Identification and Estimation of Dynamic Games when Players' Beliefs Are Not in Equilibrium

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Abstract

This paper deals with the identification and estimation of dynamic games when players' beliefs about other players' actions are biased, i.e., beliefs do not represent the probability distribution of the actual behavior of other players conditional on the information available. First, we show that an exclusion restriction, typically used to identify empirical games, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs in dynamic games with either finite or infinite horizon. We use this result to construct a simple Likelihood Ratio test of equilibrium beliefs. Second, we prove that this exclusion restriction, together with consistent estimates of beliefs at two points in the support of the variable involved in the exclusion restriction, is sufficient for nonparametric point-identification of players' belief functions as well as useful functions of payoffs. Third, we propose a simple two-step estimation method. We illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains. The key conditions for the identification of beliefs and payoffs in our application are the following: (a) the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit because the existence of sunk entry costs and economies of density in these costs; and (b) firms' beliefs are unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores, though beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs. Furthermore, imposing the restriction of unbiased beliefs generates a substantial attenuation bias in the estimate of competition effects.

Keywords: Dynamic games; Rational behavior; Rationalizability; Identification; Estimation; Market entry-exit.

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1 Introduction

The principle of *revealed preference* (Samuelson, 1938) is a cornerstone in the empirical analysis of decision models, either static or dynamic, single-agent problems or games. Under the principle of *revealed preference*, agents maximize expected payoffs and their actions reveal information on the structure of payoff functions. This simple but powerful concept has allowed econometricians to use data on agents' decisions to identify important structural parameters for which there is very limited information from other sources. Examples of parameters and functions that have been estimated using the principle of revealed preference include, among others, consumer willingness to pay for a product, agents' degree of risk aversion, intertemporal rates of substitution, market entry costs, adjustment costs and switching costs, preference for a political party, or the benefits of a merger. In the context of empirical games, where players' expected payoffs depend on their beliefs about the behavior of other players, most applications combine the principle of revealed preference with the assumption that players' beliefs about the behavior of other players are *in equilibrium*, in the sense that these beliefs represent the probability distribution of the actual behavior of other players conditional on the information available. The assumption of equilibrium beliefs plays an important role in the identification and estimation of games, and as such, is a mainstay in the empirical game literature. Equilibrium restrictions have identification power even in models with multiple equilibria (Tamer, 2003, Aradillas-Lopez and Tamer, 2008, Bajari, Hong, and Ryan, 2010). Imposing these restrictions contributes to improved asymptotic and finite sample properties of game estimators. Moreover, the assumption of equilibrium beliefs is very useful for evaluating counterfactual policies in a strategic environment. Models where agents' beliefs are endogenously determined in equilibrium not only take into account the direct effect of the new policy on agents' behavior through their payoff functions, but also through an endogenous change in agents' beliefs.

Despite the clear benefit that the assumption of equilibrium beliefs delivers to an applied researcher, there are situations and empirical applications where the assumption is not realistic and it is of interest to relax it. There are multiple reasons why players may have biased beliefs about the behavior of other players in a game. For instance, in games with multiple equilibria, players can be perfectly rational in the sense that they take actions to maximize expected payoffs given their beliefs, but they may have different beliefs about the equilibrium that has been selected. This situation corresponds to the concept of *strategic uncertainty* as defined in Van Huyk, Battalio, and Beil (1990) and Crawford and Haller (1990), and applied by Morris and Shin (2002, 2004), and Heinemann, Nagel, and Ockenfels (2009), among others. For instance, competition in oligopoly industries is often prone to strategic uncertainty (Besanko et al., 2010). Dynamic games of oligopoly competition are typically characterized by multiple equilibria, and the selection between two possible equilibria implies that some firms are better off but others are worse off. Firm managers do not have incentives to coordinate their beliefs in the same equilibrium. They can be very secretive about their own strategies and face significant uncertainty about the strategies of their competitors.[1] Strategic uncertainty may also be an important consideration in the evaluation of a policy change in a strategic environment. Suppose that to evaluate a policy change we estimate an empirical game using data before and after a new policy is implemented. After the implementation of the new policy, some players may believe that others' market behavior will continue according to the same type of equilibrium as before the policy change, while others believe the policy change has triggered the selection of a different type of equilibrium.[2] Thus, at least for some period of time, players' beliefs will be out of equilibrium, and imposing the restriction of equilibrium beliefs may bias the estimates of the effects of the new policy.

While strategic uncertainty under multiple equilibria is a motivation for our study, generally our approach fits any situation where players have limited capacity to reason, in which case it is ideal to place no restriction on what they believe and what they believe that others believe and so on. Indeed, studies in the literature of experimental games commonly find significant heterogeneity in players' elicited beliefs, and that this heterogeneity is often one of the most important factors in explaining heterogeneity in observed behavior in the laboratory.[3] Imposing the assumption of equilibrium beliefs in these applications does not seem reasonable. Interestingly, recent empirical papers establish a significant divergence between stated or elicited beliefs and the beliefs inferred from players' actions using, for example, revealed preference-based methods (see Costa-Gomes and Weizsäcker, 2008, and Rutström and Wilcox, 2009). The results in our paper can be applied to estimate beliefs and payoffs, using either observational or laboratory data, when the researcher wants to allow for the possibility of biased beliefs but she does not have data on elicited beliefs, or data on elicited beliefs is limited to only a few states of the world.

In this paper we study nonparametric identification, estimation, and inference in dynamic discrete games of incomplete information when we assume that players are rational, in the sense that each player takes an action that maximizes his expected payoff given some beliefs, but we relax the assumption that these beliefs are in equilibrium. In the class of models that we consider, a player's belief is a probability distribution over the space of other players' actions conditional on some state variables, or the player's information set. Beliefs are biased, or not in equilibrium, if they are different from the actual probability distribution of other players' actions conditional on the state variables of the model. We consider a nonparametric specification of beliefs and treat these probability distributions as *incidental parameters* that, together with the structural parameters in payoff functions and transition probabilities, determine the stochastic process followed by players'

[1] See Morris and Shin (2002) for examples of models with strategic uncertainty and related experimental evidence.

[2] For example in a game of investment there may be high investment or low investment equilibria. Prior to the policy change the game may be in the high investment equilibrium, and after the policy change one player believes that this equilibrium will continue to prevail while the other player switches to behavior according to the low equilibrium.

[3] See Camerer (2003) and recent papers by Costa-Gomes and Weizsäcker (2008), and Palfrey and Wang (2009).

actions. Our framework includes as a particular case games where the source of biased beliefs is strategic uncertainty, i.e., every player has beliefs that correspond to an equilibrium of the game but their beliefs are not 'coordinated'. However, our identification and estimation results do not rely on this restriction and our approach is therefore not restricted to this case.

The recent literature on identification of games of incomplete information is based on two main assumptions: (i) players' beliefs are in equilibrium such that they can be identified, or consistently estimated, by simply using a nonparametric estimator of the distribution of players' actions conditional on the state variables; and (ii) there is an exclusion restriction in the payoff function such that there is a player-specific state variable which enters the payoff of the player and is excluded from the payoffs of other players, but is known to other players and thus influences their beliefs (Bajari et al., 2010, in static games of incomplete information, and Tamer, 2003, and Bajari, Hong and Ryan, 2010, in static games of complete information). When players beliefs are not in equilibrium, or when the exclusion restriction is not satisfied, the model is not identified.

In this context, this paper presents two main identification results. First, we show that the exclusion restriction alone provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs, which apply to dynamic games with either finite or infinite horizon. Under this type of exclusion restriction, the observed behavior of a player identifies a function that depends only on her beliefs about the behavior of other player, and not on her preferences. Under the null hypothesis of equilibrium beliefs, this identified function of beliefs should be equal to the same function but where we replace beliefs by the actual expected behavior of the other player. We show that this result can be used to construct a formal test of the null hypothesis of equilibrium beliefs, which a researcher can use before deciding whether or not to impose equilibrium restrictions in estimation. The test statistic is a simple Likelihood Ratio of the (restricted) probability of observing the data given equilibrium beliefs to the (unrestricted) probability of observing the data with no assumptions about beliefs. This is the core result of our paper, and also serves to highlight the fact that empirical games are often over-identified, as they assume equilibrium beliefs at every point in the support.

Second, we prove that this exclusion restriction, together with consistent estimates of beliefs at two points in the support of the *player-specific state variable* (i.e., the state variable that satisfies the exclusion restriction), is sufficient for nonparametric point-identification of players' belief functions and useful functions of players' payoffs. We provide additional conditions under which payoffs are fully identified. It is worth emphasizing that in deriving this result we impose no restrictions on the evolution of beliefs, and that the result applies to games with either finite or infinite horizon. The consistent estimates of beliefs at two points of the support may come either from an assumption of unbiased beliefs at these points in the state space, or from data on elicited beliefs for some values of the state variables. We also discuss four different approaches to select the values of the player-specific state variable where we impose the restriction of unbiased beliefs: (a) using the test of unbiased beliefs; (b) testing for the monotonicity of beliefs and using this restriction; (c) minimization of beliefs bias; and (d) most visited states.

Third, we propose a simple two-step non-parametric estimation method to recover beliefs and payoffs from the data. Given that in most applications the researcher assumes a parametric specification of the payoff function, we also illustrate how one can extend the estimation method to accommodate a parametric specification.

