industry. Quarterly data from Q1-1993 to Q4-2004 (48 quarters). Information on: shipments in physical units for each type of CPU; manufacturers' average selling prices (ASP); **production costs**; CPU characteristics (speed). All prices and costs are converted to base year 2000 dollars. Quarterly R&D investment levels, obtained from firms' annual reports.

Moore's Law. Intel cofounder Gordon Moore predicted in 1965 that the number

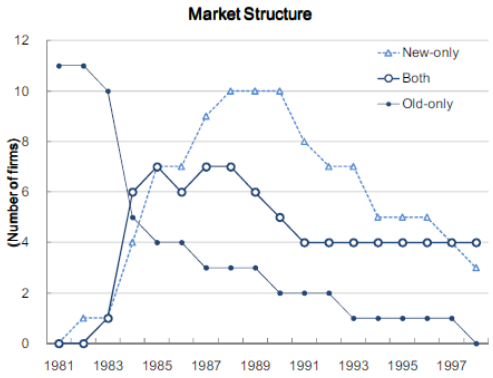


Figure 9.9: Market Structure: New/Old products

of transistors in a CPU (and therefore the CPU speed) would double every 2 years. Following figure shows "Moore's law" over the 48 quarters in the data. Quality is measured using processor speed. Quarterly % change in CPU speed is 10.2% for Intel and 11% for AMD.

Moore's Law (Frontier CPU speed)

Differential log-quality between Intel and AMD. Intel's initial quality advantage is moderate in 1993–94. Then, it becomes large in 1995-96 when Intel releases the Pentium. AMD's responded in 1997 introducing the K6 processor that narrows the gap. But parity is not achieved until the mid-2000 when AMD released the Athlon.

Model: General features. Dynamic model of an oligopoly with differentiated and durable products. Each firm j sells a single product and invests in R&D to improve its quality. If investments are successful, quality improves next quarter by a fixed proportion

Market definition:	Broad		Narrow	
Estimation method:	OLS	IV	OLS	IV
	(1)	(2)	(3)	(4)
Price ($000)	-1.66^{***}	-2.99^{***}	$-.93^{**}$	-3.28^{***}
	(.45)	(.55)	(.46)	(.63)
Diameter = 3.5-inch	$.84^{*}$	.75	1.75^{***}	$.91^{**}$
	(.46)	(.45)	(.31)	(.38)
Log Capacity (MB)	.18	$.87^{***}$	.04	1.20^{***}
	(.33)	(.27)	(.26)	(.31)
Year dummies	*Yes*	*Yes*	*Yes*	*Yes*
Region/user dummies	–	–	*Yes*	*Yes*
Adjusted R^2	.43	.33	.50	.28
Number of obs.	176	176	405	405
Partial R^2 for Price	–	.32	–	.16
P-value	–	.00	–	.00

Figure 9.10: Estimates of Demand

δ; otherwise it is unchanged: log quality $q_{jt} \in \{0, \delta, 2\delta, 3\delta, \dots\}$. Consumers: a key feature of demand for durable goods is that the value of the no-purchase option is endogenous, determined by last purchase. The distribution of currently owned products by consumers is represented by the vector Δ_t. Δ_t affects current consumer demand. [Details]

Firms and consumers are forward looking. A consumer's i state space consists of $(q^*_{it}, q_t, \Delta_t)$: - q^*_{it} = the quality of her currently owned product q^*_t; - q_t = vector of firms' current qualities q_t; - Δ_t = distribution of qualities of consumers currently owned products. Δ_t is part of the consumers' state space because it affects expectations on future prices. State space for firms is (q_t, Δ_t). Given these state variables firms simultaneously choose prices p_{jt} and investment x_{jt}.

Consumer Demand. Authors: *"We restrict firms to selling only one product because*

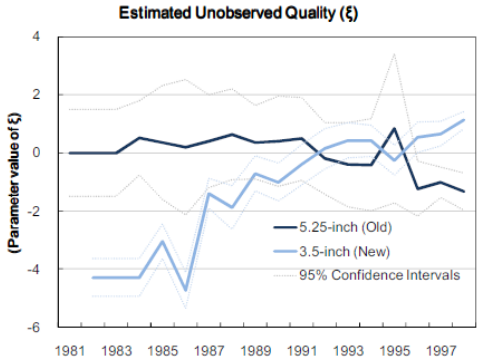


Figure 9.11: Evolution of unobserved Quality (epsi)

the computational burden of allowing multiproduct firms is prohibitive". Consumers own no more than one microprocessor at a time. Utility for a consumer i from firm j's new product with quality q_{jt} is given by: $u_{ijt} = \gamma\, q_{jt} - \alpha\ p_{jt} + \xi_j + \varepsilon_{ijt}$. Utility from the no-purchase option is: $u_{i0t} = \gamma\, q^*_{it} + \varepsilon_{i0t}$. A consumer maximizes her intertemporal utility given her beliefs about the evolution of future qualities and prices given (q_t, Δ_t).

Market shares for consumers currently owning q^* are:

$$s_{jt}(q^*) = \frac{\exp\{v_j(q_t, \Delta_t, q^*)\}}{\sum_{k=0}^{J} \exp\{v_k(q_t, \Delta_t, q^*)\}}$$

Using Δ_t to integrate over the distribution of q^* yields the market share of product j.

$$s_{jt}(q^*) = \sum_{q^*} s_{jt}(q^*)\ \Delta_t(q^*)$$

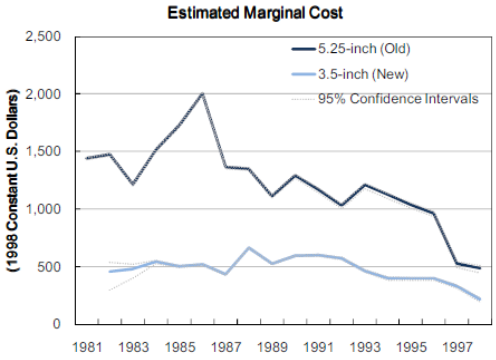


Figure 9.12: Evolution of Marginal Costs

Transition rule of Δ_t. By definition, next period Δ_{t+1} is determined by a known closed-form function of Δ_t, q_t, and s_t.

$$\Delta_{t+1} = F_\Delta(\Delta_t, q_t, s_t)$$

The period profit function is:

$$\pi_j(p_t, q_t, \Delta_t) = M \; s_j(p_t, q_t, \Delta_t) \; \left[p_{jt} - mc_j(q_{jt})\right]$$

The specification of the marginal cost is:

$$mc_j(q_{jt}) = \lambda_{0j} - \lambda_1(q_t^{\max} - q_{jt})$$

Marginal costs are smaller for non-frontier firms. Parameter λ_1 captures an spillover effect from the innovation of other firms.

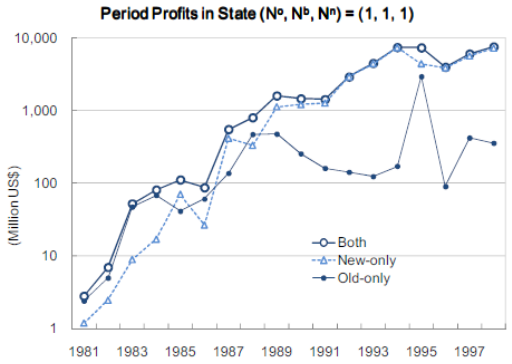


Figure 9.13: Evolution of Period Profits [keeping market structure]

Model: Firms. Innovation process. Relationship between investment in R&D (x_{jt}) and log-quality improvement ($\Delta q_{jt+1} = q_{jt+1} - q_{jt}$). Log-Quality improvement can take two values, 0 or δ. The probability that $\Delta q_{jt+1} = \delta$ is (Pakes and McGure, 1994):

$$\chi_j(x_{jt}, q_{jt}) = \frac{a_j(q_{jt})\, x_{jt}}{1 + a_j(q_{jt})\, x_{jt}}$$

$a_j(q_{jt})$ is the "investment efficiency" function. It is a decreasing function, to capture the idea of an increasing difficulty of advancing the frontier relative to catching up.

Let $W_j(q_t, \Delta_t)$ be the value function. The Bellman equation is:

$$W_j(q_t, \Delta_t) = \max_{x_{jt}, p_{jt}} \left[\pi_j(p_t, q_t, \Delta_t) - x_{jt} + \beta\, \mathbb{E}_t \left[W_j(q_{t+1}, \Delta_{t+1}) \right] \right]$$

The decision variables are continuous, and the best response function should satisfy the

Table 4: Estimates of the Dynamic Parameters

($ Billion)	Maximum Likelihood Estimates		
	(1)	(2)	(3)
Assumed order of moves:	Old-Both-New-PE	PE-New-Both-Old	PE-Old-Both-New
Fixed cost of operation (ϕ)	0.1474	0.1472	0.1451
	[−0.02, 0.33]	[−0.02, 0.33]	[−0.03, 0.33]
Incumbents' sunk cost (κ^{inc})	1.2439	1.2370	1.2483
	[0.51, 2.11]	[0.50, 2.10]	[0.51, 2.11]
Entrants' sunk cost (κ^{ent})	2.2538	2.2724	2.2911
	[1.74, 2.85]	[1.76, 2.87]	[1.78, 2.89]
Log likelihood	−112.80	−112.97	−113.46

Figure 9.14: Estimates of Dynamic Parameters

F.O.C.

$$\frac{\partial \pi_{jt}}{\partial p_{jt}} + \beta \frac{\partial \mathbb{E}_t \left[W_{j,t+1} \right]}{\partial p_{jt}} = 0$$

$$\frac{\partial \pi_{jt}}{\partial x_{jt}} - 1 + \beta \frac{\partial \mathbb{E}_t \left[W_{j,t+1} \right]}{\partial x_{jt}} = 0$$

Markov Perfect Equilibrium. (1) firms' and consumers' equilibrium strategies depend only on current payoff relevant state variables (q_t, Δ_t). (2) consumers have rational expectations about firms' policy functions. (3) each firm has rational expectations about competitors' policy functions and about the evolution of the ownership distribution.

Estimation. Marginal cost parameters (λ_0, λ_1) are estimated in a first step because the dataset includes data on marginal costs. The rest of the structural parameters, $\theta = (\gamma,$

Figure 5: Fit of Market Structure Dynamics

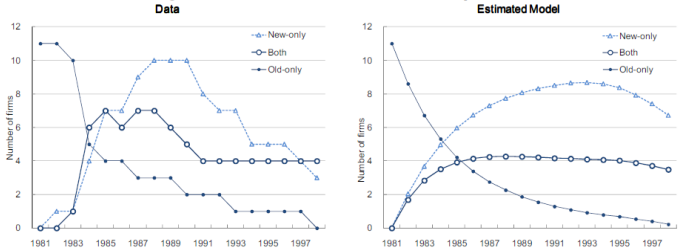


Figure 9.15: Estimated Model: Goodness of fit

α, ξ_{intel}, ξ_{amd}, $a_{0,intel}$, $a_{0,amd}$, a_1). Demand: γ, α, ξ_{intel}, ξ_{amd}; Investment innovation efficiency: $a_{0,intel}$, $a_{0,amd}$, a_1. θ is estimated using Indirect Inference or Simulated Method of Moments (SMM).

Moments to match: Mean of innovation rates $q_{j,t+1} - q_{jt}$ for each firm. Mean R&D intensities $x_{jt}/revenue_{jt}$ for each firm. Mean of differential quality $q_{intel,t} - q_{amd,t}$, and share of quarters with $q_{intel,t} \geq q_{amd,t}$. Mean of gap $q_t^{\max} - \overline{\Delta}_t$. Average prices, and OLS estimated coefficients of the regressions of p_{jt} on $q_{intel,t}$, $q_{amd,t}$, and average $\overline{\Delta}_t$. OLS estimated coefficients of the regression of $s_{intel,t}$ on $q_{intel,t} - q_{amd,t}$.

Empirical and predicted moments

Demand: Dividing γ by α: consumers are willing to pay \$21 for enjoying during 1 quarter a $\delta = 20\%$ increase in log quality. Dividing $\xi_{intel} - \xi_{amd}$ by α: consumers are willing to pay \$194 for Intel over AMD. The model needs this strong brand effect to

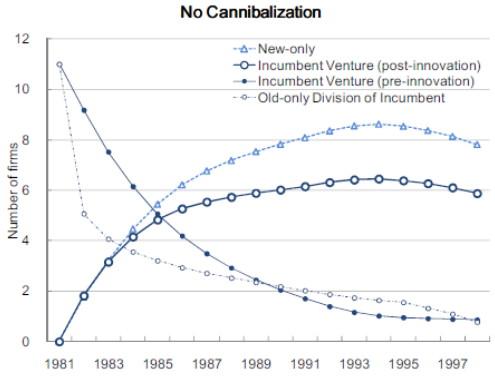


Figure 9.16: Counterfactual: Removing Cannibalization

explain the fact that AMD's share never rises above 22 percent in the period during which AMD had a faster product. Intel and AMD's innovation efficiencies are estimated to be .0010 and .0019, respectively, as needed for AMD to occasionally be the technology leader while investing much less.

Counterfactuals

From current duopoly (1) to Intel Monopoly (3) Innovation rate increases from 0.599 to 0.624. Mean quality upgrade increases 261% to 410%. Investment in R&D: increases by 1.2B per quarter: more than doubles. Price increases in \$102 (70%). Consumer surplus declines in \$121M (4.2%). Industry profits increase in \$159M. Social surplus increases in \$38M (less than 1%)

From current duopoly (1) to symmetric duopoly (2) Innovation rate declines from 0.599 to 0.501. Mean quality declines from 261% to 148%. Investment in R&D:

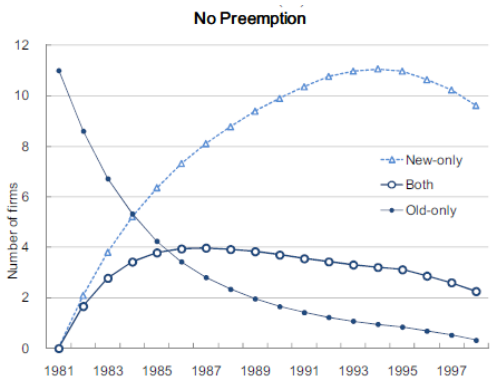


Figure 9.17: Counterfactual: Removing Preemption

declines by 178M per quarter. Price declines in \$48 (24%). Consumer surplus increases in \$34M (1.2%). Industry profits decline in \$8M. Social surplus increases in \$26M (less than 1%)

From current scenario (1) to myopic pricing. It reduces prices, increases CS, and reduces firms' profits. Innovation rates and investment in R&D decline dramatically. Why? Higher prices induce firms to innovate more rapidly. Prices are higher with dynamic pricing because firms want to preserve future demand.

The finding that innovation by a monopoly exceeds that of a duopoly reflects two features of the model: the monopoly must innovate to induce consumers to upgrade; the monopoly is able to extract much of the potential surplus from these upgrades because of its substantial pricing power. If there were a steady flow of new consumers into the market, such that most demand were not replacement, the monopoly would reduce

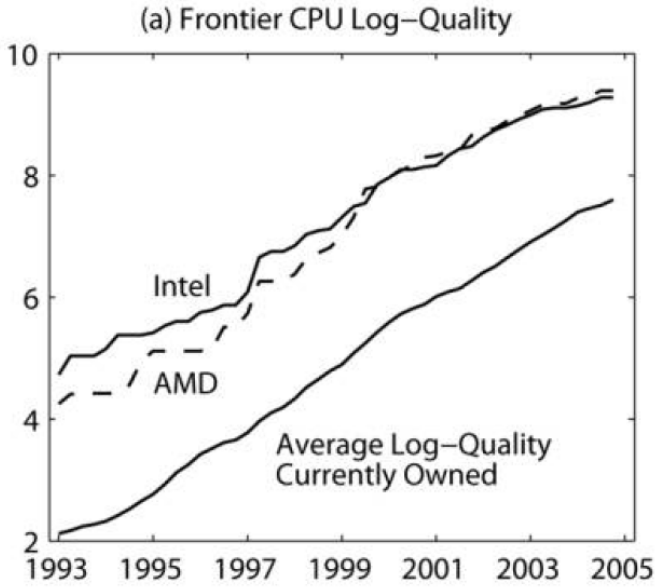


Figure 9.18: Moore's Law (Frontier CPU speed)

innovation below that of the duopoly.

Counterfactuals: Foreclosure. In 2009, Intel paid AMD $1.25 billion to settle claims that Intel's anticompetitive practices foreclosed AMD from many consumers. To study the effect of such practices on innovation, prices, and welfare, the authors perform a series of counterfactual simulations in which they vary the portion of the market to which Intel has exclusive access. Let ζ be the proportion of foreclosure market. Intel market share becomes: $s_j^* = \zeta\ \widehat{s}_j + (1-\zeta)\ s_j$, where s_j is the market share when AMD is competing, and $\widehat{s}_j$ is the market share when Intel competes only with the outside alternative.

Counterfactuals: Foreclosure

Margins monotonically rise steeply. Innovation exhibits an inverted U with a peak at $\zeta = 0.5$. Consumer surplus is actually higher when AMD is barred from a portion

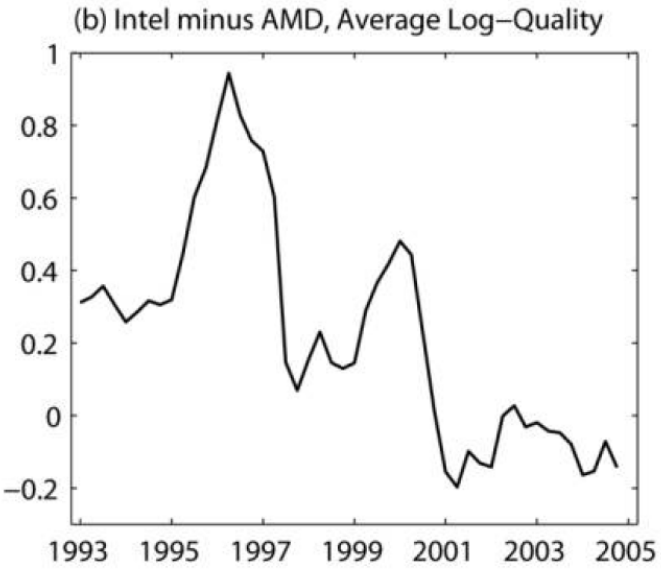


Figure 9.19: Differential log-quality between Intel and AMD

of the market, peaking at 40% foreclosure. This finding highlights the importance of accounting for innovation in antitrust policy: the decrease in consumer surplus from higher prices can be more than offset by the compounding effects of higher innovation rates.

Counterfactuals: Product substitutability

Innovation in the monopoly exhibits an inverted U as substitutability increases. Innovation in the duopoly increases as substitutability increases until Var( ) becomes too small for firms with similar qualities to coexist. - Beyond this "shakeout" threshold, the laggard eventually concedes the market as evidenced by the sharp increase in the quality difference. Duopoly innovation is higher than monopoly innovation when substitutability is near the shakeout threshold.

Summary of results. The rate of innovation in product quality would be 4.2%

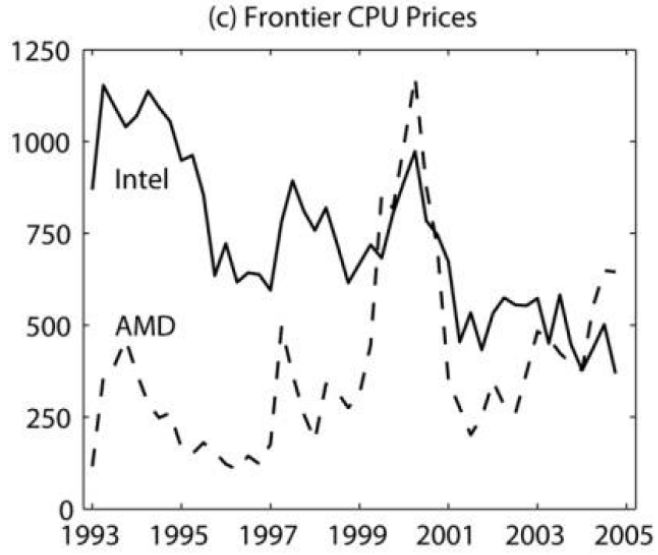


Figure 9.20: Frontier CPU Prices

higher if Intel were a monopolist, consistent with Schumpeter.

Without AMD, higher margins spur Intel to innovate faster to generate upgrade sales. As in **coase_1972**'s (**coase_1972**) conjecture, product durability can limit welfare losses from market power. This result, however, depends on the degree of competition from past sales. If first-time purchasers were to arrive sufficiently faster than we observe, innovation in an Intel monopoly would be lower, not higher, since upgrade sales would be less important.

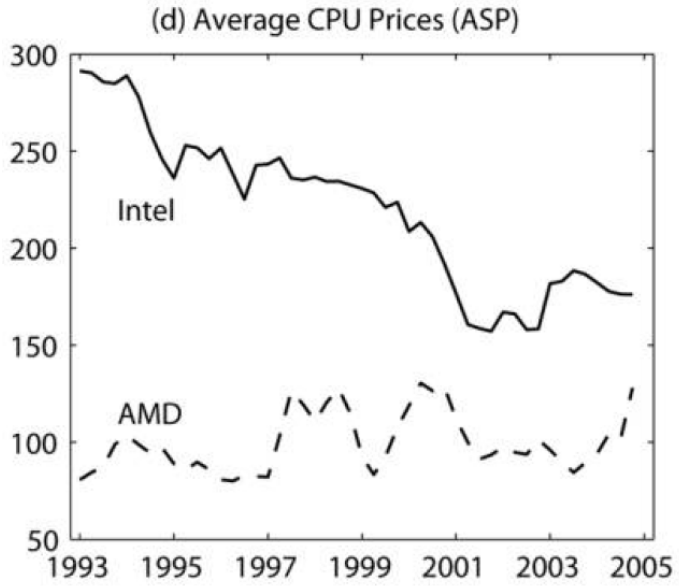


Figure 9.21: Average CPU Prices

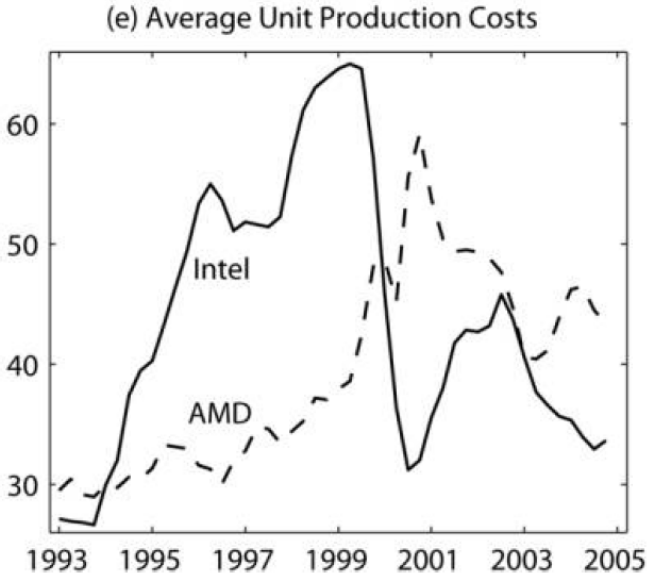


Figure 9.22: Average Unit Production Costs

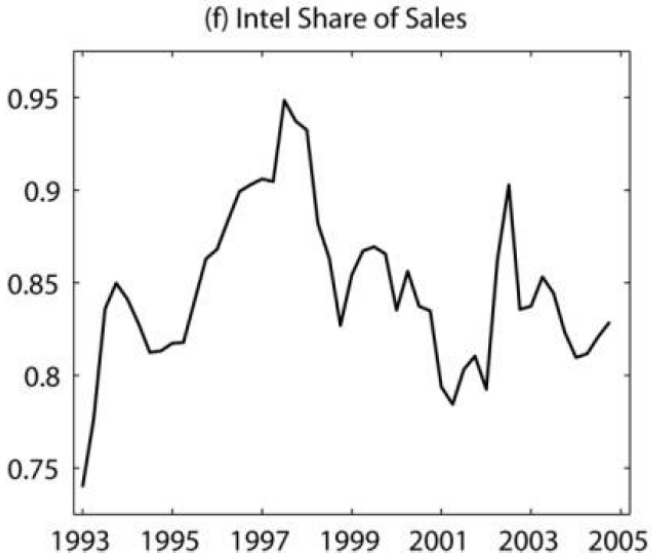


Figure 9.23: Intel Share of Sales

TABLE 1
EMPIRICAL AND SIMULATED MOMENTS

Moment	Actual	Actual Standard Error	Fitted
Intel price equation:			
Average Intel price	219.7	5.9	206.2
$q_{\text{Intel},t} - q_{\text{AMD},t}$	47.4	17.6	27.3
$q_{\text{Intel},t} - \bar{\Delta}_t$	94.4	31.6	43.0
AMD price equation:			
Average AMD price	100.4	2.3	122.9
$q_{\text{Intel},t} - q_{\text{AMD},t}$	−8.7	11.5	−22.3
$q_{\text{AMD},t} - \bar{\Delta}_t$	16.6	15.4	5.9
Intel share equation:			
Constant	.834	.007	.846
$q_{\text{Intel},t} - q_{\text{AMD},t}$	.055	.013	.092
Potential upgrade gains:			
Mean $(\bar{q}_t - \hat{\Delta}_t)$	1.146	.056	1.100
Mean innovation rates:			
Intel	.557	.047	.597
AMD	.610	.079	.602
Relative qualities:			
Mean $q_{\text{Intel},t} - q_{\text{AMD},t}$	1.257	.239	1.352
Mean $\mathcal{I}(q_{\text{Intel},t} \geq q_{\text{AMD},t})$	.833	.054	.929
Mean R&D/revenue:			
Intel	.114	.004	.101
AMD	.203	.009	.223

Figure 9.24: Empirical and predicted moments

TABLE 2
PARAMETER ESTIMATES

Parameter	Estimate	Standard Error
Price, α	.0131	.0017
Quality, γ	.2764	.0298
Intel fixed effect, ξ_{Intel}	−.6281	.0231
AMD fixed effect, ξ_{AMD}	−3.1700	.0790
Intel innovation, $a_{0,\text{Intel}}$	.0010	.0002
AMD innovation, $a_{0,\text{AMD}}$	.0019	.0002
Spillover, a_1	3.9373	.1453
Stage 1 marginal cost equation:		
Constant, λ_0	44.5133	1.1113
max (0, $q_{\text{competitor},t} - q_{\text{own},t}$), λ_1	−19.6669	4.1591

Figure 9.25: Parameter Estimates

TABLE 3
INDUSTRY OUTCOMES UNDER VARIOUS SCENARIOS

	AMD-INTEL DUOPOLY (1)	SYMMETRIC DUOPOLY (2)	MONOPOLY (3)	NO SPILLOVER DUOPOLY (4)	MYOPIC PRICING: AMD-Intel (5)	MYOPIC PRICING: Monopoly (6)
Industry profits ($ billions)	408	400	567	382	318	322
Consumer surplus (CS)	2,978	3,012	2,857	3,068	2,800	2,762
CS as share of monopoly CS	1.042	1.054	1.000	1.074	.980	.967
Social surplus (SS)	3,386	3,412	3,424	3,450	3,118	3,084
SS as share of planner SS	.929	.906	.940	.916	.828	.819
Margins, $(p - mc)/mc$	3.434	2.424	5.672	3.478	2.176	2.216
Price	194.17	146.73	296.98	157.63	140.06	143.16
Frontier innovation rate	.599	.501	.624	.438	.447	.438
Industry investment ($ millions)	830	652	1,672	486	456	787
Mean quality upgrade (%)	261	148	410	187	175	181
Intel or leader share	.164	.135	.143	.160	.203	.211
AMD or laggard share	.024	.125		.091	.016	

Figure 9.26: Counterfactuals

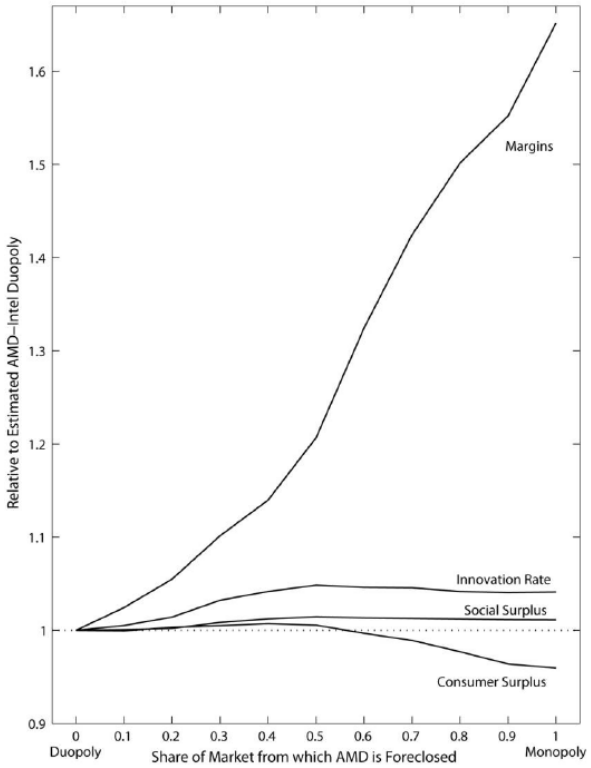


FIG. 6.—Foreclosing AMD from the market

Figure 9.27: Counterfactuals: Foreclosure

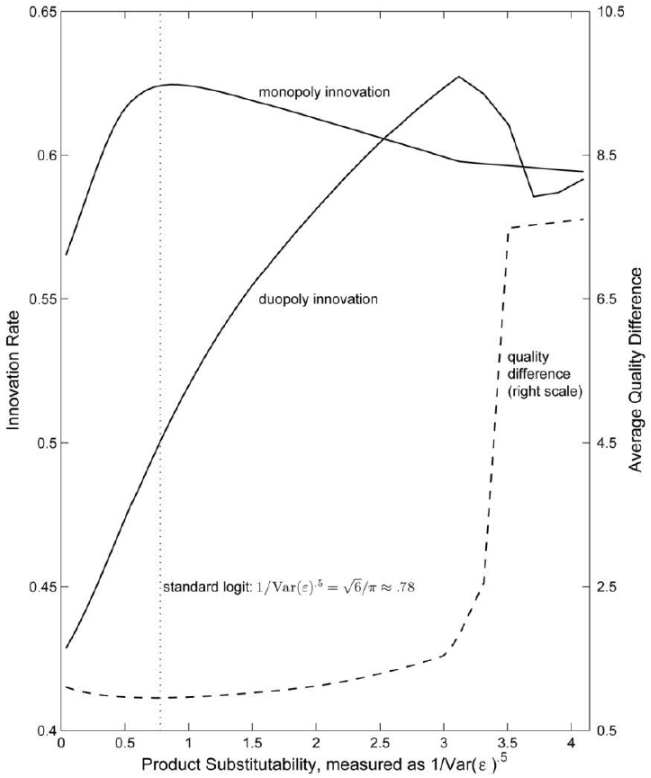


Figure 9.28: Counterfactuals: Product substitutability

10. Auctions

10.1 Introduction

Auctions are common mechanisms for selling goods and services such as agricultural products (for instance, fish, livestock, wine), natural resources (for instance, timber, oil and gas drilling rights), government contracts (procurement auctions), money in interbank markets, treasury bonds, electricity, or art work. More recently, internet auctions (for instance, eBay) have become a popular way of selling a diverse range of products.