Finally, we illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains. We use Monte Carlo experiments for two primary purposes. First, we study the properties of the test of equilibrium beliefs under two different data generating processes, one where beliefs are in equilibrium (the null is true) and one where they are not (the null is false). These experiments suggest that our test has strong power to reject the null hypothesis when it is false, and has size close to the true probability of type I error when the null is true. Second, we use the experiments to study the key trade-off that a researcher faces when deciding whether or not to impose equilibrium restrictions: the estimation bias induced by imposing equilibrium restrictions when they are not true against the higher variance associated with ignoring equilibrium restrictions when they are true. To study this issue, we estimate beliefs and payoffs with and without equilibrium restrictions in the same two DGP's that we use for assessing the test. The experiments show a significant bias when a researcher wrongly imposes equilibrium restrictions. There is also a substantial loss in efficiency and an increase in finite sample bias when we do not impose equilibrium restrictions and they do hold in the DGP. This underscores the importance of testing for equilibrium beliefs before deciding on an estimation strategy, as it is costly to ignore equilibrium restrictions when they hold, and costly to impose them when they do not.

To illustrate our model and methods in the context of an empirical application, we consider a dynamic game of store location between McDonalds and Burger King. There has been very little work on the bounded rationality of firms, as most empirical studies on bounded rationality have concentrated on individual behavior.[4] The key conditions for the identification of beliefs and payoffs in our application are the following. The first condition is an exclusion restriction in a firm's profit function that establishes that the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit because of the existence of sunk entry costs and economies of density in these costs. The second condition restricts firms' beliefs to be unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores. However, beliefs are unrestricted, and

[4] An exception is the paper by Goldfarb and Xiao (2011) that studies entry decisions in the US local telephone industry and finds significant heterogeneity in firms' beliefs about other firms' strategic behavior.

potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs for Burger King. More specifically, we find that this firm underestimated the probability of entry of McDonalds in markets that were relatively far away from McDonalds' network of stores. Furthermore, imposing the restriction of unbiased beliefs generates a substantial attenuation bias in the estimate of competition effects.

This paper builds on the literature on estimation of dynamic games of incomplete information (see Aguirregabiria and Mira, 2007, Bajari, Benkard and Levin, 2007, Pakes, Ostrovsky and Berry, 2007, and Pesendorfer and Schmidt-Dengler, 2008). All the papers in this literature assume that the data come from a Markov Perfect Equilibrium. We relax that assumption.

Our research is also related to Aradillas-Lopez and Tamer (2008) who study the identification power of the assumption of equilibrium beliefs in simple static games using the notion of *level-k rationality* to construct informative bounds around players' behavior. In relaxing the assumption of Nash equilibrium, they assume that players are level-k rational with respect to their beliefs about their opponents' behavior, a concept which derives from the notion of rationalizability (Bernheim, 1984, and Pearce, 1984). Their approach is especially useful in the context of static two-player games with binary or ordered decision variables under the condition that payoffs are supermodular in the actions of the two players. These conditions yield a sequence of closed form bounds on players' choice probabilities that grow tighter as the level of rationality k gets larger.[5] However, the derivation of bounds on choice probabilities in dynamic games is significantly more complicated. Even in simple two-player binary-choice dynamic games with supermodular payoffs, the value function is not supermodular at every value of the state variables (Aguirregabiria, 2008). As such, obtaining bounds that shrink monotonically as the level of rationality of players increases is not possible in a dynamic game, and the assumption of level-k rationality is of limited use. We are instead totally agnostic about the level of players' rationality.

Our paper also complements the growing literature on the use of data on subjective expectations in microeconometric decision models, especially the contributions of Walker (2003), Manski (2004), Delavande (2008), Van der Klaauw and Wolpin (2008), and Pantano and Zheng (2013). It is commonly the case that data on elicited beliefs has the form of unconditional probabilities, or probabilities that are conditional only on a strict subset of the state variables in the postulated model. In this context, the framework that we propose in this paper can be combined with the incomplete data on elicited beliefs in order to obtain nonparametric estimates of the complete conditional probability distribution describing an individual's beliefs. Most of these previous empirical papers on biased beliefs consider dynamic single-agent models and beliefs about exogenous future events. We extend that literature by looking at dynamic games and biased beliefs about other

[5] Other papers on the estimation of static games under rationalizability are Kline and Tamer (2012), Uetake and Watanabe (2013), An (2010), and Gillen (2010).

players' behavior.

The rest of the paper includes the following sections. Section 2 presents the model and basic assumptions. In section 3, we present our identification results. Section 4 describes estimation methods and testing procedures. Section 5 presents our Monte Carlo experiments. The empirical application is described in section 6. We summarize and conclude in section 7.

2 Model

2.1 Basic framework

This section presents a dynamic game of incomplete information where N players make discrete choices over T periods. We use indexes $i, j \in \{1, 2, ...N\}$ to represent players, and the index $-i$ to represent all players other than i. T can be finite or infinite, and time is discrete and is indexed by $t \in \{1, 2, ..., T\}$. Every period t, players simultaneously choose one out of A alternatives from the choice set $\mathcal{Y} = \{0, 1, ..., A-1\}$. Let $Y_{it} \in \mathcal{Y}$ represent the choice of player i at period t. Each player makes this decision to maximize his expected and discounted payoff, $\mathbb{E}_t(\sum_{s=0}^{T} \beta^s \Pi_{i,t+s})$, where $\beta \in (0,1)$ is the discount factor, and Π_{it} is his payoff at period t. The one-period payoff function has the following structure:

$$\Pi_{it} = \pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) \tag{1}$$

$\pi_{it}(.)$ is a real-valued function. $\mathbf{Y}_{-it}$ represents the current action of the other players. $\mathbf{X}_t$ is a vector of state variables which are common knowledge for all players. $\boldsymbol{\varepsilon}_{it} \equiv (\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(A-1))$ is a vector of private information variables for firm i at period t.

The vector of common knowledge state variables is $\mathbf{X}_t$, and it evolves over time according to the transition probability function $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$ where $\mathbf{Y}_t \equiv (Y_{1t}, Y_{2t}, ..., Y_{Nt})$. The vector of private information shocks $\boldsymbol{\varepsilon}_{it}$ is independent of $\mathbf{X}_t$ and independently distributed over time and players. Without loss of generality, these private information shocks have zero mean. The cumulative distribution function of $\boldsymbol{\varepsilon}_{it}$ is given by G_{it}, which is strictly increasing on $\mathbb{R}^A$.

EXAMPLE 1: Dynamic game of market entry and exit. Consider N firms competing in a market. Each firm sells a differentiated product. Every period, firms decide whether or not to be active in the market. Then, incumbent firms compete in prices. Let $Y_{it} \in \{0, 1\}$ represent the decision of firm i to be active in the market at period t. The profit of firm i at period t has the structure of equation (1), $\Pi_{it} = \pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it})$. We now describe the specific form of the payoff function π_{it} and the state variables $\mathbf{X}_t$ and ε_{it}. The average profit of an inactive firm, $\pi_{it}(0, \mathbf{Y}_{-it}, \mathbf{X}_t)$, is normalized to zero, such that $\Pi_{it} = \varepsilon_{it}(0)$. The profit of an active firm is $\pi_{it}(1, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(1)$ where:

$$\pi_{it}(1, \mathbf{Y}_{-it}, \mathbf{X}_t) \quad = \quad H_t \left(\theta_i^M - \theta_i^D \sum_{j \neq i} Y_{jt} \right) - \theta_{i0}^{FC} - \theta_{i1}^{FC} Z_{it} - 1\{Y_{it-1} = 0\}\ \theta_i^{EC} \tag{2}$$

The term $H_t\left(\theta_i^M - \theta_i^D \sum_{j\neq i} Y_{jt}\right)$ is the variable profit of firm i. H_t represents market size (e.g., market population) and it is an exogenous state variable. θ_i^M is a parameter that represents the per capita variable profit of firm i when the firm is a monopolist. The parameter θ_i^D captures the effect of the number of competing firms on the profit of firm i.[6] The term $\theta_{i0}^{FC} + \theta_{i1}^{FC}\ Z_{it}$ is the fixed cost of firm i, where θ_{i0}^{FC} and θ_{i1}^{FC} are parameters, and Z_{it} is an exogenous firm-specific characteristic affecting the fixed cost of the firm. The term $1\{Y_{it-1} = 0\}\ \theta_i^{EC}$ represents sunk entry costs, where $1\{.\}$ is the binary indicator function and θ_i^{EC} is a parameter. Entry costs are paid only if the firm was not active in the market at previous period. The vector of common knowledge state variables of the game is $\mathbf{X}_t = (H_t, Z_{it}, Y_{it-1} : i = 1, 2, ..., N)$. ■

Most previous literature on estimation of dynamic discrete games assumes that the data comes from a Markov Perfect Equilibrium (MPE). This equilibrium concept incorporates four main assumptions.

ASSUMPTION MOD-1 (Payoff relevant state variables): Players' strategy functions depend only on payoff relevant state variables: $\mathbf{X}_t$ *and* ε_{it}*. Also, a player's belief about the strategy of other players is a function of only common knowledge payoff relevant state variables,* $\mathbf{X}_t$*.* ■

ASSUMPTION MOD-2 (Maximization of expected payoffs): Players are forward looking and maximize expected intertemporal payoffs given beliefs. ■

ASSUMPTION MOD-3 (Unbiased beliefs on own future behavior): A player's beliefs about his own actions in the future are unbiased expectations of his actual actions in the future. ■

ASSUMPTION 'EQUIL' (Unbiased or equilibrium beliefs on other players' behavior): Strategy functions are common knowledge, and players' have rational expectations on the current and future behavior of other players. That is, players' beliefs about other players' actions are unbiased expectations of the actual actions of other players. ■

First, let us examine the implications of imposing only Assumption MOD-1.[7] The payoff-relevant information set of player i is $\{\mathbf{X}_t, \varepsilon_{it}\}$. The space of $\mathbf{X}_t$ is $\mathcal{X}$. At period t, players observe $\mathbf{X}_t$ and choose their respective actions. Let the function $\sigma_{it}(\mathbf{X}_t, \varepsilon_{it}) : \mathcal{X} \times \mathbb{R}^A \to \mathcal{Y}$ represent a strategy function for player i at period t. Given any strategy function σ_{it}, we can define a choice probability function $P_{it}(y|\mathbf{x})$ that represents the probability that $Y_{it} = y$ conditional on $\mathbf{X}_t = \mathbf{x}$ given that player i follows strategy σ_{it}. That is,

$$P_{it}(y|\mathbf{x}) \equiv \int 1\left\{\sigma_{it}\left(\mathbf{x}, \varepsilon_{it}\right) = y\right\}\ dG_{it}\left(\varepsilon_{it}\right) \tag{3}$$

[6] A more flexible specification allows for each firm j to have a different impact on the variable profit of firm i, i.e., $H_t\left(\theta_i^M - \sum_{j\neq i} \theta_{ij}^D\ Y_{jt}\right)$.