Auctions can be modelled as games of incomplete information. A seller (or a buyer, in the case of a procurement auction) is interested in selling an object. The seller faces a number of potential buyers or bidders, and she does not know their valuations of the object. A bidder knows her own valuation of the object but not other bidders' values. Each bidder submits a bid to maximize her expected payoff. The rules of the auction determine who gets the object and the price she should pay. These rules (for instance, first price sealed bids, second price), and the conditions on bidders' information and on the correlation between their valuations (for instance, independent private values, common values) are important features that determine the predictions of the model.

Consider the auction of a single object with N bidders indexed by $i \in \{1,2,...,N\}$. Bidder i's valuation for the object is $u_i = U(v_i, c)$ where $U(.,.)$ is an increasing function in both arguments; v_i represents a bidder's private signal; and c is a common value that affects the valuations of all the bidders. In an auction, the value of the vector $(v_1, v_2, ..., v_n, c)$ is a random draw from the joint cumulative distribution function $\mathbf{F}(v_1, v_2, ..., v_n, c)$ that is continuously differentiable and has compact support $[\underline{v}, \overline{v}]^2 \times [\underline{c}, \overline{c}]$. Each bidder knows her own private value v_i and the functions U and $\mathbf{F}$, but she does not know the other bidders' private values. Depending on the model, she may or may not know the common component c. The game is said to be symmetric if bidders are identical ex ante, that is, if the distribution $\mathbf{F}$ is exchangeable in its first N arguments.

Each bidder decides her bid, b_i, to maximize her expect payoff. Most of the empirical literature has focused on first-price auctions: the winner is the highest bidder (provided

it is higher than the seller's reservation price) and she pays her bid. Under this rule, the expected payoff is:

$$\pi_i^e(b_i) = \mathbb{E}\left(1\left\{b_i > b_j \; \forall j \neq i\right\} \; [U(v_i, c) - b_i]\right) \tag{10.1}$$

where $1\{.\}$ is the indicator function. This literature assumes that bids come from a Bayesian Nash equilibrium (BNE). This BNE is described as a vector of N strategy functions $\{s_i(v_i) : i = 1, 2, ..., N\}$ such that each bidder's strategy maximizes her expected payoff taking as given the strategy functions of the other bidders:

$$s_i(v_i) = \arg\max_{b_i} \; \mathbb{E}\left(1\left\{b_i > s_j(v_j) \; \forall j \neq i\right\} \; [U(v_i, c) - b_i]\right) \tag{10.2}$$

where the expectation is taken over the joint distribution of $\{v_j : j \neq i\}$ (and c, if this is not common knowledge). This BNE can be described as the solution of a system of differential equations.

Most empirical applications of structural auction models have focused on the *Independent Private Values* (IPV) model. This model assumes that valuations depend only on private information signals, $U(v_i, c) = v_i$, and they are independently and identically distributed, that is, $\mathbf{F}(v_1, v_2, ..., v_n) = \prod_{i=1}^{N} F(v_i)$. It also imposes the restriction that the data comes from a symmetric BNE: $s_i(v_i) = s(v_i)$ for every bidder i. A BNE of the IPV model can be described as a strategy function s that solves the differential equation:

$$s(v_i) = v_i - \frac{F(v_i) \; s'(v_i)}{(N-1) \; f(v_i)} \tag{10.3}$$

subject to the boundary condition boundary $s(\underline{v}) = \underline{v}$, and where f is the density function of the distribution F. This differential equation has a unique solution that has the following closed-form expression:

$$b_i = s(v_i) = v_i - \frac{1}{[F(v_i)]^{N-1}} \int_{\underline{v}}^{v_i} [F(u)]^{N-1} \, du \tag{10.4}$$

Auction data is widely available. In many countries, procurement auction data must be publicly available by law. Empirical researchers have used these data to answer different empirical questions such as detecting collusion among bidders, testing different auction models, or designing auction rules that maximize seller's revenue or total welfare.

The first empirical papers on auctions focused on testing important predictions of the model, without estimating the structural parameters (Hendricks and Porter ,1988; **hendricks_porter_1994** ,**hendricks_porter_1994**; **porter_1995** ,**porter_1995**). The papers by **paarsch_1992** (**paarsch_1992**, **paarsch_1997**) and **laffont_ossard_1995** (**laffont_ossard_1995**) present the first structural estimations of auction models.

In the structural estimation of auction models, the researcher has some information on bids and uses this information and the equilibrium conditions to estimate the distribution of bidders' valuations. Auction data may come in different forms, and this has important implications for the identification and estimation of the model. In an ideal situation, the researcher has a random sample of T independent auctions (indexed by t) of the same type of object from the same population of bidders, and she observes the bids

of each of the N_t bidders at every auction t in the sample. Such ideal situations are quite uncommon. For instance, often the researcher observes only the winning bid. It is also common that there is heterogeneity across the T auctions (for instance, different environments, or non identical objects) such that it is not plausible to assume that the same distribution of bidders' valuations, F, applies to the T auctions. In that case, it is useful to have observable auction characteristics, X_t, such that the researcher may assume that two auctions with the same observable characteristics have the same distributions of valuations: $F_t(v|X_t) = F(v|X_t)$ for every auction t. In general, an auction dataset can be described as:

$$Data = \left\{ b_t^{(n)}, X_t : n = 1, ..., \overline{N}_t; t = 1, 2, ..., T \right\} \tag{10.5}$$

where $b_t^{(1)}$ is the largest bid, $b_t^{(2)}$ is the second largest, and so on; and $\overline{N}_t$ is the number of bids the researcher observes in auction t. When the dataset includes only information on winning bids, we have that $\overline{N}_t = 1$ for any auction t.

Tree planting procurement auctions in British Columbia. **paarsch_1992** (**paarsch_1992**) studies first price sealed-bid auctions of tree planting contracts operated by the Forest Service (government agency) in the province of British Columbia, Canada. The object of an auction is described by the number and type of trees to plant and the location. The bidding variable is the price per tree, and the winner of the auction is the firm with the lowest price. The dataset consists of 144 auctions in the same forest region between 1985 and 1988 with information on all the bids. Paarsch estimates structurally independent private value models and common value models under different parametric specifications of the distribution of firms' costs. All the specifications of private value models are rejected. The estimated common value models are consistent with observed bidders' behavior. More specifically, there is evidence consistent with bidders' concern for the *winner's curse* and with bid functions that increase with the number of bidders.

The first empirical applications on structural auction models consider parametric specifications of the distribution of valuations (**paarsch_1992** ,**paarsch_1992**, **paarsch_1997**; **laffont_ossard_1995** ,**laffont_ossard_1995**; **baldwin_marshall_1997** ,**baldwin_marshall_1997**). However, the more recent literature has focused on the nonparametric identification and estimation of this distribution. Guerre, Perrigne, and Vuong (2000) obtained an important identification result and estimation method in this literature. They show that equation (10.4), that characterizes the equilibrium of the model, implies that a bidder's valuation is a known function of her bid and the distribution of observed bids. Let $G(b)$ and $g(b)$ be the distribution and the density function of bids, respectively, implied by the equilibrium of the model. Since the equilibrium bidding strategy, $s(v_i)$, is strictly increasing, we have that $v_i = s^{-1}(b_i)$ and $G(b_i) = F(s^{-1}(b_i))$, and this implies that $g(b_i) = f(v_i)/s'(v_i)$. Solving these expressions into the differential equation (10.3), we get:

$$v_i = \xi\left(b_i, G\right) = b_i + \frac{G(b_i)}{(N-1)\, g(b_i)} \tag{10.6}$$

Based on this equation, the distribution of valuations can be estimated from the data using a two-step procedure. Suppose for the moment that the data consists of a random sample of independent and identical auctions with information on all bids. Then, the distribution function G can be consistently estimated at any value $b \in [\underline{b}, \overline{b}]$ using the estimator

$\widehat{G}(b) = (NT)^{-1}\sum_{t=1}^{T}\sum_{n=1}^{N} 1\{b_t^{(n)} \leq b\}$, and the density function can be estimated using the kernel method, $\widehat{g}(b) = (NT\ h_g)^{-1}\ \sum_{t=1}^{T}\sum_{n=1}^{N}\ K[(b_t^{(n)} - b)/h_g]$, where h_g is the bandwidth parameter. In a second-step, we can use equation (10.6) to construct the estimated pseudo-values $\widehat{v}_t^{(n)} = \xi\left(b_t^{(n)}, \widehat{G}\right)$ and use them to obtain a kernel estimator of the density of values: for any $v \in [\underline{v}, \overline{v}]$, $\widehat{f}(v) = (NT\ h_f)^{-1}\ \sum_{t=1}^{T}\sum_{n=1}^{N}\ K[(\widehat{v}_t^{(n)} - v)/h_f]$. GPV show the consistency and asymptotic normality of this estimator and its speed of convergence. The also show that the estimator can be easily generalized to datasets where only the winning bid, $b_t^{(1)}$, is observed.

Athey and Haile (2002) provide a comprehensive study on the nonparametric identification of auction models. For IPV models, they show that the asymmetric IPV model is identified from data of winning bids if the identity of the winner is observed. When the distribution of values depends on observable auction characteristics, $F(v|X_t)$, they show that this distribution is identified from data of winning bids, both in the symmetric and the asymmetric IPV model. They also provide conditions for the identification of the affiliated private value model and the common values model.

In some applications, especially in procurement auctions, there may be substantial heterogeneity across auctions after controlling for the observable characteristics. Not controlling for this heterogeneity can generate important biases in the estimated distributions of valuations. **krasnokutskaya_2011** (**krasnokutskaya_2011**) and Asker (2010) propose and estimate auction models of IPV with unobserved auction heterogeneity.

In Krasnokutskaya's model, bidders' valuations have a multiplicative structure: $u_{it} = v_{it} * c_t$, where v_{it} is private information of bidder i at auction t, and c_t is common knowledge to all the bidders in auction t. She provides sufficient conditions for the nonparametric identification of the distribution of the two components, and proposes an estimation method.[1] Krasnokutskaya applies her method to data from Michigan highway procurement auctions. She finds that, after conditioning on observable auction characteristics (for instance, number of bidders and project size), private information explains only 34% of the sample variation in winning bids. The remaining sample variation comes from unobserved heterogeneity from the point of view of the researcher. Estimates of the model that ignore this unobserved heterogeneity provide substantial biases in the average and the variance of firms' costs, and underestimate firms' mark-ups.

Asker (2010) considers a similar model where bidders' valuations have a multiplicative structure between IPVs and common knowledge auction heterogeneity. He applies this model to estimate the damages and efficiency costs of a "bidding ring" (cartel) in the US market for collectible stamps. Like Krasnokutskaya, he finds that accounting for unobserved auction heterogeneity has an important impact on the estimated model and its economic implications. The model without unobserved heterogeneity over-estimates the cartel's damages to the seller by more than 100%, and under-estimates the efficiency loss from the cartel by almost 50%.

The recent literature on structural auction models has extended the standard model in different important directions. **bajari_hortacsu_2003** (**bajari_hortacsu_2003**), **li_zheng_2009** (**li_zheng_2009**), **athey_levin_2011** (**athey_levin_2011**), **marmer_shneyerov_2013** (**marmer_shneyerov_2013**), and **gentry_li_2014** (**gentry_li_2014**) study endogenous

[1] A key (and very intuitive) identification condition is that the researcher observes multiple bids for each auction. Data with only winning bids is not sufficient.

entry of bidders (and sellers). Jofre-Bonet and Pesendorfer (2003) estimate a dynamic structural model of procurement auctions where firms have capacity constraints and are forward-looking. **groeger_2014** (**groeger_2014**) estimates a dynamic model of entry in procurement auctions where firms have sunk entry costs. **lu_perrigne_2008** (**lu_perrigne_2008**), **guerre_perrigne_2009** (**guerre_perrigne_2009**), **campo_2011** (**campo_2011**) incorporate bidders' risk aversion, provide conditions for nonparametric identification, and propose estimation methods. Finally, **lewis_bajari_2011** (**lewis_bajari_2011**) and **takahashi_2014** (**takahashi_2014**) study procurement auctions where the winner is determined by a scoring rule that weighs both the price and the quality in a firm's bid.

11. Appendix 1: Random Utility Mode

11.1 Introduction

Consider a discrete choice Random Utility Model (RUM) where the optimal choice, a^*, is defined as:

$$a^* = \arg\max_{a \in A} \{u_a + \varepsilon_a\}$$

$A = \{1, 2, ..., J\}$ is the set of feasible choice alternative. $u = (u_1, u_2, ..., u_J)$ is the vector with the deterministic or constant components of the utility. $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_J)$ is the vector with the stochastic or random component of the utility. The vector ε has a joint CDF Gthat is continuous and strictly increasing with respect to the Lebesgue measure in the Euclidean space.

This note derives closed form expressions for the distribution of the maximum utility (that is, $\max_{a \in A} \{u_a + \varepsilon_a\}$), the expected maximum utility (that is, $\mathbb{E}(\max_{a \in A} \{u_a + \varepsilon_a\} | u)$), and the choice probabilities (that is, $\Pr(a^* = a|u)$) under three different assumptions on the distribution of the vector ε: (1) iid Extreme Value distribution (MNL model); (2) nested Extreme Value distribution (NL model); and (3) Ordered Generalized Extreme Value distribution (OGEV model).

The following definitions and properties are used in the note.

Definition: A random variable X has a Double Exponential or Extreme Value distribution with location parameter μ and dispersion parameter σ if its CDF is:

$$G(X) = \exp\left\{-\exp\left(-\left[\frac{X-\mu}{\sigma}\right]\right)\right\}$$

Definition: Maximum utility. Let v^* be the random variable that represents the maximum utility, that is, $v^* \equiv \max_{a \in A} \{u_a + \varepsilon_a\}$. This maximum utility is a random variable because it depends on the vector of random variables ε.

Definition: Social Surplus function (McFadden). The social surplus function $S(u)$ is the expected value of the maximum utility conditional on the vector of constants u, that is, $S(u) \equiv \mathbb{E}(\max_{a \in A} \{u_a + \varepsilon_a\} | u)$.

Definition: Conditional choice probabilities (CCPs). The conditional choice probability $P(a)$ is the probability that alternative a is optimal choice, that is, $P(a|u) \equiv Pr(a^* = a|u)$.

Definition: Conditional choice expected utilities (CCEU). The conditional choice expected utility $e(a,u)$ is the expected value of $u_a + \varepsilon_a$ conditional on the vector u and on alternative a being the optimal choice, that is, $e(a,u) \equiv \mathbb{E}(u_a + \varepsilon_a | u, a^* = a)$.

Williams-Daly-Zachary (WDZ) Theorem. Let $S(u)$ be the function that represents the expected maximum utility conditional on the vector of constants u, that is, $S(u) \equiv \mathbb{E}(\max_{a \in A} \{u_a + \varepsilon_a\} | u)$. Then, the conditional choice probabilities (CCPs), $Pr(a^* = a|u)$, can be obtained as the partial derivatives of the function $S(u)$, that is,

$$Pr(a^* = a|u) = \frac{\partial S(u)}{\partial u_a}$$

Proof: (reference here). By definition of $S(u)$, $\frac{\partial S(u)}{\partial u_a} = \frac{\partial}{\partial u_a} \int \max_{j \in A} \{u_j + \varepsilon_j\} dG(\varepsilon)$. Given the assumptions on the CDF of ε, we have that $\frac{\partial}{\partial u_a} \int \max_{j \in A} \{u_j + \varepsilon_j\} dG(\varepsilon) = \int \frac{\partial}{\partial u_a} \max_{j \in A} \{u_j + \varepsilon_j\} dG(\varepsilon)$. Therefore,

$$\begin{aligned} \frac{\partial S(u)}{\partial u_a} &= \int \frac{\partial}{\partial u_a} \max_{j \in A} \{u_j + \varepsilon_j\} dG(\varepsilon) \\ &= \int 1\{u_a + \varepsilon_a \geq u_j + \varepsilon_j \text{ for any } j \in A\} dG(\varepsilon) \\ &= Pr(a^* = a|u) \end{aligned}$$

Theorem. For any distribution of ε, any value of the vector u, and any choice alternative a, the conditional choice expected utility $e(a,u)$ is equal to the social surplus function $S(u)$, that is, $e(a,u) = S(u)$ for any (a,u). Furthermore, this implies that $\mathbb{E}(\varepsilon_a \mid u, a^* = a) = S(u) - u_a$.

Proof: (reference here). By definition, $e(a,u) = u_a + \mathbb{E}(\varepsilon_a \mid u, a^* = a)$. Taking into account that the random variable v^* represents maximum utility, we have that the event $\{a^* = a\}$ is equivalent to the event $\{v^* = u_a + \varepsilon_a\}$. Therefore,

$$\begin{aligned} e(a,u) &= u_a + \mathbb{E}(\varepsilon_a \mid u,\ v^* = u_a + \varepsilon_a) \\ &= u_a + \mathbb{E}(v^* - u_a \mid u) \\ &= \mathbb{E}(v^*|u) = S(u) \end{aligned}$$

*Inversion Theorem. *** Representation of $S(u)$* in terms of CCPs and utilities only. How to do it in general? ****

11.2 Multinomial logit (MNL)

Suppose that the random variables in the vector ε are independent and identically distributed with double exponential distribution with zero location and dispersion σ. That is, for every alternative a, the CDF of ε_a is $G(\varepsilon_a) = \exp\{-\exp(-\frac{\varepsilon_a}{\sigma})\}$.

(a) Distribution of the Maximum Utility

Let v^* be the random variable that represents the maximum utility, that is, $v^* \equiv \max_{a \in A} \{u_a + \varepsilon_a\}$. This maximum utility is a random variable because it depends on the vector of random variables ε. By definition, the cumulative probability distribution of v^* is:

$$\begin{aligned} H(v) \equiv \Pr(v^* \leq v) &= \prod_{a \in A} \Pr(\varepsilon_a \leq v - u_a) \\ &= \prod_{a \in A} \exp\left\{-\exp\left(-\frac{v-u_a}{\sigma}\right)\right\} \\ &= \exp\left\{-\exp\left(-\frac{v}{\sigma}\right) U\right\} \end{aligned} \tag{11.1}$$

where $U \equiv \sum_{a \in A} \exp\left(\frac{u_a}{\sigma}\right)$. We can also write $H(v) = \exp\left\{-\exp\left(-\frac{v - \sigma \ln U}{\sigma}\right)\right\}$. This expression shows that the maximum utility v^* is a double exponential random variable with dispersion parameter σ and location parameter $\sigma \ln U$. Therefore, the maximum of a vector of i.i.d. double exponential random variables is also a double exponential random variable. This is the reason why this family of random variables is also called "extreme value". The density function of v^* is:

$$h(v) \equiv H'(v) = H(v) \frac{U}{\sigma} \exp\left(-\frac{v}{\sigma}\right)$$

(b) Expected maximum utility

By definition, $S(u) = \mathbb{E}(v^*|u)$. Therefore,

$$S(u) = \int_{-\infty}^{+\infty} v^* \, h(v^*) \, dv^* = \int_{-\infty}^{+\infty} v^* \exp\left\{-\exp\left(-\frac{v^*}{\sigma}\right) U\right\} \frac{U}{\sigma} \exp\left(-\frac{v^*}{\sigma}\right) dv^*$$

We apply the change in variable: $z = \exp(-v^*/\sigma)$, such that $v^* = -\sigma \ln(z)$, and $dv^* = -\sigma(dz/z)$. Then,

$$\begin{aligned} S(u) &= \int_{+\infty}^{0} -\sigma \ln(z) \, \exp\{-z\,U\} \, \frac{U}{\sigma} \, z \left(-\sigma \frac{dz}{z}\right) \\ &= -\sigma U \int_{0}^{+\infty} \ln(z) \, \exp\{-z\,U\} \, dz \end{aligned}$$

Using Laplace transformation we have that $\int_0^{+\infty} \ln(z) \, \exp\{-z\,U\} \, dz = \frac{\ln(U) + \gamma}{U}$, where γ is Euler's constant. Therefore, the expected maximum utility is:

$$S = \sigma U \left(\frac{\ln(U) + \gamma}{U}\right) = \sigma \, (\, \ln(U) + \gamma \,)$$

(c) Choice probabilities

By Williams-Daly-Zachary (WDZ) theorem, the optimal choice probabilities can be obtained by differentiating the surplus function. Therefore, for the MNL model,

$$\begin{aligned} P(a|u) &= \sigma \frac{\partial \ln(U)}{\partial u_a} = \sigma \frac{\partial U}{\partial u_a} \frac{1}{U} \\ &= \exp\left(\frac{u_a}{\sigma}\right) \frac{1}{U} = \frac{\exp(u_a/\sigma)}{\sum_{j \in A} \exp(u_j/\sigma)} \end{aligned}$$

(d) Conditional choice expected utilities

As shown in general, $e(a,u) = S(u)$. This implies that $\mathbb{E}(\varepsilon_a \mid u, a^* = a) = S(u) - u_a$. For the case of the i.i.d. double exponential ε we have that:

$$\mathbb{E}(\varepsilon_a|u,a^*=a) = \sigma\ (\ \ln(U) + \gamma\ ) - u_a$$

(e) Function relating $\mathbb{E}(\varepsilon_a \mid u, a^* = a)$ and CCPs.

In some applications we are interested in the function that relates the expected value $\mathbb{E}(\varepsilon_a|u,a^*=a)$ with conditional choice probabilities $\{P(j|u) : j = 1,2,...,J\}$. From the expression for $P(a|u)$ in the MNL model, we have that $\ln P(a|u) = u_a/\sigma - \ln U$, and therefore $\ln(U) = u_a/\sigma - \ln P_a$. Solving this expression in the previous formula for the expectation $\mathbb{E}(\varepsilon_a|u,a^*=a)$ we get:

$$\mathbb{E}(\varepsilon_a|u,a^*=a) = \sigma\ (u_a/\sigma - \ln P(a|u) + \gamma) - u_a = \sigma\,(\gamma - \ln P(a|u))$$

11.3 Nested logit (NL)

Suppose that the random variables in the vector ε have the following joint CDF:

$$G(\varepsilon) = \exp\left\{ -\sum_{r=1}^{R} \left[\sum_{a\in A_r} \exp\left(-\frac{\varepsilon_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}} \right\}$$

where $\{A_1, A_2, ..., A_R\}$ is a partition of A, and δ, σ_1, σ_2, ..., σ_R are positive parameters, with $\delta \leq 1$.

(a) Distribution of the Maximum Utility

$$\begin{aligned}
H(v) \equiv \Pr(v^* \leq v) &= \Pr(\varepsilon_a \leq v - u_a : for\ any\ a \in A) \\
&= \exp\left\{ -\sum_{r=1}^{R} \left[\sum_{a\in A_r} \exp\left(-\frac{v-u_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}} \right\} \\
&= \exp\left\{ -\exp\left(-\frac{v}{\delta}\right) \sum_{r=1}^{R} \left[\sum_{a\in A_r} \exp\left(\frac{u_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}} \right\} \\
&= \exp\left\{ -\exp\left(-\frac{v}{\delta}\right) U \right\}
\end{aligned}$$

where:

$$U \equiv \sum_{r=1}^{R} \left[\sum_{a\in A_r} \exp\left(\frac{u_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}} = \sum_{r=1}^{R} U_r^{1/\delta}$$

where $U_r \equiv \left[\sum_{a\in A_r} \exp\left(\frac{u_a}{\sigma_r}\right)\right]^{\sigma_r}$. The density function of v^* is:

$$h(v) \equiv H'(v) \ = \ H(v)\,\frac{U}{\delta}\,\exp\left(-\frac{v}{\delta}\right)$$

(b) Expected maximum utility

By definition, $S(u) = \mathbb{E}(v^*)$. Therefore,

$$S(u) \;=\; \int_{-\infty}^{+\infty} v^*\, h(v^*)\, dv^* \;=\; \int_{-\infty}^{+\infty} v^*\, \exp\left\{-\exp\left(-\frac{v^*}{\delta}\right) U\right\} \frac{U}{\delta}\, \exp\left(-\frac{v^*}{\delta}\right) dv^*$$

Let's apply the following change in variable: $z = \exp(-v^*/\delta)$, such that $v^* = -\delta \ln(z)$, and $dv^* = -\delta(dz/z)$. Then,

$$S \;=\; \int_{+\infty}^{0} -\delta \ln(z)\, \exp\{-z\,U\}\, \frac{U}{\delta}\, z\, \left(-\delta \frac{dz}{z}\right) \;=\; -\delta U \int_{+\infty}^{0} \ln(z)\, \exp\{-z\,U\}\, dz$$

And using Laplace transformation:

$$S \;=\; \delta U \left(\frac{\ln(U) + \gamma}{U}\right) \;=\; \delta\, (\, \ln(U) + \gamma\,)$$

where γ is the Euler's constant.

(c) Choice probabilities

By Williams-Daly-Zachary (WDZ) theorem, choice probabilities can be obtained differentiating the surplus function. For the NL model:

$$\begin{aligned} P(a|u) \;&=\; \delta\, \frac{\partial \ln(U)}{\partial u_a} = \delta \frac{\partial U}{\partial u_a} \frac{1}{U} = \\ &= \delta \frac{\sigma_{ra}}{\delta} \left[\sum_{j \in A_{ra}} \exp\left(\frac{u_j}{\sigma_{ra}}\right)\right]^{\frac{\sigma_{ra}}{\delta} - 1} \frac{1}{\sigma_{ra}} \exp\left(\frac{u_a}{\sigma_{ra}}\right) \frac{1}{U} \\ &= \frac{\exp\left(u_a/\sigma_{ra}\right)}{\sum_{j \in A_{ra}} \exp\left(u_j/\sigma_{ra}\right)} \frac{\left[\sum_{j \in A_{ra}} \exp\left(u_j/\sigma_{ra}\right)\right]^{\frac{\sigma_{ra}}{\delta}}}{\sum_{r=1}^{R} \left[\sum_{j \in A_r} \exp\left(u_j/\sigma_r\right)\right]^{\frac{\sigma_r}{\delta}}} \end{aligned}$$

The first term is $q(a|r_a)$ (that is, probability of choosing a given that we are in group A_{ra}), and the second term is $Q(r_a)$ (that is, probability of selecting the group A_{ra}).

(d) Conditional choice expected utilities

As shown in general, $e(a,u) = S(u)$. This implies that $\mathbb{E}(\varepsilon_a \mid u, a^* = a) = S(u) - u_a$. Given that for the NL model $S(u) = \delta\,(\ln U + \gamma)$ we have that:

$$\mathbb{E}(\varepsilon_a | u, a^* = a) = \delta\gamma + \delta \ln U - u_a$$

(e) Function relating $\mathbb{E}(\varepsilon_a \mid u, a^* = a)$ and CCPs.

To write $\mathbb{E}(\varepsilon_a | u, a^* = a)$ in terms of choice probabilities, note that from the definition of $q(a|r_a)$ and $Q(r_a)$, we have that:

$$\ln q(a|r_a) \;=\; \frac{u_a - \ln U_{ra}}{\sigma_{ra}} \;\Rightarrow\; \ln U_{ra} \;=\; u_a - \sigma_{ra} \ln q(a|r_a)$$

and

$$\ln Q(r_a) = \frac{\ln U_{ra}}{\delta} - \ln U \Rightarrow \ln U = \frac{\ln U_{ra}}{\delta} - \ln Q(r_a)$$

Combining these expressions, we have that:

$$\ln U = \frac{u_a - \sigma_{ra} \ln q(a|r_a)}{\delta} - \ln Q(r_a)$$

Therefore,

$$\begin{aligned} e_a &= \delta\gamma + \delta\left(\frac{u_a - \sigma_{ra}\ln q(a|r_a)}{\delta} - \ln Q(r_a)\right) - u_a \\ &= \delta\gamma - \sigma_{ra}\ \ln q(a|r_a) - \delta\ \ln Q(r_a) \end{aligned}$$

11.4 Ordered GEV (OGEV)

Suppose that the random variables in the vector ε have the following joint CDF:

$$G(\varepsilon) = \exp\left\{-\sum_{r=1}^{J+M}\left[\sum_{a\in B_r} W_{r-a}\exp\left(-\frac{\varepsilon_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}}\right\}$$

where:

(1) M is a positive integer;

(2) $\{B_1, B_2, ..., B_{J+M}\}$ are $J+M$ subsets of A, with the following definition:

$$B_r = \{a \in A : r - M \leq a \leq r\}$$

For instance, if $A = \{1,2,3,4,5\}$ and $M = 2$, then $B_1 = \{1\}$, $B_2 = \{1,2\}$, $B_3 = \{1,2,3\}$, $B_4 = \{2,3,4\}$, $B_5 = \{3,4,5\}$, $B_6 = \{4,5\}$, and $B_7 = \{5\}$.

(3) δ, and σ_1, σ_2, ..., σ_{J+M} are positive parameters, with $\delta \leq 1$;

(4) W_0, W_1, ..., W_M are constants (weights) such that: $W_m \geq 0$, and $\sum_{m=0}^{M} W_m = 1$.