[7] Fershtman and Pakes (2012) study dynamic games where a player's private information is serially correlated, e.g., time-invariant private information. In this context, the whole past history of a rival's decisions contains information about the 'type' (private information) of that rival. These authors propose a framework and a new equilibrium concept (Experience-based equilibrium) to deal with this dimensionality problem.

It is convenient to represent players' behavior using these *Conditional Choice Probability* (CCP) functions. When the variables in $\mathbf{X}_t$ have a discrete support, we can represent the CCP function $P_{it}(.)$ using a finite-dimensional vector $\mathbf{P}_{it} \equiv \{P_{it}(y|\mathbf{x}) : y \in \mathcal{Y}, \mathbf{x} \in \mathcal{X}\} \in [0,1]^{A|\mathcal{X}|}$. Throughout the paper we use either the function $P_{it}(.)$ or the vector $\mathbf{P}_{it}$ to represent the *actual behavior* of player i at period t.

Without imposing Assumption 'Equil' ('Equilibrium Beliefs'), a player's beliefs about the behavior of other players do not necessarily represent the actual behavior of the other players. Therefore, we need functions other than $\sigma_{jt}(.)$ and $P_{jt}(.)$ to represent players i's beliefs about the strategy of other players. To maximize expected intertemporal payoffs at some period t, a player needs to form beliefs about other players' behavior not only at period t but also at any other period $t+s$ in the future. Let $B_{i,t+s}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}_{t+s})$ be the probability function that represents player i's belief at period t about the other players' actions at period $t+s$ conditional on the common knowledge state variables at that period. That is, in the beliefs function $B_{i,t+s}^{(t)}$ the index t represents the time period in which beliefs are formed, and the index $t+s$, with $s \geq 0$, represents the time period when the event that is the object of these beliefs occurs. In principle, this belief function may vary with t due to players' learning and forgetting, or to other factors that cause players' beliefs to change over time. When $\mathcal{X}$ is a discrete and finite space, we can represent function $B_{i,t+s}^{(t)}(.)$ using a finite-dimensional vector $\mathbf{B}_{i,t+s}^{(t)} \equiv \{B_{i,t+s}^{(t)}(\mathbf{y}_{-i}|\mathbf{x})) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}, \mathbf{x} \in \mathcal{X}\} \in [0,1]^{A^{N-1}|\mathcal{X}|}$. Using this notation, Assumption 'Equil' can be represented in vector form as $\mathbf{B}_{i,t+s}^{(t)} = \Pi_{j\neq i}\mathbf{P}_{j,t+s}$ for every player i, every t, and $s \geq 0$.

The following assumption replaces the assumption of 'Equilibrium Beliefs' and summarizes our minimum conditions on players' beliefs.[8]

ASSUMPTION MOD-4: It is common knowledge that players' private information $\boldsymbol{\varepsilon}_{it}$ is independently distributed across players. This condition implies that a player's beliefs should satisfy the restriction that other players' actions are independent conditional on common knowledge state variables: $B_{i,t+s}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}) = \Pi_{j\neq i}\, B_{ij,t+s}^{(t)}(y_j|\mathbf{x})$, where $B_{ij,t+s}^{(t)}(y_j|\mathbf{x})$ represents the beliefs of player i on the behavior of player j. ■

Assumption MOD-4 can be seen as natural implication of Assumption MOD-1 and the assumption that private information variables are independent across players. If a player knows that other players' strategy functions depend only on payoff relevant state variables $\mathbf{X}_t$ and $\boldsymbol{\varepsilon}_{it}$ (i.e., Assumption MOD-1) and that private information variables $\boldsymbol{\varepsilon}_{it}$ are independent across players, then this

[8] Assumptions MOD-1 and MOD-4 establish independence between players' private information in a single market. While the model may potentially have multiple equilibria, which is a source of biased beliefs, coordination on an equilibrium can not generate correlation in actions because behavior is conditional on the equilibrium being played. In other words, if players are playing the same equilibrium, once we condition on that equilibrium and the state variables, their actions remain independent. When we look at data from multiple markets, players' actions and beliefs can be correlated across markets. Our model with unobserved market-specific heterogeneity (section 3.2.8) allows for this correlation.

player's beliefs should satisfy the independence condition $B^{(t)}_{i,t+s}(\mathbf{y}_{-i}|\mathbf{x}) = \Pi_{j\neq i}\ B^{(t)}_{ij,t+s}(y_j|\mathbf{x})$. Note also that assumption 'Equil" implies assumption MOD-4 but, of course, it is substantially stronger.

Assumption MOD-4 substantially reduces the dimension of the beliefs function in games with more than two players. For example, for a given player, and given value of $\mathbf{X}$, t, and $t+s$, the number of free beliefs decreases to $(N-1)(A-1)$ from $A^{N-1}-1$.

Our identification results, that we present in section 3, allow the belief functions $B^{(t)}_{ij,t+s}$ to vary freely both over t (i.e., over the period when these beliefs of player i are formed) and over $t+s$ (i.e., over the period of player j's behavior). In particular, our model and identification results allow players to update their beliefs and learn (or not) over time t. As we explain in section 3, this does not mean that we can identify beliefs $B^{(t)}_{ij,t+s}$ at every pair of periods t and $t+s$ with $s \geq 0$. We establish the identification of the payoff function and of contemporaneous beliefs $B^{(t)}_{ijt}$. However, our identification results do not impose restrictions on beliefs $B^{(t)}_{ij,t+s}$ for $s > 0$.

ASSUMPTION MOD-5: The state space $\mathcal{X}$ is discrete and finite, and $|\mathcal{X}|$ represents its dimension or number of elements. ■

For the rest of the paper, we maintain Assumptions MOD-1 to MOD-5 but we do not impose the restriction of Equilibrium Beliefs. We assume that players are rational, in the sense that they maximize expected and discounted payoffs given their beliefs on other players' behavior. These assumptions are standard in the literature of empirical dynamic games and are a strict subset of the assumptions required for Markov Perfect Equilibrium (MPE). Our departure from the literature is that we do not impose assumption EQUIL. In this sense, we significantly relax the MPE assumption. Our approach is agnostic about the formation of players' beliefs. Beliefs are allowed to evolve freely over t and $t+s$ in the DGP. Our assumptions are neither weaker nor stronger than rationalizability. In particular, rationalizability not only imposes assumption MOD-2, i.e., players' are rational in the sense that they maximize expected payoffs given their beliefs, but also that this rationality is common knowledge among the players. We do not impose any restriction on common knowledge rationality. However, rationalizability does not require assumptions MOD-3 and MOD-4.

2.2 Best response mappings

We say that a strategy function σ_{it} (and the associated CCP function P_{it}) is *rational* if for every possible value of $(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}) \in \mathcal{X} \times \mathbb{R}^A$ the action $\sigma_{it}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it})$ maximizes player i's expected and discounted value given his beliefs on the opponent's strategy. Given his beliefs at period t, $\mathbf{B}^{(t)}_i = \{B^{(t)}_{ij,t+s} : s \geq 0\}$, player i's best response at period t is the optimal solution of a single-agent dynamic programming (DP) problem. This DP problem can be described in terms of: (i) a discount factor, β; (ii) a sequence of expected one-period payoff functions, $\{\pi^{\mathbf{B}^{(t)}}_{i,t+s}(y_{it+s}, \mathbf{x}_{t+s}) + \varepsilon_{it+s}(y_{it+s}) : s = 0,$

1, ..., $T-t\}$, where

$$\pi^{\mathbf{B}(t)}_{it+s}(y_{it+s}, \mathbf{x}_{t+s}) \equiv \sum_{\mathbf{y}_{-i}\in\mathcal{Y}^{N-1}} \pi_{it+s}(y_{it+s}, \mathbf{y}_{-i}, \mathbf{x}_{t+s})\ B^{(t)}_{it+s}(\mathbf{y}_{-i}|\mathbf{x}_{t+s})\ ; \tag{4}$$

and (iii) a sequence of transition probability functions $\{f^{\mathbf{B}(t)}_{it+s}(\mathbf{x}_{t+s+1}|y_{it+s}, \mathbf{x}_{t+s}) : s = 0, 1, ..., T-t\}$, where

$$f^{\mathbf{B}(t)}_{it+s}(\mathbf{x}_{t+s+1}|y_{it+s}, \mathbf{x}_{t+s}) \quad \equiv \quad \sum_{\mathbf{y}_{-i}\in\mathcal{Y}^{N-1}} f_{t+s}(\mathbf{x}_{t+s+1}|y_{it+s}, \mathbf{y}_{-i}, \mathbf{x}_{t+s})\ B^{(t)}_{it+s}(\mathbf{y}_{-i}|\mathbf{x}_{t+s}) \tag{5}$$