(a) Distribution of the Maximum Utility

$$\begin{aligned} H(v) \equiv \Pr(v^* \leq v) &= \Pr(\varepsilon_a \leq v - u_a : for\ any\ a \in A) \\ &= \exp\left\{-\sum_{r=1}^{J+M}\left[\sum_{a\in B_r} W_{r-a}\exp\left(-\frac{v-u_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}}\right\} \\ &= \exp\left\{-\exp\left(-\frac{v}{\delta}\right)\sum_{r=1}^{J+M}\left[\sum_{a\in B_r} W_{r-a}\exp\left(\frac{u_a}{\sigma_r}\right)\right]^{\frac{\sigma_r}{\delta}}\right\} \\ &= \exp\left\{-\exp\left(-\frac{v}{\delta}\right)U\right\} \end{aligned}$$

where:

$$U \equiv \sum_{r=1}^{J+M} \left[\sum_{a \in B_r} W_{r-a} \exp\left(\frac{u_a}{\sigma_r}\right) \right]^{\frac{\sigma_r}{\delta}} = \sum_{r=1}^{J+M} U_r^{1/\delta}$$

where $U_r \equiv \left[\sum_{a \in B_r} W_{r-a} \exp\left(\frac{u_a}{\sigma_r}\right) \right]^{\sigma_r}$. The density function of v^* is:

$$h(v) \equiv H'(v) \quad = \quad H(v) \, \frac{U}{\delta} \, \exp\left(-\frac{v}{\delta}\right)$$

(b) Expected maximum utility

By definition, $S(u) = \mathbb{E}(v^*|u)$. Therefore,

$$S(u) \quad = \quad \int_{-\infty}^{+\infty} v^* \, h(v^*) \, dv^* \quad = \quad \int_{-\infty}^{+\infty} v^* \, \exp\left\{ -\exp\left(-\frac{v^*}{\delta}\right) U \right\} \, \frac{U}{\delta} \, \exp\left(-\frac{v^*}{\delta}\right) \, dv^*$$

Let's apply the following change in variable: $z = \exp(-v^*/\delta)$, such that $v^* = -\delta \ln(z)$, and $dv^* = -\delta(dz/z)$. Then,

$$S \quad = \quad \int_{+\infty}^{0} -\delta \ln(z) \, \exp\{-z\,U\} \, \frac{U}{\delta} \, z \, \left(-\delta \frac{dz}{z}\right) \quad = \quad -\delta U \int_{0}^{+\infty} \ln(z) \, \exp\{-z\,U\} \, dz$$

And using Laplace transformation:

$$S \quad = \quad \delta U \left(\frac{\ln U + \gamma}{U}\right) \quad = \quad \delta \left(\ln U + \gamma\right) = \delta\gamma + \delta \, \ln \left[\sum_{r=1}^{J+M} \left[\sum_{a \in B_r} W_{r-a} \exp\left(\frac{u_a}{\sigma_r}\right) \right]^{\frac{\sigma_r}{\delta}} \right]$$

where γ is the Euler's constant.

(c) Choice probabilities

By Williams-Daly-Zachary (WDZ) theorem, choice probabilities can be obtained differentiating the surplus function.

$$P(a|u) = \frac{1}{U} \sum_{r=a}^{a+M} \left[\sum_{j \in B_r} W_{r-j} \exp\left(\frac{u_j}{\sigma_r}\right) \right]^{\frac{\sigma_r}{\delta}-1} W_{r-a} \exp\left(\frac{u_a}{\sigma_r}\right) = \sum_{r=a}^{a+M} q(a|r) \; Q(r)$$

where:

$$\begin{aligned} q(a|r) \quad &= \quad \frac{W_{r-a} \, \exp\left(u_a/\sigma_r\right)}{\sum_{j \in B_r} W_{r-j} \exp\left(u_j/\sigma_r\right)} = \frac{\exp\left(u_a/\sigma_r\right)}{\exp(\ln U_r/\sigma_r)} \\ Q(r) \quad &= \quad \frac{\exp\left(\ln U_r/\delta\right)}{\sum_{j=1}^{J+M} \exp\left(\ln U_j/\delta\right)} = \frac{\exp\left(\ln U_r/\delta\right)}{U} \end{aligned}$$

(d) Conditional choice expected utilities

As shown in general, $e(a,u) = S(u)$. This implies that $\mathbb{E}(\varepsilon_a \mid u, a^* = a) = S(u) - u_a$. Given that for the OGEV model $S(u) = \delta\left(\ln U + \gamma\right)$ we have that:

$$\mathbb{E}(\varepsilon_a|u, a^* = a) = \delta\gamma + \delta \ln U - u_a$$

(e) Function relating $\mathbb{E}(\varepsilon_a \mid u, a^* = a)$ and CCPs.

To write $\mathbb{E}(\varepsilon_a|u, a^* = a)$ in terms of choice probabilities, note that from the definition of $q(a|r)$ and $Q(r)$, we have that: ....

TBC

12. Appendix 2

12.1 Problem set #1

Context. At the end of year 2002, the federal government of the Republic of Greenishtan introduced a new environmental regulation on the cement industry, one of the major polluting industries. The most important features of this regulation is that new plants, in order to operate in the industry, should pass an environmental test and should install new equipment that contributes to reduce pollutant emissions. Industry experts consider that this new law increased the fixed cost of operating in this industry. However, these experts disagree in the magnitude of the effect. There is also disagreement with respect to whether the new law affected variable costs, competition, prices, and output. You have been hired by the Ministry of Industry as an independent researcher to study and to evaluate the effects of this policy on output, prices, firms' profits, and consumer welfare.

Data. To perform your evaluation, you have a panel dataset with annual information on the industry for the period 1998-2007. The Stata datafile `eco2901_problemset_01_2011.dta` contains panel data from 200 local markets (census tracts) over 10 years (1998-2007) for the cement industry in the Republic of Greenishtan. The local markets in this dataset have been selected following criteria similar to the ones in **bresnahan_reiss_1991** (**bresnahan_reiss_1991**). This is the list of variables in the dataset:

Variable name		Description
`market`	:	Code of local market
`year`	:	Year
`pop`	:	Population of local market
`income`	:	Per capita income in local market
`output`	:	Annual output produced in the local market
`price`	:	Price of cement in local market
`pinput`	:	Price index of intermediate inputs in local market
`nplant`	:	Number of cement plants in local market at current year

Model. To answer our empirical questions, we consider a model in the spirit of the model by Bresnahan and Reiss that we have seen in class. The main difference with respect to that model is that we specify the demand function and the cost function in the industry and make it explicit the relationship between these primitives and the profit of a plant.

Demand of cement in market m at period t. We assume that cement is an homogeneous product and consider the following inverse demand function:

$$\ln P_{mt} = \alpha_0^D + \alpha_1^D \ln POP_{mt} + \alpha_2^D \ln INC_{mt} - \alpha_3^D \ln Q_{mt} + \varepsilon_{mt}^D$$

where $\alpha^{D\prime}s$ are demand parameters, Q_{mt} represents output, POP_{mt} is population, INC_{mt} is per capita income, P_{mt} is price, and ε_{mt}^D is a component of the demand that is unobserved to the researcher.

Production costs. Let q_{mt} be the amount of output of a cement plant in market m and period t. The production cost function is $C_{mt}(q_{mt}) = FC_{mt} + MC_{mt}\, q_{mt}$, where FC_{mt} and MC_{mt} are the fixed cost function and the marginal cost, respectively. We consider the following specification for FC_{mt} and MC_{mt}:

$$\begin{aligned} FC_{mt} &= \exp\left\{X_{mt}\ \alpha^{FC} + \varepsilon_{mt}^{FC}\right\} \\ MC_{mt} &= \exp\left\{X_{mt}\ \alpha^{MC} + \varepsilon_{mt}^{MC}\right\} \end{aligned}$$

where X_{mt} is the vector $(1, \ln POP_{mt}, \ln INC_{mt}, \ln PINPUT_{mt})$, where $PINPUT_{mt}$ is the index price of inputs (energy and limestone); α^{FC} and α^{MC} are vectors of parameters; and ε_{mt}^{FC} and ε_{mt}^{MC} are components of the fixed cost and the marginal cost, respectively, that are unobserved to the researcher. The main reason why we consider an exponential function in the specification of FC_{mt} and MC_{mt} is to impose the natural restriction that costs should be always positive.

Entry costs and scrapping value. For simplicity, we consider a static model and therefore we assume that there are not sunk entry costs.

Unobservables. Let ε_{mt} be the vector of unobservables $\varepsilon_{mt} \equiv (\varepsilon_{mt}^D, \varepsilon_{mt}^{MC}, \varepsilon_{mt}^{FC})$. We allow for serial correlation in these unobservables. In particular, we assume that each of these unobservables follows an AR(1) process. For $j \in \{D, MC, FC\}$:

$$\varepsilon_{mt}^j = \rho^j\ \varepsilon_{mt-1}^j + u_{mt}^j$$

where $\rho^j \in [0,1)$ is the autorregressive parameter, and the vector $u_{mt} = (u^D_{mt}, u^{MC}_{mt}, u^{FC}_{mt})$ is i.i.d. over markets and over time with a joint normal distribution with zero means and variance-covariance matrix Ω.

Question 1 [20 points]. **(a)** Propose an estimator of the demand parameters and explain the assumptions under which the estimator is consistent. **(b)** Obtain estimates and standard errors. **(c)** Test the null hypothesis of "no structural break" in demand parameters after year 2002.

Question 2 [20 points]. **(a)** Describe how to use the Cournot equilibrium conditions to estimate the parameters in the marginal cost function. Explain the assumptions under which the estimator is consistent. **(b)** Obtain estimates and standard errors. **(c)** Test the null hypothesis of "no structural break" in the variable cost parameters after year 2003.

Question 3 [30 points]. Assume that $\rho^{FC} = 0$. **(a)** Describe how to estimate the parameters in the fixed cost function. Show that these costs are identified in dollar amounts (that is, not only up to scale). Explain the assumptions under which the estimator is consistent. How does the estimation of fixed costs change if $\rho^{FC} \neq 0$? Explain. **(b)** Obtain estimates and standard errors. **(c)** Test the null hypothesis of "no structural break" in the fixed cost parameters after year 2003.

Question 4 [30 points]. Now, we use our estimates to evaluate the effects of the policy change. Suppose that we attribute to the new policy the estimated change in the parameters of the cost function, but not the estimated change in the demand parameters.

(a) [10 points] Given the estimated parameters "after 2002", calculate the equilibrium values of the endogenous variables $\{P_{m,2003}, Q_{m,2003}, N_{m,2003}\}$ for every local market in 2003, that is, for every value of the exogenous variables $(X_{m,2003}, \varepsilon_{m,2003})$. Obtain also firms' profits, consumer welfare, and total welfare.

(b) [10 points] Now, consider the counterfactual scenario where demand parameters are the ones "after 2002" but cost parameters are the ones "before 2003". For this scenario, calculate the "counterfactual" equilibrium values of the endogenous variables $\{P^*_{m,2003}, Q^*_{m,2003}, N^*_{m,2003}\}$ for every local market in 2003. Also obtain the counterfactual values for firms' profits, consumer welfare, and total welfare.

(c) [10 points] Obtain the effects of the policy one the number of firms, output, prices, firms' profits, consumer welfare, and total welfare. Comment the results. Present two-way graphs of these effects with the logarithm of population in the horizontal axis and the estimated on a certain endogenous variable in the vertical axis. Comment the results. What are the most important effects of this policy?

12.2 Problem set #2

The Stata datafile `eco2901_problemset_01_chiledata_2010.dta` contains a panel dataset of 167 local markets in Chile with annual information over the years 1994 to 1999 and for five retail industries: Restaurants ('Restaurantes,' product code 63111); Gas stations ('Gasolineras,' product code 62531); Bookstores ('Librerias,' product code 62547); Shoe Shops ('Calzado,' product code 62411); and Fish shops

('Pescaderias,' product code 62141). The 167 "isolated" local markets in this dataset have been selected following criteria similar to the ones in **bresnahan_reiss_1991** (**bresnahan_reiss_1991**). This is the list of variables in the dataset with a brief description of each variable:

`comuna_code`	:	Coder of local market
`comuna_name`	:	Name of local market
`year`	:	Year
`procode`	:	Code of product/industry
`proname`	:	Name of product/industry
`pop`	:	Population of local market (in # people)
`areakm2`	:	Area of local market (in square Km)
`expc`	:	Annual expenditure per capita in all retail products in the local market
`nfirm`	:	Number of firms in local market and industry at current year
`nfirm_1`	:	Number of firms in local market and industry at previous year
`entries`	:	Number of new entrants in local market and industry during current year
`exits`	:	Number of exiting firms in local market and industry during current year

Consider the following static entry model in the spirit of **bresnahan_reiss_1991** (**bresnahan_reiss_1991**). The profit of an active firm in market m at year t is:

$$\Pi_{mt} = S_{mt}\ v(n_{mt}) - F_{mt}$$

where S_{mt} is a measure of market size; n_{mt} is the number of firms active in the market; $v$is the *variable profit per capita* and it is a decreasing function; and F_{mt} represents fixed operating costs in market m at period t. The function $v$is nonparametrically specific. The specification of market size is:

$$S_{mt} = POP_{mt}\ \exp\left\{\beta_0^S + \beta_1^S\ expc_{mt} + \varepsilon_{mt}^S\right\}$$

where POP_{mt} is the population in the local market; $expc_{mt}$ is per capita sales in all retail industries operating in the local market; β_0^S and β_1^S are parameters; and ε_{mt}^S is an unobservable component of market size. The specification of the fixed cost is:

$$F_{mt} = \exp\left\{\beta^F + \varepsilon_{mt}^F\right\}$$

where β^F is a parameter, and ε_{mt}^F in an unobservable component of the fixed cost. Define the unobservable $\varepsilon_{mt} \equiv \varepsilon_{mt}^S - \varepsilon_{mt}^F$. And let $X_{mt} \equiv (\ln POP_{mt}, expc_{mt})$ be the vector with the observable characteristics of the local market. We assume that ε_{mt} is independent of X_{mt} and iid over (m,t) $N(0,\sigma^2)$.

Question 1. [10 points] Show that the model implies the following probability distribution for the equilibrium number of firms: let $n_{\max}$ be the maximum value of n_{mt}, then for any $n \in \{0,1,...,n_{\max}\}$:

$$\begin{aligned}
\Pr(n_{mt} = n \mid X_{mt}) &= \Pr\left(cut(n) \leq X_{mt}\begin{bmatrix} \frac{1}{\sigma} \\ \frac{\beta_1^S}{\sigma} \end{bmatrix} + \frac{\varepsilon_{mt}}{\sigma} \leq cut(n+1)\right) \\
&= \Phi\left(cut(n+1) - X_{mt}\begin{bmatrix} \frac{1}{\sigma} \\ \frac{\beta_1^S}{\sigma} \end{bmatrix}\right) - \Phi\left(cut(n) - X_{mt}\begin{bmatrix} \frac{1}{\sigma} \\ \frac{\beta_1^S}{\sigma} \end{bmatrix}\right)
\end{aligned}$$

where $cut(0)$, $cut(1)$, $cut(2)$, ... are parameters such that for $n \in \{1,2,...,n_{\max}\}$, $cut(n) \equiv (\beta^F - \beta_0^S - \ln v(n))/\sigma$, and $cut(0) \equiv -\infty$, and $cut(n_{\max}+1) \equiv -\infty$.

Question 2. [20 points] Given the Ordered Probit structure of the model, estimate the vector of parameters $\{1/\sigma,\ \beta_1^S/\sigma,\ cut(1),\ cut(2),\ ...,\ cut(n_{\max})\}$ for each of the five industries separately. Given these estimates, obtain estimates of the parameters $\dfrac{v(n+1)}{v(n)}$ for $n \in \{1,2,...,n_{\max}\}$. Present a figure of the estimated function $\dfrac{v(n+1)}{v(n)}$ for each of the five industries. Interpret the results. Based on these results, what can we say about *the nature of competition* in each of these industries?