Let $V^{\mathbf{B}(t)}_{it+s}(\mathbf{x}_{t+s}, \boldsymbol{\varepsilon}_{it+s})$ be the value function for player i's DP problem given his beliefs at period t. By Bellman's principle, the sequence of value functions $\{V^{\mathbf{B}(t)}_{it+s} : s \geq 0\}$ can be obtained recursively using the following *Bellman equation*:

$$V^{\mathbf{B}(t)}_{it}(\mathbf{x}_t, \boldsymbol{\varepsilon}_{it}) \quad = \quad \max_{y_{it}\in\mathcal{Y}} \left\{\ v^{\mathbf{B}(t)}_{it}(y_{it}, \mathbf{x}_t) + \varepsilon_{it}(y_{it})\ \right\} \tag{6}$$

where $v^{\mathbf{B}(t)}_{it}(y_{it}, \mathbf{x}_t)$ is the *conditional choice value function*

$$v^{\mathbf{B}(t)}_{it}(y_{it}, \mathbf{x}_t) \quad \equiv \quad \pi^{\mathbf{B}(t)}_{it}(y_{it}, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \int V^{\mathbf{B}(t)}_{it+1}(\mathbf{x}_{t+1}, \boldsymbol{\varepsilon}_{it+1})\ dG_{it}(\boldsymbol{\varepsilon}_{it+1})\ f^{\mathbf{B}(t)}_{it}(\mathbf{x}_{t+1}|y_{it}, \mathbf{x}_t) \tag{7}$$

Given his beliefs, the best response function of player i at period t is the optimal decision rule of this DP problem. This best response function can be represented using the following threshold condition:

$$\{Y_{it} = y\} \text{ iff } \left\{\varepsilon_{it}(y') - \varepsilon_{it}(y) \leq v^{\mathbf{B}(t)}_{it}(y, \mathbf{x}_t) - v^{\mathbf{B}(t)}_{it}(y', \mathbf{x}_t) \text{ for any } y' \neq y\right\} \tag{8}$$

The *best response probability* (*BRP*) *function* is a probabilistic representation of the best response function. More precisely, it is the best response function integrated over the distribution of $\boldsymbol{\varepsilon}_{it}$. In this model, the *BRP* function given $\mathbf{X}_t = \mathbf{x}$ is:

$$\begin{aligned}
\Pr(Y_{it} = y|\mathbf{x}) \quad &= \quad \int 1\left\{\varepsilon_{it}(y') - \varepsilon_{it}(y) \leq v^{\mathbf{B}(t)}_{it}(y, \mathbf{x}) - v^{\mathbf{B}(t)}_{it}(y', \mathbf{x}) \text{ for any } y' \neq y\right\}\ dG_{it}(\boldsymbol{\varepsilon}_{it}) \\
&= \quad \Lambda_{it}\left(y;\ \widetilde{\mathbf{v}}^{\mathbf{B}(t)}_{it}(\mathbf{x})\right)
\end{aligned}$$

where $\Lambda_{it}(y;\ .)$ is the CDF of the vector $\{\varepsilon_{it}(y') - \varepsilon_{it}(y) : y' \neq y\}$ and $\widetilde{\mathbf{v}}^{\mathbf{B}}_{it}(\mathbf{x})$ is the $(A-1)\times 1$ vector of value differences $\{\widetilde{v}^{\mathbf{B}(t)}_{it}(y, \mathbf{x}) : y = 1, 2, ..., A-1\}$ with $\widetilde{v}^{\mathbf{B}(t)}_{it}(y, \mathbf{x}) \equiv v^{\mathbf{B}(t)}_{it}(y, \mathbf{x}) - v^{\mathbf{B}(t)}_{it}(0, \mathbf{x})$. For instance, if $\varepsilon_{it}(y)$'s are iid Extreme Value type 1, the best response function has the well–known logit form:

$$\frac{\exp\left\{\widetilde{v}^{\mathbf{B}(t)}_{it}(y, \mathbf{x})\right\}}{\sum_{y'\in\mathcal{Y}} \exp\left\{\widetilde{v}^{\mathbf{B}(t)}_{it}(y', \mathbf{x})\right\}} \tag{9}$$

Therefore, under Assumptions MOD-1 to MOD-3 the actual behavior of player i, represented by the CCP function $P_{it}(.)$, satisfies the following condition:

$$P_{it}(y|\mathbf{x}) = \Lambda_{it}\left(y;\ \widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})\right) \tag{10}$$

This equation summarizes all the restrictions that Assumptions MOD-1 to MOD-3 impose on players' choice probabilities. The right hand side of equation (10) is the best response function of a rational player. The concept of Markov Perfect Equilibrium (MPE) is completed with assumption 'Equil' ('Equilibrium Beliefs'). Under this assumption, players' beliefs are in equilibrium, i.e., $\mathbf{B}_{it+s}^{(t)} = \Pi_{j\neq i}\mathbf{P}_{jt+s}$ for every player i and every period $t+s$ with $s \geq 0$. A MPE can be described as a sequence of CCP vectors, $\{\mathbf{P}_{it} : i = 1, 2, ..., N;\ t = 1, 2, ..., T\}$ such that for every player i and time period t, we have that $P_{it}(y|\mathbf{x}) = \Lambda_{it}\left(y;\ \widetilde{\mathbf{v}}_{it}^{\mathbf{P}}(\mathbf{x})\right)$. In this paper, we do not impose this equilibrium restriction.

As we mentioned in section 2.1, our model incorporates a concept of rationality in dynamic games that is related but different to the concept of rationalizability. The notion of rationalizability, well-defined as a solution concept in static games, has no counterpart in the solution of dynamic games. Although Pearce (1984) provides an extension of the notion of rationalizability in static games to extensive form games, two problems with this notion exist. First, the rationalizable outcome may not be a sequential equilibrium (see the example on page 1044 of Pearce, 1984). Second, as shown by Battigalli (1997, page 44), in some extensive form games, allowing for the possibility that rationality is not common knowledge provides an incentive for players to strategically manipulate the beliefs of other players.

There are also substantial computational problems in the implementation of rationalizability in dynamic games. Aradillas-Lopez and Tamer (2008) consider a static, two-player, binary-choice game of incomplete information that is a specific case of our framework. Under the assumption that players' payoffs are supermodular (or submodular) in players' decisions, they derive bounds around players' conditional choice probabilities that are robust to the values of players' beliefs when they are level-k rational, and show that the bounds become tighter as k increases. To extend this approach to dynamic games, one needs to calculate lower and upper bounds of choice probabilities with respect to beliefs not only on the opponents' current decisions but also on their decisions in the future. In general, intermtemporal value functions are not supermodular or submodular in players' decisions at every state, even in the simpler dynamic games. This complicates very substantially the computation of these bounds. We discuss this issue in more detail in the appendix.

3 Identification

3.1 Conditions on Data Generating Process

Suppose that the researcher has panel data with realizations of the game over multiple geographic *locations* and time periods.[9] We use the letter m to index locations. The researcher observes a random sample of M locations with information on $\{y_{imt}, \mathbf{x}_{mt}\}$ for every player $i \in \{1, 2, ..., N\}$ over periods $t \in \{1, 2, ..., T_{data}\}$, where T_{data} denotes the number of periods observed in the data. We emphasize here that the researcher may not observe all the periods in the model, which itself can be finite or infinite horizon. That is, in general we have that $T_{data} \leq T \leq \infty$, and our identification results apply to both $T_{data} = T$ and $T_{data} < T$.

We assume that T_{data} is small and the number of local markets, M, is large. For the identification results in this section we assume that M is infinite. We first study identification in a model where the only unobservable variables for the researcher are the private information shocks $\{\varepsilon_{imt}\}$, which are assumed to be independently and identically distributed across players, markets, and over time. We relax this assumption in section 3.2.7 where we allow for time-invariant market-specific state variables that are common knowledge to all the players but unobservable to the researcher.

We want to use this sample to estimate the structural 'parameters' or functions of the model: i.e., payoffs $\{\pi_{it}, \beta\}$; transition probabilities $\{f_t\}$; distribution of unobservables $\{\Lambda_{it}\}$; and beliefs $\{B_{it+s}^{(t)}\}$ for $i \in \{1, 2, ..., N\}$ and $t \in \{1, 2, ..., T_{data}\}$. Beliefs are allowed to evolve arbitrarily with t and $t+s$ in the DGP. For primitives other than players' beliefs, we make some assumptions that are standard in previous research on identification of static games and of dynamic structural models with rational or equilibrium beliefs.[10] We assume that the distribution of the unobservables, Λ_{it}, is known to the researcher up to a scale parameter. We study identification of the payoff functions π_{it} up to scale, but for notational convenience we omit the scale parameter.[11] Following the standard approach in dynamic decision models, we assume that the discount factor, β, is known to the researcher. Finally, the transition probability functions $\{f_t\}$ are nonparametrically identified. Therefore, we concentrate on the identification of the payoff functions π_{it} and the belief functions $B_{it+s}^{(t)}$ and assume that $\{f_t, \Lambda_{it}, \beta\}$ are known.

Let $\mathbf{P}_{imt}^0$ be the vector of CCPs with the true (population) conditional probabilities $\Pr(Y_{imt} = y|i, m, t, \mathbf{X}_{mt} = \mathbf{x})$ for player i in market m at period t. Similarly, let $\mathbf{B}_{im}^{(t)0}$ be the vector of probabilities with the true values of player i's beliefs in market m at period t about behavior in *all* future periods, i.e., $\mathbf{B}_{im}^{(t)0} \equiv \{B_{im,t+s}^{(t)0} : s \geq 0\}$. Finally, let $\boldsymbol{\pi}^0 \equiv \{\pi_{it}^0 : i = 1, 2;\ t = 1, 2, ..., T\}$ be the true payoff functions in the population. Assumption *ID-1* summarizes our conditions on the Data

[9] In the context of empirical applications of games in IO, a geographic location is a *local market*.

[10] See Bajari and Hong (2005), or Bajari et al (2010), among others.

[11] Aguirregabiria (2010) and Norets and Tang (2014) provide conditions for the nonparametric identification of the distribution of the unobservables in single-agent binary-choice dynamic structural models. Those conditions can be applied to identify the distribution of the unobservables in our model.

Generating Process.

ASSUMPTION ID-1. (A) For every player i, $\mathbf{P}^0_{imt}$ *is his best response at period t given his beliefs* $\mathbf{B}^{(t)0}_{im}$ *and the payoff functions* $\boldsymbol{\pi}^0$*. (B) A player has the same beliefs in two markets with the same observable characteristics* $\mathbf{X}$*, i.e., for every two markets m and m′ with* $\mathbf{X}_{m,t+s} = \mathbf{X}_{m',t+s} = \mathbf{x}$*, we have that* $B^{(t)0}_{im,t+s}(\mathbf{y}_{-i}|\mathbf{x}) = B^{(t)0}_{im',t+s}(\mathbf{y}_{-i}|\mathbf{x}) = B^{(t)0}_{i,t+s}(\mathbf{y}_{-i}|\mathbf{x})$. ■

Assumption *ID-1* (A) establishes that players are rational in the sense that their actual behavior is the best response given their beliefs. Assumption *ID-1* (B) is common in the literature on estimation of games under the restriction of equilibrium beliefs (e.g., Bajari, Benkard, and Levin, 2007, or Bajari et al, 2010). Note that beliefs can vary across markets according to the state variables in $\mathbf{X}_{mt}$. This assumption allows players to have different belief functions in different markets as long as these markets have different values of time-invariant observable exogenous characteristics. For instance, beliefs could be a function of "market type," which are determined by some market specific time-invariant observable characteristics. If the number of market types is small (more precisely, if it does not increase with M), then we can allow players' beliefs to be completely different in each market type. In section 3.2.8, we relax Assumption ID-1(B) by introducing time-invariant common- knowledge state variables that are unobservable to the researcher. In that extended version of the model, players' beliefs can vary across markets that are observationally equivalent to the researcher.

In dynamic games where beliefs are in equilibrium, Assumption *ID-1* effectively allows the researcher to identify player beliefs. Under this assumption, conditional choice probabilities are identified, and if beliefs are in equilibrium, the belief of player i about the behavior of player j is equal to the conditional choice probability function of player j. When beliefs are not in equilibrium, Assumption *ID-1* is not sufficient for the identification of beliefs. However, assumption *ID-1* still implies that CCPs are identified from the data. This assumption implies that for any player i, any period t, and any value of $(y, \mathbf{x})$, we have that $P^0_{imt}(y|\mathbf{x}) = P^0_{it}(y|\mathbf{x}) = \Pr(Y_{imt} = y|\mathbf{X}_{mt} = \mathbf{x})$, and this conditional probability can be estimated consistently using the M observations of $\{Y_{imt}, \mathbf{X}_{mt}\}$ in our random sample of these variables. This in turn, as we will show, is important for the identification of beliefs themselves.

For notational simplicity, we omit the market subindex m for the rest of this section.

ASSUMPTION ID-2 (Normalization of payoff function): The one-period payoff function π_{it} *is normalized to zero for* $y_{it} = 0$*, i.e.,* $\pi_{it}(0, \mathbf{y}_{-it}, \mathbf{x}_t) = 0$ *for any value* $(\mathbf{y}_{-it}, \mathbf{x}_t)$. ■

Assumption *ID-2* establishes a normalization of the payoff that is commonly adopted in many discrete choice models: the payoff to one of the choice alternatives, say alternative 0, is normalized to zero.[12] The particular form of normalization of payoffs does not affect our identification results

[12] As is well-known, in discrete choice models preferences can be identified only up to an affine transformation.

as long as the normalization imposes $A^{N-1}|\mathcal{X}|$ restrictions on each payoff function π_{it}.

3.2 Identification of payoff and belief functions

In this subsection we examine different types of restrictions on payoffs and beliefs that can be used to identify dynamic games.[13] The main point that we want to emphasize here is that restrictions that apply either only to beliefs or only to payoffs are not sufficient to identify this class of models. For instance, the assumption of equilibrium beliefs alone can identify beliefs but it is not enough to identify the payoff function. We also show that an exclusion restriction that has been commonly used to identify the payoff function can be exploited to relax the assumption of equilibrium beliefs.

3.2.1 Identification of value differences from choice probabilities

Let $\mathbf{P}_{it}(\mathbf{x})$ be the $(A-1)\times 1$ vector of CCPs $(P_{it}(1|\mathbf{x}),\ ...,\ P_{it}(A-1|\mathbf{x}))$, and let $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})$ be the $(A-1)\times 1$ vector of differential values $(\widetilde{v}_{it}^{\mathbf{B}(t)}(1,\mathbf{x}),\ ...,\ \widetilde{v}_{it}^{\mathbf{B}(t)}(A-1,\mathbf{x}))$. The model restrictions can be represented using the best response conditions $\mathbf{P}_{it}(\mathbf{x}) = \mathbf{\Lambda}\left(\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})\right)$, where $\mathbf{\Lambda}(\mathbf{v})$ is the vector-valued function $(\Lambda(1|\mathbf{v}),\ \Lambda(2|\mathbf{v}),\ ...,\Lambda(A-1|\mathbf{v}))$. Given these conditions, and our normalization assumption ID-2, we want to identify payoffs and beliefs.

For all our identification results, a necessary first step consists of the identification of the vector of value differences $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})$ from the vector of CCPs $\mathbf{P}_{it}(\mathbf{x})$. The following Theorem, due to Hotz and Miller (1993, Proposition 3), establishes this identification result.

THEOREM (Hotz-Miller inversion Theorem). If the distribution function $G_{it}(\boldsymbol{\varepsilon})$ is continuously differentiable over the whole Euclidean space, then, for any $(i,t,\mathbf{x})$, the mapping $\mathbf{P}_{it}(\mathbf{x}) = \mathbf{\Lambda}\left(\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})\right)$ is invertible such that there is a one-to-one relationship between the $(A-1)\times 1$ vector of CCPs $\mathbf{P}_{it}(\mathbf{x})$ and the $(A-1)\times 1$ vector of value differences $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})$. ■

Let $\mathbf{q}(\mathbf{P}) \equiv (q(1,\mathbf{P}),\ q(2,\mathbf{P}),\ ...,\ q(A-1,\mathbf{P}))$ be the inverse mapping of $\mathbf{\Lambda}$ such that if $\mathbf{P} = \mathbf{\Lambda}(\mathbf{v})$ then $\mathbf{v} = \mathbf{q}(\mathbf{P})$. Therefore, $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x}) = \mathbf{q}(\mathbf{P}_{it}(\mathbf{x}))$. For instance, for the multinomial logit case with $\Lambda(y|\mathbf{v}) = \exp\{v(y)\}/\sum_{y'\in\mathcal{Y}}\exp\{v(y')\}$, the inverse function $\mathbf{q}(\mathbf{P}_{it}(\mathbf{x}))$ is $q(y,\mathbf{P}_{it}(\mathbf{x})) = \ln(P_{it}(y|\mathbf{x})) - \ln(P_{it}(0|\mathbf{x}))$.

We assume the researcher knows the distribution of private information and so identification is not fully nonparametric in nature. However, the assumption that Λ is known to the researcher can be relaxed to achieve full nonparametric identification. This has been proved before by Aguirregabiria (2010) and Norets and Tang (2014) in the context of single-agent dynamic structural models based on previous results by Matzkin (1992) for the binary choice case, and Matzkin (1993) for the

[13] Some of the discussion in this section is similar in spirit to Pesendorfer and Schmidt-Dengler's (2008) discussion on underidentification in dynamic games. However, they take beliefs as given (they are identified from CCPs in their setting).

multinomial case.[14] As we do not consider this to be a focus of this paper, for the sake of simplicity, we assume throughout that the distribution Λ is known.

Given that CCPs are identified and that the distribution function Λ and the inverse mapping $\mathbf{q}(.)$ are known (up to scale) to the researcher, we have that the differential values $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})$ are identified. Then, hereafter, we treat $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}(t)}(\mathbf{x})$ as an identified object. To underline the identification of the value differences from inverting CCPs, we will often use $q(y, \mathbf{P}_{it}(\mathbf{x}))$, or with some abuse of notation $q_{it}(y, \mathbf{x})$, instead of $\widetilde{v}_{it}^{\mathbf{B}(t)}(y, \mathbf{x})$.