Question 3. [20 points] Repeat the same exercise as in Question 3 but using the following specification of the unobservable ε_{mt}:

$$\varepsilon_{mt} = \gamma_t + \delta_m + u_{mt}$$

where γ_t are time effects that can be captured by using time-dummies; δ_m are fixed market effects that can be captured by using market-dummies; and u_{mt} is independent of X_{mt} and iid over (m,t) $N(0,\sigma^2)$. Comment the results.

Now, consider the following static entry model of incomplete information. There are N_{mt} potential entrants in market m at period t. The profit of an active firm in market m at year t is:

$$\Pi_{imt} = S_{mt}\ v(n_{mt}) - F_{imt}$$

Market size, S_{mt}, has the same specification as in Question 2. The firm-specific fixed cost, F_{mt}, has the following specification:

$$F_{imt} = \exp\left\{\beta^F + \varepsilon_{mt}^F + \xi_{imt}\right\}$$

The random variables ε_{mt}^S, ε_{mt}^F, and ξ_{imt} are unobservable to the researcher. From the point of view of the firms in the market, the variables ε_{mt}^S and ε_{mt}^F are common knowledge, while ξ_{imt} is private information of firm i. We assume that ξ_{imt} is independent of X_{mt} and iid over (m,t) $N(0,\sigma_\xi^2)$.

The number of potential entrants, N_{mt}, is assumed to be proportional to population: $N_{mt} = \lambda\ POP_{mt}$, where the parameter λ is industry specific.

Question 4. [5 points] Consider the following estimator of the number of potential entrants:

$$\hat{N}_{mt} = integer\left\{\max_{\text{over all}\{m',t'\}}\left[\frac{entrants_{m't'} + incumbents_{m't'}}{POP_{m't'}}\right] POP_{mt}\right\} \tag{12.1}$$

where $entrants_{m't'}$ and $incumbents_{m't'}$ are the number of new entrants and the number of incumbents, respectively, in market m' at period t'. Show that $\hat{N}_{mt}$ is a consistent estimator of $N_{mt} = \lambda\ POP_{mt}$.

Question 5. [15 points] Let $P(X_{mt}, \varepsilon_{mt})$ be the equilibrium probability of entry given the common knowledge variables $(X_{mt}, \varepsilon_{mt})$. And let $G(n|X_{mt}, \varepsilon_{mt})$ be the distribution of the number of active firms in equilibrium conditional on $(X_{mt}, \varepsilon_{mt})$ and given that one of the firms is active with probability one. (i) Obtain the expression of the probability distribution $G(n|X_{mt}, \varepsilon_{mt})$ in terms of the probability of entry $P(X_{mt}, \varepsilon_{mt})$. (ii) Derive the expression for the expected profit of an active firm in terms of the probability of entry. (iii) Obtain the expression of the equilibrium mapping that defines implicitly the equilibrium probability of entry $P(X_{mt}, \varepsilon_{mt})$.

NOTE: *For Questions 6 and 7, consider the following approximation to the function* $\ln \mathbb{E}(v(n_{mt}) \mid X_{mt}, \varepsilon_{mt}, 1 \text{ sure})$*:*

$$\ln \mathbb{E}(v(n_{mt})|X_{mt}, \varepsilon_{mt}, 1\text{sure}) \simeq \ln v(1) + \sum_{n=1}^{N_{mt}} G(n|X_{mt}, \varepsilon_{mt}) \left[\frac{v(n) - v(1)}{v(1)}\right]$$

This is a first order Taylor approximation to $\ln \mathbb{E}(v(n_{mt})|X_{mt}, \varepsilon_{mt}, 1sure)$ *around the values* $v(1) = v(2) = ... = v(N)$*, that is, no competition effects. The main advantage of using this approximation for estimation is that it is linear in the parameters* $\left[\frac{v(n)-v(1)}{v(1)}\right]$.

Question 6. [20 points] Suppose that $\varepsilon_{mt} \equiv \varepsilon_{mt}^S - \varepsilon_{mt}^F$ is just an aggregate time effect, $\varepsilon_{mt} = \gamma_t$. Use a two-step pseudo maximum likelihood method to estimate the vector of parameters:

$$\theta \equiv \left\{ \frac{1}{\sigma_\xi}, \frac{\beta_1^S}{\sigma_\xi}, \frac{\ln v(1) + \beta_0^S - \beta^F}{\sigma_\xi}, \frac{v(n) - v(1)}{\sigma_\xi \; v(1)} : n = 2, 3, ... \right\}$$

for each of the five industries separately. Given these estimates, obtain estimates of the parameters $\frac{v(n+1)}{v(n)}$ for $n \in \{1, 2, ..., n_{\max}\}$. Present a figure of the estimated function $\frac{v(n+1)}{v(n)}$ for each of the five industries. Interpret the results. Based on these results, what can we say about *the nature of competition* in each of these industries? Compare these results to those from the estimation of the *BR-91* models in Questions 2 and 3.

Question 7. [10 points] Repeat the same exercise as in Question 7 but using the following specification of the unobservable ε_{mt}:

$$\varepsilon_{mt} = \gamma_t + \delta_m$$

where γ_t are time effects that can be captured by using time-dummies; and δ_m are fixed market effects that can be captured by using market-dummies. Comment the results. Compare these results to those in Questions 2, 3, and 6.

12.3 Problem set #3

This problem set describes a dynamic game of entry/exit in an oligopoly market. To answer the questions below, you have to write computer code (for instance, GAUSS,

MATLAB) for the solution, simulation and estimation of the model. Please, submit the program code together with your answers.

Consider the UK fast food industry during the period 1991-1995, as analyzed by Toivanen and Waterson (2005). During this period, the industry was dominated by two large retail chains: McDonalds (MD) and Burger King (BK). The industry can be divided into isolated/independent local markets. Toivanen and Waterson consider local districts as the definition of local market (of which there are almost 500 in UK). At each local market these retail chains decide whether to have an outlet or not.

We index firms by $i \in \{MD, BK\}$ and time (years) by t. The current profit of firm i in a local market is equal to variable profits, VP_{it}, minus fixed costs of operating an outlet, FC_{it}, and minus the entry cost of setting up an outlet by first time, EC_{it}. Variable profits are $VP_{it} = (p_{it} - c_i)q_{it}$, where p_{it} represents the price, c_i is firm i's marginal cost (that is, the marginal cost of an average meal in chain i), and q_{it} is the quantity sold (that is, total number of meals served in the outlet at year t). The demand for an outlet of firm i in the local market is:

$$q_{it} = \frac{S_t \ \exp\{w_i - \alpha \ p_{it}\}}{1 + \exp\{w_i - \alpha \ p_{it}\} + a_{jt} \ \exp\{w_j - \alpha \ p_{jt}\}}$$

S_t represents the size of the local market at period t (that is, total number of restaurant meals over the year). w_i and w_j are the average willingness to pay for products i and j, respectively. α is a parameter. And a_{jt} is the indicator of the event "firm j is active in the local market at period t". Every period t, the active firms compete in prices. There is not dynamics in consumers demand or in variable costs, and therefore price competition is static. Fixed costs and entry costs have the following form:

$$\begin{aligned} FC_{it} &= FC_i + \varepsilon_{it} \\ EC_{it} &= (1 - a_{i,t-1}) \ EC_i \end{aligned}$$

The fixed cost is paid every year that the firm is active in the market. The entry cost, or setup cost, is paid only if the firm was not active at previous year (if $a_{i,t-1} = 0$). Both fixed costs and entry costs are firm-specific. The entry cost is time invariant. ε_{it} represents a firm-idiosyncratic shock in firm i's fixed cost that is iid over firms and over time with a distribution $N(0, \sigma^2)$. We also assume that ε_{it} is private information of firm i. If a firm is not active in the market, its profit is zero. For notational simplicity I "normalize" the variance of ε_{it} to be 1, though it should be understood that the structural parameters in the profit function are identified up to scale.

QUESTION 1. [5 POINTS] Consider the static model of price competition. Show that equilibrium price-cost margins, $p_{it} - c_i$, and equilibrium market shares, q_{it}/S_t, do not depend on market size S_t. Therefore, we can write the equilibrium variable profit function as:

$$VP_{it} = \left(1 - a_{jt}\right) \ S_t \ \theta_i^M + a_{jt} \ S_t \ \theta_i^D$$

where θ_i^M and θ_i^D represent the equilibrium variable profits per-capita (per-meal) when firm i is a monopolist and when it is a duopolist, respectively.

The payoff-relevant information of firm i at period t is $\{x_t, \varepsilon_{it}\}$ where $x_t \equiv \{S_t, a_{1,t-1}, a_{2,t-1}\}$. Let $P_j(x_t)$ represents firm $i's$ belief about the probability that firm j will be active in the market given state x_t. Given this belief, the expected profit of firm i at period t is:

$$\begin{aligned}\pi_{it}^P &= \left(1 - P_j(x_t)\right)\ S_t\ \theta_i^M + P_j(x_t)\ S_t\ \theta_i^D - FC_i - (1 - a_{i,t-1})\ EC_i - \varepsilon_{it} \\ &= Z_{it}^P\ \theta_i - \varepsilon_{it}\end{aligned}$$

where $Z_{it}^P \equiv \left(\left(1 - P_j(x_t)\right) S_t, P_j(x_t) S_t, -1, -(1 - a_{i,t-1})\right)$ and $\theta_i \equiv \left(\theta_i^M, \theta_i^D, FC_i, EC_i\right)'$.

For the rest of this problem set, we consider the following values for the profit parameters:

$$\theta_{MD}^M = 1.5 \quad ; \quad \theta_{MD}^D = 0.7 \quad ; \quad FC_{MD} = 6 \quad ; \quad EC_{MD} = 6$$

$$\theta_{BK}^M = 1.2 \quad ; \quad \theta_{BK}^D = 0.3 \quad ; \quad FC_{BK} = 4 \quad ; \quad EC_{BK} = 4$$

MD's product has higher quality (even after adjusting for marginal costs) than BK's. This implies that MD has higher variable profits than BK, either under monopoly or under duopoly. However, MD has also higher costs of setting up and operating an outlet.

Market size S_t follows a discrete Markov process with support $\{4,5,6,7,8,9\}$ and transition probability matrix:

$$F_S = \begin{bmatrix} 0.9 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.8 & 0.1 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.8 & 0.1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.1 & 0.8 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.1 & 0.8 & 0.1 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.9 \end{bmatrix}$$

A. STATIC (MYOPIC) ENTRY-EXIT GAME

We first consider a static (not forward-looking) version of the entry-exit game. A Bayesian Nash Equilibrium (BNE) in this game can be described as a pair of probabilities, $\{P_{MD}(x_t), P_{BK}(x_t)\}$ solving the following system of equations:

$$\begin{aligned}P_{MD}(x_t) &= \Phi\left(Z_{MDt}^P\ \theta_{MD}\right) \\ P_{BK}(x_t) &= \Phi\left(Z_{BKt}^P\ \theta_{BK}\right)\end{aligned}$$

where $\Phi(.)$ is the CDF of the standard normal.

QUESTION 2. [10 POINTS] For every possible value of the state x_t (that is, 24 values) obtain all the BNE of the static entry game.

Hint: Define the functions $f_{MD}(P) \equiv \Phi\left(Z_{MDt}^P\ \theta_{MD}\right)$ and $f_{BK}(P) \equiv \Phi\left(Z_{BKt}^P\ \theta_{BK}\right)$. Define also the function $g(P) \equiv P - f_{MD}(f_{BK}(P))$. A BNE is zero of the function $g(P)$. You can search for all the zeroes of $g(P)$ in different ways, but in this case

the simpler method is to consider a discrete grid for P in the interval $[0,1]$, for instance, uniform grid with 101 points.

For some values of the state vector x_t, the static model has multiple equilibria. To answer Questions 3 to 5, assume that, in the population under study, the "equilibrium selection mechanism" always selects the equilibrium with the higher probability that MD is active in the market.

Let X be the set of possible values of x_t. And let $\mathbf{P}^0 \equiv \{P^0_{MD}(x), P^0_{BK}(x) : x \in X\}$ be the equilibrium probabilities in the population. Given $\mathbf{P}^0$ and the transition probability matrix for market size, F_S. We can obtain the steady-state distribution of x_t. Let $p^*(x_t)$ be the steady-state distribution. By definition, for any $x_{t+1} \in X$:

$$\begin{aligned} p^*(x_{t+1}) &= \sum_{x_t \in X} p^*(x_t)\ \Pr(x_{t+1}|x_t) \\ &= \sum_{x_t \in X} p^*(x_t)\ F_S(S_{t+1}|S_t) \\ &\quad \left[P^0_{MD}(x_t)\right]^{a_{MDt+1}} \left[1 - P^0_{MD}(x_t)\right]^{1-a_{MDt+1}} \left[P^0_{BK}(x_t)\right]^{a_{BKt+1}} \left[1 - P^0_{BK}(x_t)\right]^{1-a_{BKt+1}} \end{aligned}$$

QUESTION 3. [10 POINTS] Compute the steady-state distribution of x_t in the population.

QUESTION 4. [20 POINTS] Using the values of P^0, F_S and p^* obtained above, simulate a data set $\{x_{mt} : t = 0,1,...,T; m = 1,2,...,M\}$ for $M = 500$ local markets and $T+1 = 6$ years with the following features: (1) local markets are independent; and (2) the initial states x_{m0} are random draws from the steady-state distribution p^*. Present a table with the mean values of the state variables in x_t and with the sample frequencies for the following events: (1) MD is a monopolist; (2) BK is a monopolist; (3) duopoly; (4) MD is active given that (conditional) she was a monopolist at the beginning of the year (the same for BK); (5) MD is active given that BK was a monopolist at the beginning of the year (the same for BK); (6) MD is active given that there was a duopoly at the beginning of the year (the same for BK); and (7) MD is active given that there were no firms active at the beginning of the year (the same for BK).

QUESTION 5. [20 POINTS] Use the simulated data in Question 4 to estimate the structural parameters of the model. Implement the following estimators: (1) two-step PML using a frequency estimator of $\mathbf{P}^0$ in the first step; (2) two-step PML using random draws from a U(0,1) for $\mathbf{P}^0$ in the first step; (3) 20-step PML using a frequency estimator of $\mathbf{P}^0$ in the first step; (4) 20-step PML using random draws from a U(0,1) for P^0 in the first step; and (5) NPL estimator based on 10 NPL fixed points (that is, 10 different initial $P's$). Comment the results.

QUESTION 6. [30 POINTS] Suppose that the researcher knows that local markets are heterogeneous in their market size, but she does not observed market size S_{mt}. Suppose that the researcher assumes that market size is constant over time but it varies across markets, and it has a uniform distribution with discrete support $\{4,5,6,7,8,9\}$. Obtain the NPL estimator under this assumption (use 20 NPL fixed points). Comment the results.

QUESTION 7. [30 POINTS] Use the previous model (both the true model and the model estimated in Question 5) to evaluate the effects of a value added tax. The value added tax is paid by the retailer and it is such that the parameters θ_i^M and θ_i^D are reduced by 10%. Obtain the effects of this tax on average firms' profits, and on the probability distribution of market structure.

B. DYNAMIC ENTRY-EXIT GAME

Now, consider the dynamic (forward-looking) version of the entry-exit game. A Markov Perfect Equilibrium (MPE) in this game can be described as a vector of probabilities $\mathbf{P} \equiv \{P_i(x_t) : i \in \{MD, BK\}, x_t \in X\}$ such that, for every (i, x_t):

$$P_i(x_t) = \Phi\left(\tilde{Z}_{it}^P \, \theta_{MD} + \tilde{e}_{it}^P\right)$$

where $\tilde{Z}_{it}^P$ and $\tilde{e}_{it}^P$ are defined in the class notes.

QUESTION 8. [20 POINTS] Obtain the MPE that we obtain when we iterate in the equilibrium mapping starting with an initial $\mathbf{P} = \mathbf{0}$. Find other MPEs.

QUESTION 9. [10 POINTS] Compute the steady-state distribution of x_t in the population.

QUESTION 10. [20 POINTS] The same as in Question 4 but using the dynamic game and the MPE in Question 8.

QUESTION 11. [20 POINTS] The same as in Question 5 but using the dynamic game and the MPE in Question 8.

QUESTION 12. [30 POINTS] The same as in Question 6 but using the dynamic game and the MPE in Question 8.

QUESTION 13. [30 POINTS] The same as in Question 7 but using the dynamic game and the MPE in Question 8.

12.4 Problem set #4

QUESTION 1 (25 POINTS): This question deals with the paper by Hendel and Nevo (2006).

(a) Explain the implications on estimated elasticities and market power of ignoring (when present) consumer forward-looking behavior and dynamics in the demand of differentiated storable products. Discuss how the biases depend on the stochastic process of prices (for instance, Hi-Lo pricing versus a more stable price).

(b) Describe the main issues in the estimation of Hendel-Nevo model. Discuss the assumptions that they make to deal with these issues.

QUESTION 2 (25 POINTS): The geographic definition of a local market is an important modelling decision in empirical models of market entry.

(a) Explain the implications on the empirical predictions of these model of using a definition of local that is too broad or too narrow.

(b) Explain the approach in Seim (2006). Discuss its advantages and limitations.

QUESTION 3 (50 POINTS): There is a significant number of empirical applications of static and dynamic models of entry in local markets which find the following empirical regularity: after conditioning on observable market characteristics (for instance, population, income, age) there is a positive correlation between the entry decisions of potential entrants. Three main hypotheses have been proposed to explain this evidence: (1) spillover effects in consumer traffic; (2) information externalities (see **caplin_leahy_1998** [**caplin_leahy_1998**] and Toivanen and Waterson [2005]; and (3) market characteristics which are observable for the firms but unobservable to the researcher.

(a) Explain how these hypotheses can explain the empirical evidence.

(b) Discuss why it is important to distinguish between these hypothesis. Do they have different policy implications?

(c) Consider the data and the empirical application in Toivanen and Waterson (2005). Explain how it is possible to identify empirically the contribution of the three hypotheses.

(d) Consider the dynamic game of entry-exit in the Problem Set of this course. Explain how to extend this model to incorporate information externalities as in **caplin_leahy_1998** (**caplin_leahy_1998**). Discuss identification issues.

12.5 Problem set #5

Consider a market with N firms who can potentially operate in it. We index firms by $i \in \{1,2,...,N\}$. Firms produce and sell a differentiated product. There are S consumers and each consumer buys at most one unit (per period) of this differentiated product. A consumer (indirect) utility of buying firm $i's$ product is:

$$U_i = w_i - p_i + \varepsilon_i$$

w_i is the "quality" of product i which is valued in the same way by every consumer. p_i is the price. And $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_N\}$ are consumer specific preferences which are i.i.d. with a type 1 extreme value distribution with dispersion parameter α. The utility of not buying any of these products is normalized to zero. For simplicity, we consider that there are only two levels of quality, high and low: $w_i \in \{w_L, w_H\}$, with $w_L < w_H$. Firms choose endogenously their qualities and prices. They also decide whether to operate in the market or not. Let n_L and n_H be the number of active firms with low and high quality products, respectively. Then, the demand for an active firm with quality w_i and price p_i is:

$$q_i = \frac{S \exp\left\{\frac{w_i - p_i}{\alpha}\right\}}{1 + n_L \exp\left\{\frac{w_L - p_L}{\alpha}\right\} + n_H \exp\left\{\frac{w_H - p_H}{\alpha}\right\}}$$

where we have imposed the (symmetric) equilibrium restriction that firms with the same quality charge the same price. Inactive firms get zero profit. The profit of an active firm is:

$$\Pi_i = (p_i - c(w_i))\, q_i - F(w_i)$$

where $c(w_i)$ and $F(w_i)$ are the (constant) marginal cost and the fixed cost of producing a product with quality w_i. Each firm decides: (1) whether or not to operate in the market; (2) the quality of its product; and (3) its price. The game that firms play is a sequential game with the following two steps:

Step 1: Firms make entry and quality decisions. This determines n_L and n_H.

Step 2: Given (n_L, n_H), firms compete in prices a la Bertrand.

We start describing the Bertrand equilibrium at step 2 of the game.

QUESTION 1. [10 POINTS] Show that the best response functions of the Bertrand game in step 2 have the following form.

$$p_L = c_L + \alpha \left[1 - \frac{\exp\left\{\frac{w_L - p_L}{\alpha}\right\}}{1 + n_L \exp\left\{\frac{w_L - p_L}{\alpha}\right\} + n_H \exp\left\{\frac{w_H - p_H}{\alpha}\right\}}\right]^{-1}$$

$$p_H = c_H + \alpha \left[1 - \frac{\exp\left\{\frac{w_H - p_H}{\alpha}\right\}}{1 + n_L \exp\left\{\frac{w_L - p_L}{\alpha}\right\} + n_H \exp\left\{\frac{w_H - p_H}{\alpha}\right\}}\right]^{-1}$$

ANSWER:

Note that equilibrium prices depend on (n_L, n_H).

QUESTION 2. [30 POINTS] Write a computer program that computes equilibrium prices in this Bertrand game. For given values of the structural parameters (for instance, $\alpha = 1$, $w_L = 2$, $w_H = 4$, $c_L = 1$, $c_H = 2$) calculate equilibrium prices for every possible combination of (n_L, n_H) given that $N = 4$. Present the results in a table.

n_L	n_H	p_L	p_H
1	0	?	?
0	1	?	?
1	1	?	?
2	0	?	?
...	...	...	...

Now, consider the game at step 1. It is useful to define the indirect variable profit function that results from the Bertrand equilibrium in step 2 of the game. Let $\Pi_L(n_L, n_H)$ and $\Pi_H(n_L, n_H)$ be this indirect variable profit, that is, $\Pi_L(n_L, n_H) = (p_L - c_L)q_L$ and $\Pi_H(n_L, n_H) = (p_H - c_H)q_H$, where prices and quantities are equilibrium ones.

QUESTION 3. [10 POINTS] Show that: $\Pi_L(n_L, n_H) = \alpha S q_L/(S - q_L)$ and $\Pi_H(n_L, n_H) = \alpha S q_H/(S - q_H)$.

Let $n_{L(-i)}$ and $n_{H(-i)}$ be the number of low and high quality firms, respectively, excluding firm i. Let's use $w_i = \varnothing$ to represent no entry. And let $b(n_{L(-i)}, n_{H(-i)})$ be the best response mapping of a firm at step 1 of the game.

QUESTION 4. [10 POINTS] Show that the best response function $b(n_{L(-i)}, n_{H(-i)})$ can be described as follows:

$$b(n_{L(-i)}, n_{H(-i)}) = \begin{cases} \varnothing \quad if \quad \left[\begin{array}{c} \{\Pi_L(n_{L(-i)}+1, n_{H(-i)}) - F_L < 0\} \\ \text{and } \{\Pi_H(n_{L(-i)}, n_{H(-i)}+1) - F_H < 0\} \end{array} \right] \\ w_L \quad if \quad \left[\begin{array}{c} \{\Pi_L(n_{L(-i)}+1, n_{H(-i)}) - F_L \geq 0\} \\ \text{and } \{\Pi_L(n_{L(-i)}+1, n_{H(-i)}) - F_L > \Pi_H(n_{L(-i)}, n_{H(-i)}+1) - F_H\} \end{array} \right] \\ w_H \quad if \quad \left[\begin{array}{c} \{\Pi_H(n_{L(-i)}, n_{H(-i)}+1) - F_H \geq 0\} \\ \text{and } \{\Pi_L(n_{L(-i)}+1, n_{H(-i)}) - F_L \leq \Pi_H(n_{L(-i)}, n_{H(-i)}+1) - F_H\} \end{array} \right] \end{cases}$$

Now, suppose that a component of the fixed cost is private information of the firm: that is, $F_i(w_L) = F_L + \xi_{iL}$ and $F_i(w_H) = F_H + \xi_{iH}$, where F_L and F_H are parameters and ξ_{iL} and ξ_{iH} are private information variables which are iid extreme value distributed

across firms. In this Bayesian game a firm's strategy is a function of her own private information $\xi_i \equiv (\xi_{iL},\xi_{iH})$ and of the common knowledge variables (that is, parameters of the model and market size S). Let $\omega(\xi_i,S)$ be a firm's strategy function. A firm's strategy can be also described in terms of two probabilities: $P_L(S)$ and $P_H(S)$, such that:

$$P_L(S) \equiv \int I\{\omega(\xi_i,S) = w_L\}\, dF_\xi(\xi_i)$$

$$P_H(S) \equiv \int I\{\omega(\xi_i,S) = w_H\}\, dF_\xi(\xi_i)$$

where $I\{.\}$ is the indicator function and F_ξ is the CDF of ξ_i.

QUESTION 5. [20 POINTS] Show that a Bayesian Nash Equilibrium (BNE) in this game is a pair (P_L,P_H) that is a solution to the following fixed problem:

$$P_L = \frac{\exp\{\Pi_L^e(P_L,P_H) - F_L\}}{1+\exp\{\Pi_L^e(P_L,P_H) - F_L\} + \exp\{\Pi_H^e(P_L,P_H) - F_H\}}$$

$$p_H = \frac{\exp\{\Pi_H^e(P_L,P_H) - F_H\}}{1+\exp\{\Pi_L^e(P_L,P_H) - F_L\} + \exp\{\Pi_H^e(P_L,P_H) - F_H\}}$$

with:

$$\Pi_L^e(P_L,P_H) = \sum_{n_{L(-i)},n_{H(-i)}} \Pi_L(n_{L(-i)}+1,n_{H(-i)})\; T(n_{L(-i)},n_{H(-i)}|N-1,P_L,P_H)$$

$$\Pi_H^e(P_L,P_H) = \sum_{n_{L(-i)},n_{H(-i)}} \Pi_H(n_{L(-i)},n_{H(-i)}+1)\; T(n_{L(-i)},n_{H(-i)}|N-1,P_L,P_H)$$

where $T(x,y|n,p_1,p_2)$ is the PDF of a trinomial distribution with parameters (n,p_1,p_2).

QUESTION 6. [50 POINTS] Write a computer program that computes the BNE in this entry/quality game. Consider $N = 4$. For given values of the structural parameters, calculate the equilibrium probabilities $(P_L(S),P_H(S))$ for a grid of points for market size S. Present a graph for $(P_L(S),P_H(S))$ (on the vertical axis) on S (in the horizontal axis). Does the proportion of high quality firms depend on market size?

QUESTION 7. [30 POINTS] Define the function $\lambda(S) \equiv P_H(S)/P_L(S)$ that represents the average ratio between high and low quality firms in the market. Repeat the same exercise as in Question 1.6. but for three different values of the ratio F_H/F_L. Present a graph of $\lambda(S)$ on S for the three values of F_H/F_L. Comment the results.

QUESTION 8. [50 POINTS] A regulator is considering a policy to encourage the production of high quality products. The policy would provide a subsidy of 20% of the additional fixed cost of producing a high quality product. That is, the new fixed cost of producing a high quality product would be $F_H^* = F_H - 0.20*(F_H - F_L)$. Given a parametrization of the model, obtain the equilibrium before and after the policy and calculate the effect of the policy on: (1) prices; (2) quantities; (3) firms' profits; (4) average consumers' surplus; and (5) total surplus.

Suppose that the researcher observes a random sample of M isolated markets, indexed by m, where these N firms compete. More specifically, the researcher observes:

$$Data = \{S_m,\ n_{Hm},\ n_{Lm},\ q_{Hm},\ q_{Lm},\ p_{Hm},\ p_{Lm} : m = 1,2,...,M\}$$

For instance, consider data from the hotel industry in a region where "high quality" is defined as four stars or more (low quality as three stars or less). We incorporate two sources of market heterogeneity in the econometric model (that is, unobservables for the researcher).

(A) Consumers' average valuations: $w_{Lm} = w_L + \eta_m^{(wL)}$ and $w_{Hm} = w_H + \eta_m^{(wH)}$, where $\eta_m^{(wL)}$ and $\eta_m^{(wH)}$ are zero mean random variables.
(B) Marginal costs: $c_{Lm} = c_L + \eta_m^{(cL)}$ and $c_{Hm} = c_H + \eta_m^{(cH)}$, where $\eta_m^{(cL)}$ and $\eta_m^{(cH)}$ are zero mean random variables.

We assume that the vector of unobservables $\eta_m \equiv \{\eta_m^{(wL)}, \eta_m^{(wH)}, \eta_m^{(cL)}, \eta_m^{(cH)}\}$ is iid over markets and independent of market size S_m. We also assume that these variables are common knowledge. We want to use these data to estimate the structural parameters $\theta = \{\alpha, w_j, c_j, F_j : j = L,H\}$.

QUESTION 9. [30 POINTS] Show that the econometric model can be described in terms of three sets of equations.

(1) Demand equations: For $j \in \{L,H\}$ let s_{jm} be the market share q_{jm}/S_m. Then:

$$\ln\left(\frac{s_{jm}}{1 - s_{Lm} - s_{Hm}}\right) = \frac{w_j}{\alpha} - \frac{1}{\alpha}\, p_{jm} + \frac{\eta_m^{(wj)}}{\alpha} \qquad \text{if } n_{jm} > 0$$

(2) Price equations: For $j \in \{L,H\}$:

$$p_{jm} = c_j + \alpha\left(\frac{1}{1 - s_{jm}}\right) + \eta_m^{(cj)} \qquad \text{if } n_{jm} > 0$$

(3) Entry/Quality choice: Suppose that from the estimation of (1) and (2) we can obtain consistent estimates of η_m as residuals. After that estimation, we can treat η_m as "observable" (though we should account for estimation error). Then,

$$\Pr(n_{Lm}, n_{Hm} | S_m, \eta_m) = T(n_{Lm}, n_{Hm} | N, P_L(S_m, \eta_m), P_H(S_m, \eta_m))$$

where $P_L(S_m, \eta_m), P_H(S_m, \eta_m)$ are equilibrium probabilities in market m.

QUESTION 10. [30 POINTS] Discuss in detail the econometric issues in the estimation of the parameters $\{w_L, w_H, \alpha\}$ from the demand equations: for $j \in \{L, H\}$:

$$\ln\left(\frac{s_{jm}}{1-s_{Lm}-s_{Hm}}\right) = \frac{w_j}{\alpha} - \frac{1}{\alpha}\, p_{jm} + \frac{\eta_m^{(wj)}}{\alpha} \qquad \text{if } n_{jm} > 0$$

Propose and describe in detail a method that provides consistent estimates of $\{w_L, w_H, \alpha\}$.