3.2.2 Identification of payoffs and beliefs without exclusion restrictions

We can represent the relationship between value differences and payoffs and beliefs using a recursive system of linear equations. For every period t and $(y_i, \mathbf{x}) \in [\mathcal{Y} - \{0\}] \times \mathcal{X}$, the following equation holds:

$$q_{it}(y_i, \mathbf{x}) \quad = \quad \mathbf{B}_{it}^{(t)}(\mathbf{x})' \left[\boldsymbol{\pi}_{it}(y_i, \mathbf{x}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}) \right] \tag{11}$$

where $\mathbf{B}_{it}^{(t)}(\mathbf{x})$, $\boldsymbol{\pi}_{it}(y_i, \mathbf{x})$, and $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$ are vectors with dimension $A^{N-1} \times 1$. $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ is the vector of contemporaneous beliefs at period t, $\{B_{it}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}\}$; $\boldsymbol{\pi}_{it}(y_i, \mathbf{x})$ is a vector of payoffs $\{\pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{x}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}\}$; $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x})$ is a vector of continuation value differences $\{c_{it}^{\mathbf{B}(t)}(y_i, \mathbf{y}_{-i}, \mathbf{x}) - c_{it}^{\mathbf{B}(t)}(0, \mathbf{y}_{-i}, \mathbf{x}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}\}$, and $c_{it}^{\mathbf{B}(t)}(\mathbf{y}_t, \mathbf{x}_t)$ is the *continuation value function* that provides the expected and discounted value of *future* payoffs given future beliefs, current state, and current choices of *all* players:

$$c_{it}^{\mathbf{B}(t)}(\mathbf{y}_t, \mathbf{x}_t) \equiv \beta \int V_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1}, \boldsymbol{\varepsilon}_{it+1}) \; dG_{it}(\boldsymbol{\varepsilon}_{it+1}) \; f_t(\mathbf{x}_{t+1}|\mathbf{y}_t, \mathbf{x}_t) \tag{12}$$

The system of equations (11) summarizes all the restrictions of the model. At first glance, this system seems to have a recursive structure such that in a dynamic game with finite horizon and $T_{data} = T$ one could proceed with a backwards induction argument, i.e., at the last period T, the continuation values $\widetilde{\mathbf{c}}_{iT}^{\mathbf{B}(T)}$ are zero such that, under some restrictions, the observed behavior at period T could identify payoffs and beliefs at period T; then, given beliefs and payoffs at period T, one could argue that the continuation values at $T-1$, $\widetilde{\mathbf{c}}_{iT-1}^{\mathbf{B}(T-1)}$, are known. However, without further restrictions on the evolution of beliefs over time, this recursive argument does not hold. For instance, at period $T-1$ the continuation values $\widetilde{\mathbf{c}}_{iT-1}^{\mathbf{B}(T-1)}$ depend on beliefs formed at period $T-1$ about the opponents' behavior at period T, i.e., $B_{i,T}^{(T-1)}$, but the beliefs identified from the observed behavior at period T are beliefs formed at period T about the opponents' behavior at period T, i.e., $B_{i,T}^{(T)}$. In general, beliefs $B_{i,T}^{(T-1)}$and $B_{i,T}^{(T)}$ will be different even though both are beliefs about the behavior of other players at period T, e.g., players may learn over time.

[14] This identification result is based on the assumption that there is a special state variable(s) that enters additively in the index $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ and that has full support variation over the Euclidean space.

A backwards induction approach to the identification of beliefs and payoffs is possible with finite horizon, $T_{data} = T$, and the additional restriction that beliefs are not updated over time (e.g., there is no learning), i.e., if we impose the restriction that, for any period $t+s$, $B_{it+s}^{(t)}$ is invariant with respect to t. Our identification results do not impose this restriction.

The under-identification of dynamic games under the assumption of equilibrium beliefs but no further restriction on payoffs has been noted and studied in previous papers such as Aguirregabiria and Mira (2002), Pesendorfer and Schmidt-Dengler (2003), and Bajari et al. (2010), who propose different versions of the exclusion restriction in Assumption ID-3(i) below to deal with this identification problem. Given this, it is not surprising that the model is not identified when we leave beliefs unrestricted. Table 1 formally presents the number of parameters, restrictions, and over- or under- identifying restrictions when beliefs are unrestricted (column A) and when beliefs are restricted to be unbiased (column B). As a simple example, consider a binary choice game with two players. For player i at period t, there are 2 payoff functions (one for each choice of j), one belief and one continuation value difference to identify for each value of $\mathbf{X}$, for a total of $4|\mathcal{X}|$ unknown parameters. As there are only $|\mathcal{X}|$ restrictions implied by players' behavior in the system of equations (11), the model is clearly underidentified without equilibrium beliefs. Assuming equilibrium beliefs adds only $|\mathcal{X}|$ further restrictions, and the model is still underidentified.

3.2.3 Identification with exclusion restrictions

Assumption *ID-3* presents nonparametric restrictions on the payoff function that, combined with the assumption of equilibrium beliefs, are typically used for identification in games with equilibrium beliefs (e.g., Bajari et al., 2010).

ASSUMPTION ID-3 (Exclusion Restriction): The vector of state variables $\mathbf{X}_t$ can be partitioned into two subvectors, $\mathbf{X}_t = (\mathbf{S}_t, \mathbf{W}_t)$. The vectors $\mathbf{S}_t$ and $\mathbf{W}_t \in \mathcal{W}$ satisfy the following conditions:

(i) $\mathbf{S}_t = (S_{1t}, S_{2t}, ..., S_{Nt}) \in \mathcal{S}^N$ where $S_{it} \in \mathcal{S}$ represents state variables that enter into the payoff function of player i but not the payoff function of any of the other players.

$$\pi_{it}\left(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{S}_{-it}, \mathbf{W}_t\right) = \pi_{it}\left(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{S}'_{-it}, \mathbf{W}_t\right) \quad \text{for any } \mathbf{S}'_{-it} \neq \mathbf{S}_{-it} \quad (13)$$

(ii) The number of states in the support set $\mathcal{S}$ is greater or equal than the number of actions A, i.e., $|\mathcal{S}| \geq A$.

(iii) The joint distribution of $(S_{it}, \mathbf{S}_{-it}, \mathbf{W}_t)$, over the population of M markets where we observe these variables, has a strictly positive probability at every point in the joint support set $\mathcal{S}^N \times \mathcal{W}$.

(iv) The transition probability of the state variable S_{it} is such that the value of $S_{i,t+1}$ does not depend on $(S_{it}, \mathbf{S}_{-it})$ once we condition on Y_{it} and $\mathbf{W}_t$, i.e., $\Pr(S_{i,t+1}|Y_{it}, \mathbf{S}_t, \mathbf{W}_t) = \Pr(S_{i,t+1}|Y_{it}, \mathbf{W}_t)$. ■

The exclusion restriction in assumptions ID-3(i)-(ii) appears naturally in many applications of dynamic games of oligopoly competition in Industrial Organization. The incumbent status, capacity, capital stock, or product quality of a firm at period $t-1$ are state variables that enter in the firm's payoff function at period t because there are investment and adjustment costs that depend on these lagged variables. The firm's payoff function at period t depends also on the competitors' values of these variables at period t, but it does not depend on the competitors' values of these variables at $t-1$.

Importantly, the assumption that some of the variables which enter player j's payoff function are excluded from player i's payoffs *does not* mean that player i does not condition his *behavior* on those excluded variables. Each player conditions his behavior on all the (common knowledge) state variables that affect the payoff of a player in the game, even if these variables are excluded from his own payoff.

Assumption *ID-3(iii)* is a condition on the joint cross-sectional distribution of the state variables $(S_{it}, \mathbf{S}_{-it}, \mathbf{W}_t)$ over the sample of M markets where we observe these variables. Since the state variables $(S_{it}, \mathbf{S}_{-it}, \mathbf{W}_t)$ follow a Markov process, we can see Assumption ID-3(iii) as a condition on the ergodic distribution of these variables.

Assumption *ID-3(iv)* restricts the transition probability, or transition rule, of the state variable S_{it}. An important class of models that satisfies condition (iv) is when the state variable S_{it} is the lagged decision, such that the transition rule for this state variable is $S_{i,t+1} = Y_{it}$, that trivially satisfies condition (iv). This is an important class of models because many dynamic games of oligopoly competition belong to this class, e.g., market entry/exit, technology adoption, and some dynamic games of quality or capacity competition, among others. Condition (iv) rules out some interesting models too. In Example 2, we discuss two types of dynamic games of quality competition, one that satisfies condition (iv) and the other does not. Similarly, Example 3 presents dynamic games of machine replacement with and without condition (iv).

EXAMPLE 2: Consider a quality ladder dynamic game (e.g., Pakes and McGuire, 1994). The player-specific state variable S_{it} is the firm's quality at previous period and it has support set $\mathcal{S} = \{0, 1, ..., |\mathcal{S}|\}$. The decision variable Y_{it} is the change in the firm's quality at period t. The choice set has the following restrictions: $Y_{it} \in \{-1, 0, 1\}$ when $0 < S_{it} < |\mathcal{S}|$; $Y_{it} \in \{0, 1\}$ when $S_{it} = 0$; and $Y_{it} \in \{-1, 0\}$ when $S_{it} = |\mathcal{S}|$. The transition rule for the quality state variable is $S_{it+1} = S_{it} + Y_{it}$. Given this transition rule, it is clear that this model does not satisfy condition (iv). Now, consider a different dynamic game of quality competition with the same state variable S_{it} but now the decision variable Y_{it} is the firm's quality at period t, that has also support $\{0, 1, ..., |\mathcal{S}|\}$. In this model, the transition rule for quality is $S_{it+1} = Y_{it}$ that satisfies condition (iv). To capture the idea (implicit in the quality ladder model) that it is very costly to change the level of quality in more than one unit per period, this model can include in the profit function a convex adjustment cost function $AC(Y_{it} - S_{it})$ such that $AC(0) = 0$, $AC(x) > 0$ for $|x| \neq 0$, and $AC''(x) > 0$. The quality ladder model could be seen as this model when $AC(x) = \infty$ for $|x| > 1$. When the adjustment cost function is finite-valued, this model is not exactly the quality ladder model but it is clear that if the adjustment cost is large enough for quality changes $|x| > 1$, the two models are observationally very similar. ■

EXAMPLE 3: Consider a game version of a machine replacement model of investment. In this class of models, the firm-specific endogenous state variable S_{it} is the age of firm i's machine, with support $\mathcal{S} = \{1, 2, ..., |\mathcal{S}|\}$, and the decision $Y_{it} \in \{0, 1\}$ is a binary variable that represents replacing the machine by a new one (if $Y_{it} = 1$) or keeping the old machine (if $Y_{it} = 0$). The transition rule of the endogenous state variable is $S_{it+1} = \max\{(1 - Y_{it})\ S_{it} + 1,\ |\mathcal{S}|\}\}$, that does not satisfy condition (iv). This standard machine replacement model assumes that there is not a market for used capital such that a firm can choose between only two options: keep its existing machine or buy a brand new machine. Now, consider an alternative model that assumes that there is a market for used machines where it is possible to buy machines of any age, from age 0 to age $|\mathcal{S}|$. In this model, the decision variable $Y_{it} \in \{0, 1, ..., |\mathcal{S}|\}$ represents the age of the machine that the firm chooses at period t. The transition rule of the state variable is $S_{it+1} = \max\{Y_{it} + 1,\ |\mathcal{S}|\}$, that satisfies condition (iv). The transaction costs of using the second hand market can be captured by including an adjustment cost function $AC(Y_{it} - S_{it})$. ■

Table 2 illustrates how the exclusion restriction in assumption ID-3 reduces the degree of under-identification. Again, consider a binary choice two player game. As explained above, without assumption ID-3, there are a total of $4|\mathcal{X}|$ parameters to identify, $2|\mathcal{X}|$ from payoffs, $|\mathcal{X}|$ from beliefs and $|\mathcal{X}|$ from continuation value differences. Under the assumption of equilibrium beliefs, both beliefs and continuation values are identified. Using assumption ID-3, the number of unknown payoff parameters is reduced from $2|\mathcal{X}| = 2|\mathcal{S}|^2|\mathcal{W}|$ to $2|\mathcal{S}|\ |\mathcal{W}|$. Given the $|\mathcal{X}| = |\mathcal{S}|^2|\mathcal{W}|$ restrictions from observed behavior, as long as $|\mathcal{S}| \geq 2$, payoffs are also identified.

Clearly, without any restriction on beliefs, the exclusion restriction is not enough to identify the model. However, when the number of states in the set $\mathcal{S}$ is strictly greater than the number of possible actions, the restrictions implied by equilibrium conditions overidentify payoffs. We show now that these overidentifying restrictions provide a test of the null hypothesis of unbiased beliefs.

3.2.4 Tests of equilibrium beliefs and monotonicity in beliefs

Though Assumptions ID-1 to ID-3 are not sufficient for the identification of payoffs and beliefs, they provide enough restrictions to test the null hypothesis of unbiased beliefs. We present here our test in a game with two players but the test can be extended to any number of players.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ is $(S_i, S_j, \mathbf{W})$, and players' actions are Y_i and Y_j. Let s_j^0 be an arbitrary value in the set $\mathcal{S}$ of the player-specific state variable. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s_j^0\}$ such that they satisfy two conditions: (1) each of the sets has $A-1$ elements; and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A+1$, it is always possible to construct two subsets that satisfy these conditions. Given these sets, we can define the $(A-1)\times(A-1)$ matrices of contemporaneous beliefs $\Delta\mathbf{B}_{it}^{(a)}(s_i,\mathbf{w})$ and $\Delta\mathbf{B}_{it}^{(b)}(s_i,\mathbf{w})$, where $\Delta\mathbf{B}_{it}^{(a)}(s_i,\mathbf{w})$ has elements $\{B_{it}^{(t)}(y_j,s_i,s_j,\mathbf{w}) - B_{it}^{(t)}(y_j,s_i,s_j^0,\mathbf{w})$: for $y_j \in \mathcal{Y}-\{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$, and $\Delta\mathbf{B}_{it}^{(b)}(s_i,\mathbf{w})$ has the same definition but for subset $\mathcal{S}^{(b)}$. Similarly, we can define matrices $\Delta\mathbf{Q}_{it}^{(a)}(s_i,\mathbf{w})$ and $\Delta\mathbf{Q}_{it}^{(b)}(s_i,\mathbf{w})$, with elements $\{q_{it}(y_i,s_i,s_j,\mathbf{w})-\ q_{it}(y_i,s_i,s_j^0,,\mathbf{w})$: for $y_i \in \mathcal{Y}-\{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$, and matrices $\Delta\mathbf{P}_{jt}^{(a)}(s_i,\mathbf{w})$ and $\Delta\mathbf{P}_{jt}^{(b)}(s_i,\mathbf{w})$, with elements $\{P_{jt}(y_j|s_i,s_j,\mathbf{w})-\ P_{jt}(y_j|s_i,s_j^0,\mathbf{w})$: for $y_j \in \mathcal{Y}-\{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$.

PROPOSITION 1: Suppose that assumptions MOD1 to MOD5 and ID-1 to ID-3 hold. Then:

(1.1) The $(A-1)\times(A-1)$ *matrix of contemporaneous beliefs* $\Delta\mathbf{B}_{it}^{(a)}(s_i,\mathbf{w})\left[\Delta\mathbf{B}_{it}^{(b)}(s_i,\mathbf{w})\right]^{-1}$ *is identified from the CCPs of player* i as $\Delta\mathbf{Q}_{it}^{(a)}(s_i,\mathbf{w})\left[\Delta\mathbf{Q}_{it}^{(b)}(s_i,\mathbf{w})\right]^{-1}$;

(1.2) Under the assumption of unbiased beliefs, $\Delta\mathbf{B}_{it}^{(a)}(s_i,\mathbf{w})\left[\Delta\mathbf{B}_{it}^{(b)}(s_i,\mathbf{w})\right]^{-1}$ *is also identified from the CCPs of the other player,* j, as $\Delta\mathbf{P}_{jt}^{(a)}(s_i,\mathbf{w})\left[\Delta\mathbf{P}_{jt}^{(b)}(s_i,\mathbf{w})\right]^{-1}$;

(1.3) Combining (1.1) and (1.2), the assumption of unbiased contemporaneous beliefs for player i *implies the following* $(A-1)^2$ *restrictions between CCPs:*

$$\Delta\mathbf{Q}_{it}^{(a)}(s_i,\mathbf{w})\left[\Delta\mathbf{Q}_{it}^{(b)}(s_i,\mathbf{w})\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(s_i,\mathbf{w})\left[\Delta\mathbf{P}_{jt}^{(b)}(s_i,\mathbf{w})\right]^{-1} = \mathbf{0} \qquad \blacksquare$$

Proof. In the Appendix.

Under the conditions of Proposition 1, for every value of $(S_i, \mathbf{W})$ we can use player i's CCPs to construct $(A-1)^2$ values that according to the model depend only on the beliefs of player i and not on payoffs, i.e., the observed behavior of player i identifies these functions of beliefs. Of course, if we assume that beliefs are unbiased, we know that these beliefs are equal to the choice probabilities of the other player, and therefore we have a completely different form, with different data, to identify these functions of beliefs. If the hypothesis of equilibrium beliefs is correct, then both approaches should give us the same result. Therefore, the restriction provides a natural approach to test for the null hypothesis of equilibrium or unbiased beliefs.

We describe in detail how to implement a formal test for equilibrium beliefs in section 4.3 below.

EXAMPLE 4: Suppose that the dynamic game has two players making binary choices: $N=2$ and $A=2$. Then, subsets $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have only one element each: $\mathcal{S}^{(a)} = \{s^{(a)}\}$ and $\mathcal{S}^{(b)} = \{s^{(b)}\}$ with $s^{(a)} \neq s^0$, $s^{(b)} \neq s^0$, and $s^{(a)} \neq s^{(b)}$. By Proposition 1, for a given selection of $(s^0, s^{(a)}, s^{(b)})$, and a given value of $(S_i, \mathbf{W})$, the hypothesis of unbiased beliefs implies one testable restriction.

The restriction has this form:

$$\frac{q_{it}(1, s_i, s^{(a)}, \mathbf{w}) - q_{it}(1, s_i, s^0, \mathbf{w})}{q_{it}(1, s_i, s^{(b)}, \mathbf{w}) - q_{it}(1, s_i, s^0, \mathbf{w})} - \frac{P_{jt}(1|s_i, s^{(a)}, \mathbf{w}) - P_{jt}(1|s_i, s^0, \mathbf{w})}{P_{jt}(1|s_i, s^{(b)}, \mathbf{w}) - P_{jt}(1|s_i, s^0, \mathbf{w})} = 0 \tag{14}$$

It is clear that we can estimate nonparametrically all the components of this expression and implement a test. ■

In addition to knowledge of whether beliefs are in equilibrium or not, researchers can be interested in other properties of the beliefs function. In some applications, it is economically interesting to know whether beliefs are monotone in the player-specific variable. For example, in a model of market entry with multiple number of stores, such as our empirical application in section 6, the beliefs function represents the probability of opening a new store. One of the player-specific state variables is the stock of stores opened at previous periods. The beliefs function can be increasing, decreasing, or non-monotonic in the stock of opponent's stores depending on a firm's beliefs about the opponent's degree of cannibalization and economies of density.

Suppose that the state variable S_j is ordered. Without making any further assumptions, we can in fact test whether beliefs functions are monotone with respect to this state variable. To see this, consider again the case with $A = 2$. For a given value of $(S_i, \mathbf{W})$ and a given selection of values $\{s^{(1)}, s^{(2)}, s^{(3)}\}$ such that $s^{(1)} < s^{(2)} < s^{(3)}$, define:

$$\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)}) \equiv \frac{q_{it}(1, s_i, s^{(3)}, \mathbf{w}) - q_{it}(1, s_i, s^{(2)}, \mathbf{w})}{q_{it}(1, s_i, s^{(2)}, \mathbf{w}) - q_{it}(1, s_i, s^{(1)}, \mathbf{w})} \tag{15}$$

By Proposition 1 we know that $\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)})$ is identified and is a function of player i's contemporaneous beliefs about player j:

$$\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)}) = \frac{B_{ijt}^{(t)}(1, s_i, s^{(3)}, \mathbf{w}) - B_{ijt}^{(t)}(1, s_i, s^{(2)}, \mathbf{w})}{B_{ijt}^{(t)}(1, s_i, s^{(2)}, \mathbf{w}) - B_{ijt}^{(t)}(1, s_i, s^{(1)}, \mathbf{w})} \tag{16}$$

Moreover, it is clearly the case that $\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)}) \geq 0$ if and only if the beliefs function $B_{ijt}^{(t)}$ is monotonic (either increasing or decreasing) in S_j. Therefore, in addition to a test of equilibrium beliefs, we also have a test of monotonicity versus non-monotonicity of the beliefs function.