QUESTION 11. [30 POINTS] Suppose for the moment that α has not been estimated from the demand equations. Discuss in detail the econometric issues in the estimation of the parameters $\{c_L, c_H, \alpha\}$ from the pricing equations: for $j \in \{L, H\}$:

$$p_{jm} = c_j + \alpha\left(\frac{1}{1-s_{jm}}\right) + \eta_m^{(cj)} \qquad \text{if } n_{jm} > 0$$

Propose and describe in detail a method that provides consistent estimates of $\{c_L, c_H, \alpha\}$. What if α has been estimated in a first step from the demand equations? Which are the advantages of a joint estimation of demand and supply equations?

QUESTION 12. [50 POINTS] For simplicity, suppose that the parameters $\{w_L, w_H, c_L, c_H, \alpha\}$ are known and that η_m is observable (that is, we ignore estimation error from the first step estimation). We want to estimate the fixed costs F_L and F_H using information on firms' entry/quality choices. Discuss in detail the econometric issues in the estimation of these parameters. Propose and describe in detail a method that provides consistent estimates of $\{F_L, F_H\}$.

QUESTION 13. [50 POINTS] Suppose that you incorporate a third source of market heterogeneity in the model:

> **(C) Fixed costs: $F_{Lm} = F_L + \eta_m^{(FL)}$ and $F_{Hm} = F_H + \eta_m^{(FH)}$, where $\eta_m^{(FL)}$ and $\eta_m^{(FH)}$ are zero mean random variables, and they are common knowledge to the players.**

Explain which are the additional econometric issues in the estimation of $\{F_L, F_H\}$ when we have these additional unobservables. Propose and describe in detail a method that provides consistent estimates of $\{F_L, F_H\}$ and the distribution of $\{\eta_m^{(FL)}, \eta_m^{(FH)}\}$.

QUESTION 14. [50 POINTS] Consider the econometric model without $\{\eta_m^{(FL)}, \eta_m^{(FH)}\}$. Suppose that S_m is log normally distributed and $\eta_m \equiv \{\eta_m^{(wL)}, \eta_m^{(wH)}, \eta_m^{(cL)}, \eta_m^{(cH)}\}$ has a normal distribution with zero means. Generate a random sample of $\{S_m, \eta_m\}$ with sample size of $M = 500$ markets. Given a parametrization of the model, for every value $\{S_m, \eta_m\}$ in the sample, solve the model and obtain the endogenous

variables $\{n_{Hm}, n_{Lm}, q_{Hm}, q_{Lm}, p_{Hm}, p_{Lm}\}$. Present a table with the summary statistics of these variables: for instance, mean, median, standard deviation, minimum, maximum.

QUESTION 15. [50 POINTS] Write a computer program that implements the method for the estimation of the demand that you proposed in Question 10. Apply this method to the data simulated in Question 14. Present and comment the results.

QUESTION 16. [50 POINTS] Write a computer program that implements the method for the estimation of the pricing equations that you proposed in Question 11. Apply this method to the data simulated in Question 14. Present and comment the results.

QUESTION 17. [100 POINTS] Write a computer program that implements the method for the estimation of the entry/quality choice game that you proposed in Question 12. Apply this method to the data simulated in Question 14. Present and comment the results.

QUESTION 18. [50 POINTS] Use the estimated model to evaluate the policy in question 8. Present a table that compares the average (across markets) "actual" and estimated effects of the policy on: (1) prices; (2) quantities; (3) firms' profits; (4) average consumers' surplus; and (5) total surplus.

12.6 Problem set #6

In the paper *"The Interpretation of Instrumental Variables Estimators in Simultaneous Equations Models with an Application to the Demand for Fish,"* (REStud, 2000), Angrist, Graddy and Imbens consider the following random coefficients model of supply and demand for an homogeneous product:

$$\text{Inverse Demand:} \quad p_t = x_t\,\beta^D - \left(\alpha^D + \upsilon_t^D\right)\,q_t + \varepsilon_t^D$$

$$\text{Inverse Supply:} \quad p_t = x_t\,\beta^S + \left(\alpha^S + \upsilon_t^S\right)\,q_t + \varepsilon_t^S$$

where p_t is logarithm of price; q_t is the logarithm of the quantity sold; and ε_t^D, ε_t^S, υ_t^D and υ_t^S are unobservables which have zero mean conditional on x_t. The variables υ_t^D and υ_t^S account for random shocks in the price elasticities of demand and supply. Suppose that the researcher has a sample $\{p_t, q_t, x_t : t = 1, 2, ..., n\}$ and is interested in the estimation of the demand parameters β^D and α^D.

1. Explain why instrumental variables (or 2SLS) provides inconsistent estimates of the parameters β^D and α^D.
2. Descrine an estimation method that provides consistent estimates of β^D and α^D.
3.

12.7 Problem set #7

Mitsubishi entered the Canadian automobile market in September 2002. You can consider this to be an exogenous change. Subsequently, the firm had to decide in which local markets to open dealerships. This, you should consider to be endogenous choices.

1. How could you use this type of variation to estimate a model of entry like **bresnahan_reiss_1988** (**bresnahan_reiss_1988**, 1990, **bresnahan_reiss_1991**)? What variation in the data will be useful to identify which underlying economic parameters? How would you learn about or control for the competitiveness of market operation?
2. (It is not necessary to derive any equations, although you can if it helps your exposition.).
3. Could you use the same data to estimate an entry model like Berry (1992)? How?
4. How would you use data for this industry to estimate the lower bound on concentration in the sense of Sutton?
5. Give an example of an economic question that you would be able to address with this type of variation over time —entry by a new firm— that the previous authors were unable to address using only cross sectional data.
6.

12.8 Problem set #8

In the paper *"The valuation of new goods under perfect and imperfect competition,"* Jerry Hausman estimates a demand system for ready-to eat cereals using panel data on quantities and prices for multiple markets (cities), brands and quarters. The demand system is (Deaton-Muellbauer demand system):

$$w_{jmt} = \alpha_j^0 + \alpha_m^1 + \alpha_t^2 + \sum_{k=1}^{J} \beta_{jk} \ln(p_{kmt}) + \gamma_j \ln(x_{mt}) + \varepsilon_{jmt}$$

where: j, m and t are the product, market (city) and quarter subindexes, respectively; x_{mt} represents exogenous market characteristics such as population and average income. There are not observable cost shifters. The terms α_j^0, α_m^1 and α_t^2 represent product, market and time effects, respectively, which are captured using dummies. As instruments for prices, Hausman uses average prices is nearby markets. More specifically, the instrument for price p_{jmt} is z_{jmt} which is defined as:

$$z_{jmt} = \frac{1}{\#(R_m)} \sum_{\substack{m' \neq m \\ m' \in R_m}} p_{jm't}$$

where R_m is the set of markets nearby market m, and, $\#(R_m)$ is the number of elements in that set.

1. Explain under which economic assumptions, on supply or price equations, these instruments are valid.
2. Describe how Deaton-Muellbauer demand system can be used to calculate the value of a new product.
3. Comment the limitations of this approach as a method to evaluate the effects of new product on consumers' welfare and firms' profits.

4. Explain how the empirical literature on demand models in characteristics space deals with some of the limitations that you have mentioned in question (c).

5.

12.9 Problem set #9

Consider Berry-Levinshon-Pakes (BLP) model for the demand of a differentiated product. The (indirect) utility of buying product j for consumer i is:

$$U_{ij} = (\beta_1 + \omega_{1i})x_{1j} + ... + (\beta_K + \omega_{Ki})x_{Kj} - \alpha\ p_j + \xi_j + \varepsilon_{ij}$$

where α, β_1, ..., and β_K are parameters; $\omega_i \equiv (\omega_{1i}, \omega_{2i}, ..., \omega_{Ki})$ is a vector of normal random variables (with zero mean); and $\varepsilon_i \equiv (\varepsilon_{i1}, \varepsilon_{i2}, ..., \varepsilon_{iJ})$ is a vector of independent extreme value random variables.

1. Describe in detail BLP estimation method.
2. Explain why it is important to allow for consumer heterogeneity in the marginal utility with respect to product characteristics.
3. A key identifying assumption in BLP method is that unobserved product characteristics, ξ_j, are not correlated with observed product characteristics other than price, $(x_{1j}, x_{2j}, ..., x_{Kj})$. Comment on this assumption.
4. Suppose that there is only one observable product characteristic, x_j, that we can interpret as a measure of product quality. Let x_j^* is the "true" quality of product j, which is unobservable to the researcher. That is, $x_j = x_j^* + e_j$ where e_j is measurement error which is assumed independent of x_j^*. According to this model, the unobservable ξ_j is equal to $-\beta e_j$. Show that the type of instrumental variables proposed by BLP can still be valid in this model with measurement error in quality.

5.

12.10 Problem set #10

Consider an oligopoly industry in which competition takes place at the level of local markets. For concreteness, suppose that there are only two firms in the industry: firm 1 and firm 2. There are M local markets, where M is a large number. Consider the following adaptation to this industry of the simultaneous equations model in *Olley and Pakes (1996)*.

$$\text{Production Function:} \quad y_{imt} = \alpha_{Li}\, \ell_{imt} + \alpha_{Ki}\, k_{imt} + \omega_{imt} + e_{imt}$$

$$\text{Investment Function:} \quad i_{imt} = f_i\left(k_{1mt}, k_{2mt}, \omega_{1mt}, \omega_{2mt}, r_{mt}\right)$$

$$\text{Stay-in-the-market decision:} \quad \chi_{imt} = I\{\omega_{imt} \geq \omega_i^*\left(k_{1mt}, k_{2mt}, r_{mt}\right)\}$$

where: i is the firm subindexl; m is the local-market subindex; t is the time subindex; r_{mt} represents input prices in market m at period t; and all the other variables and parameters have the same interpretation as in Olley-Pakes. Following Olley-Pakes we assume that labor is a perfectly flexible input and that new investment is not productivity until next

period (that is, time-to-build). We are interested in the estimation of the production function parameters $\{\alpha_{L1}, \alpha_{K1}, \alpha_{L2}, \alpha_{K2}\}$.

1. Explain why a direct application of Olley-Pakes method to this model will not provide consistent estimates of the parameters of interest.
2. Describe how Olley-Pakes method can be adapted/extended to this industry and data to obtain a consistent estimator of $\{\alpha_{L1}, \alpha_{K1}, \alpha_{L2}, \alpha_{K2}\}$.
3. Suppose that the average productivity of labor is larger in markets where both firms are active (relative to markets where only one of the two firms is active). Mention different hypotheses that might explain this evidence. Explain how one can use the estimated model to measure the contribution of each of these hypothesis to the observed differential in the average productivity of labor.
4.

12.11 Problem set #11

Consider the following description of a hotel industry. There are N firms/hotel chains in the industry. These firms compete in independent local markets (cities). We index hotel chains by $i \in \{1,2,...,N\}$ and local markets by $m \in \{1,2,...,N\}$. The product that hotels sell is vertically differentiated. For simplicity, we consider that there are only two levels of quality, high (H) and low (L). At each local market, each firm decides whether or not to operate in the market, the quality of its product, and its price. The game that hotel chains play is a sequential game with the following two steps. Step 1: firms make entry and quality decisions. This step determines the number of low and high quality hotels in the market: n_m^L and n_m^H respectively. Step 2: Given (n_m^L, n_m^H), firms compete in prices a la Bertrand. Associated to the Bertrand equilibrium we can define the (indirect) variable profit functions $V_L(n_m^L, n_m^H, S_m)$ and $V_H(n_m^L, n_m^H, S_m)$: that is, $V_L(n_m^L, n_m^H, S_m)$ $(V_H(n_m^L, n_m^H, S_m))$ is the variable profit of a low (high) quality hotel in a market with size S_m, with n_m^L low quality hotels and with n_m^H high quality hotels. Total operating costs are: $\Pi_{Lim} = V_L(n_m^L, n_m^H, S_m) - F_L - \varepsilon_{Lim}$ and $\Pi_{Him} = V_H(n_m^L, n_m^H, S_m) - F_H - \varepsilon_{Him}$, where F_L and F_H are the fixed costs for low and high quality firms, respectively, and ε_{Lim} and ε_{Him} are private information shocks which are iid extreme value distributed across firms and markets. A firm's strategy can be described in terms of two probabilities: the probability of being active with low quality, P_L, and the probability of being active and high quality, P_H.

1. Show that a Bayesian Nash Equilibrium (BNE) in this game is a pair (P_L, P_H) that is a solution to the following fixed problem:

$$P_L = \frac{\exp\{V_L^e(P_L,P_H) - F_L\}}{1 + \exp\{V_L^e(P_L,P_H) - F_L\} + \exp\{V_H^e(P_L,P_H) - F_H\}}$$

$$P_H = \frac{\exp\{V_H^e(P_L,P_H) - F_H\}}{1 + \exp\{V_L^e(P_L,P_H) - F_L\} + \exp\{V_H^e(P_L,P_H) - F_H\}}$$

with:

$$V_L^e(P_L,P_H) = \sum_{n_{L(-i)}, n_{H(-i)}} V_L(n_{L(-i)} + 1, n_{H(-i)})\, T(n_{L(-i)}, n_{H(-i)} | N-1, P_L, P_H)$$

$$V_H^e(P_L,P_H) = \sum_{n_{L(-i)}, n_{H(-i)}} V_H(n_{L(-i)}, n_{H(-i)} + 1)\, T(n_{L(-i)}, n_{H(-i)} | N-1, P_L, P_H)$$

where $T(x,y|n,p_1,p_2)$ is the PDF of a trinomial distribution with parameters (n,p_1,p_2).

2. Suppose that the indirect profit functions $V_L(n_L,n_H,S)$ and $V_H(n_L,n_H,S)$ are known, that is, they have been estimated using price an quantity data). The researcher observes the sample $\{n_{Hm}, n_{Lm}, S_m : m = 1,2,...,M\}$. We want to estimate the fixed costs F_L and F_H using information on firms' entry/quality choices. Discuss in detail the econometric issues in the estimation of these parameters. Propose and describe in detail a method that provides consistent estimates of $\{F_L,F_H\}$.
3. Suppose that you incorporate unobserved market heterogeneity in fixed costs: $F_{Lm} = F_L + \eta_m^L$ and $F_{Hm} = F_H + \eta_m^H$, where η_m^L and η_m^H are zero mean random variables, and they are common knowledge to the players. Explain which are the additional econometric issues in the estimation of $\{F_L,F_H\}$ when we have these additional unobservables. Propose and describe in detail a method that provides consistent estimates of $\{F_L,F_H\}$ and the distribution of $\{\eta_m^L,\eta_m^H\}$.

12.12 Problem set #12

Consider an extension of Rust's machine replacement model Rust (1987) that incorporates asymmetric information in the market of machines. A firm produces at several independent plants (indexed by i) that operate independently. Each plant has a machine. The cost of operation and maintenance of a machine increases with the age of the machine. Let x_{it}.be the age of the machine at plant i and at period t. There are two types of machines according to their maintenance costs: low and high maintenance costs. When the firm's manager decides to buy a machine, she does not observe its type. However, the manager learns this type just after one year of operation. The maintenance cost is: $c_i\, x_{it} + \varepsilon_{it}(0)$ where $c_i \in \{\theta_L,\theta_H\}$ is a parameter and $\varepsilon_{it}(0)$ is a component of the maintenance cost that is unobserved for the researcher. There is a cost of replacing an old machine by a new one. This replacement cost is: $RC + \varepsilon_{it}(1)$ where RC is a parameter, and $\varepsilon_{it}(1)$ is a component of the maintenance cost that is unobserved for the researcher. The firm has decide when to replace a machine in order to minimize the present value of the sum of maintenance and replacement costs. Suppose that the researcher has a random sample of machines.

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