Note that the identification result in Proposition 1 applies both to non-stationary dynamic models (e.g., finite horizon, time-varying payoff and/or transition probability functions) and to infinite horizon stationary models. Suppose that the researcher is willing to assume that the primitives of the model, other than beliefs, satisfy the stationary conditions, i.e., infinite horizon and time-invariant payoff and transition probability functions. Then, the identification result in Proposition 1 can be used to test for convergence of beliefs to a stationary equilibrium. More specifically, the identification result can be applied to every period t in the sample such that the researcher can check whether the belief biases captured by the expression $\Delta Q_{it}^{(a)}(s_i, \mathbf{w}) \left[\Delta \mathbf{Q}_{it}^{(b)}(s_i, \mathbf{w})\right]^{-1} - \Delta P_{jt}^{(a)}(s_i, \mathbf{w})$ $\left[\Delta \mathbf{P}_{jt}^{(b)}(s_i, \mathbf{w})\right]^{-1}$ decline over time or not.

3.2.5 Identification with exclusion restrictions and partially unbiased beliefs

The following assumption presents a restriction on beliefs that is weaker than the assumption of equilibrium beliefs and that together with assumptions ID-1 to ID-3 is sufficient to nonparametrically identify beliefs and useful functions of payoffs in the model.

ASSUMPTION ID-4: Let $\mathcal{S}^{(R)} \subset \mathcal{S}$ be a subset of values in the set $\mathcal{S}$, with dimension $|\mathcal{S}^{(R)}| \equiv R$ that is strictly smaller than $|\mathcal{S}|$.

> *(i) For every state $\mathbf{x} = (\mathbf{s}, \mathbf{w})$ with $s_j \in \mathcal{S}^{(R)}$, the contemporaneous beliefs of player i on the behavior of player j, $B_{ijt}^{(t)}(y_j|\mathbf{x})$, are known to the researcher, either because beliefs are unbiased at these states, i.e., $B_{ijt}^{(t)}(y_j|\mathbf{x}) = P_{jt}(y_j|\mathbf{x})$, or because the researcher has information on elicited beliefs at these states.*
> *(ii) Let $\mathbf{P}_{-it}^{(R)}(s_i, \mathbf{w})$ be the $R^{N-1} \times A^{N-1}$ matrix with elements $\{P_{-it}(\mathbf{y}_{-i}|s_i, \mathbf{s}_{-i}, \mathbf{w}) : \mathbf{y}_{-i} \in \mathcal{Y}^{\mathbf{N-1}}, \mathbf{s}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$. For every period t and any value of $(S_i, \mathbf{W})$, this matrix has rank A^{N-1}.*

Condition (i) establishes that there are some values of the opponents' stock variables $\mathbf{S}_{-i}$ for which the researcher has direct information on beliefs, either because at these states strategic uncertainty disappears and beliefs about opponents' choice probabilities become unbiased, or alternatively, because the researcher has data on elicited beliefs for a limited number of states. Since $\mathcal{S}^{(R)}$ is a subset of the space $\mathcal{S}$, it is clear that Assumption ID-4(i) is weaker than the assumption of equilibrium beliefs, or alternatively, it is weaker than the condition of observing elicited beliefs for every possible value of the state variables. Note that the assumption does not necessarily mean that there is a subset of markets where beliefs are always in equilibrium. The assumption says that there is a subset of points in the state space such that a player's beliefs are unbiased every time that a point in that subset is reached, in any market. As such, in two markets m_1 and m_2, players may have beliefs out of equilibrium at some time period t, but the state in market m_1 may transit to a point where beliefs are unbiased at period $t+1$ while the state in market m_2 does not.

Condition (ii) is needed for the rank condition of identification. A stronger but more intuitive condition than (ii) is that $P_{-it}(\mathbf{y}_{-i}|\mathbf{x})$ is strictly monotonic with respect to $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, the actual choice probabilities of the other players depend monotonically on the state variables in $\mathbf{S}_{-i}$. Note that this intuition only applies to the case where $\mathbf{S}_{-i}$ is a scalar variable, as would be the case in a two player game.

EXAMPLE 5: For the dynamic game in Example 1, we have that $S_{it} = (Z_{it}, Y_{i,t-1})$ such that the space $\mathcal{S}$ is equal to $\mathcal{Z} \times \mathcal{Y}$, with $\mathcal{Z}$ being the space of Z_{it} and $\mathcal{Y}$ is the binary set $\{0, 1\}$. Suppose that the set $\mathcal{S}^{(R)}$ consists of a pair of values $\{z^*, 0\}$ and $\{z^*, 1\}$, where z^* is a particular point in the support $\mathcal{Z}$. Assumption ID-4 establishes that for every value of S_{it} we have that:

$$\begin{array}{rcl} B_{it}^{(t)}(1 \mid S_{it}, S_{jt} = [z^*, 0]) & = & P_{jt}(1 \mid S_{it}, S_{jt} = [z^*, 0]) \\ & & \\ B_{it}^{(t)}(1 \mid S_{it}, S_{jt} = [z^*, 1]) & = & P_{jt}(1 \mid S_{it}, S_{jt} = [z^*, 1]) \end{array} \tag{17}$$

That is, when the value of Z_j is z^*, player i has unbiased beliefs about the behavior of player j whatever is the value of S_{it} and Y_{jt-1}. In this example, $\mathbf{P}_{-it}^{(R)}(S_{it})$ is the 2×2 matrix:

$$\mathbf{P}_{jt}^{(R)}(S_{it}) = \left[\begin{array}{cc} P_{jt}(0 \mid S_{it}, S_{jt} = [z^*, 0]) \ , & P_{jt}(1 \mid S_{it}, S_{jt} = [z^*, 0]) \\ P_{jt}(0 \mid S_{it}, S_{jt} = [z^*, 1]) \ , & P_{jt}(1 \mid S_{it}, S_{jt} = [z^*, 1]) \end{array} \right] \tag{18}$$

Condition (ii) on the rank of $\mathbf{P}_{jt}^{(R)}$ is satisfied if $P_{jt}(1|S_{it}, S_{jt} = [z^*, 0]) \neq P_{jt}(1|S_{it}, S_{jt} = [z^*, 1])$, i.e., if being an incumbent in the market at previous period has a non-zero effect on the probability of being in the market at current period. This is a very weak condition that we expect to be always satisfied in a dynamic game of market entry and exit. ■

The choice of the subset $\mathcal{S}^{(R)}$ of values where we impose the restriction of unbiased beliefs seems a potentially important modelling decision. In subsection 3.2.6 below, we discuss different approaches for the selection of subset $\mathcal{S}^{(R)}$.

Proposition 2 presents our main result on the joint identification of beliefs and payoffs. It establishes the identification of contemporaneous beliefs, $B_{ijt}^{(t)}$, and of the payoff differences $\pi_{it}\left(y_i, \mathbf{y}_{-i}, s_i^a, \mathbf{w}\right) - \pi_{it}\left(y_i, \mathbf{y}_{-i}, s_i^b, \mathbf{w}\right)$ for any pair of values s_i^a and s_i^b of the player-specific state variable S_{it}. Consider the order condition of identification for payoffs and beliefs under assumptions ID-1 to ID-4. When the number of players in the game is two, this condition becomes $R \geq A$, that can be satisfied for models where the number of states $|S|$ is strictly greater than the number of actions A. In games with more than two players, we have that $R \geq A$ is not sufficient to guarantee the order condition. However, if the support of the player-specific state variable $|S|$ is large enough, then for any number of players N and any number of actions A, there is always a value of R between A and $|S|$ such that the order condition for identification is satisfied. More generally, note that the order condition can be represented as:

$$\frac{1 - (R/|\mathcal{S}|)}{1 - (A/|\mathcal{S}|)^{N-1}} \leq \frac{1}{N-1} \tag{19}$$

If $|\mathcal{S}|$ is large enough such that $R/|\mathcal{S}|$ is close enough to 1, then it is clear that for any value of N and A, it is possible to find a value of R strictly smaller than $|\mathcal{S}|$ that satisfies this condition.

PROPOSITION 2: Suppose that assumptions MOD1 to MOD5 and ID-1 to ID-4 hold, and: (i) R is large enough such that the order condition $[1 - (R/|\mathcal{S}|)] \,/\, \left[1 - (A/|\mathcal{S}|)^{N-1}\right] \leq (N-1)^{-1}$ *holds; and (ii) matrix* $\mathbf{Q}_{it}^{(R)}(s_i, \mathbf{w})$*, with dimension* $A \times R^{N-1}$ *and elements* $\{q_{it}(y_i, s_i, \mathbf{s}_{-i}, \mathbf{w}) : y_i \in \mathcal{Y}, \mathbf{s}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$*, has rank equal to A. Then, for dynamic games with either finite or infinite horizon we have that for any period t in the sample:*

(2.1) The contemporaneous beliefs functions $\{B_{ijt}^{(t)}(y_j|\mathbf{s},\mathbf{w}) : j \neq i\}$ are nonparametrically identified everywhere.
(2.2) Function $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w}) \equiv \pi_{it}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{w})$ is nonparametrically identified everywhere.
(2.3) For any two values of S_{it}, say (s^a, s^b) the payoff difference $\pi_{it}(y, \mathbf{y}_{-i}, s^a, \mathbf{w}) - \pi_{it}(y, \mathbf{y}_{-i}, s^b, \mathbf{w})$ is identified everywhere. ■

Proof. In the Appendix.

Remark 1. The condition that the rank of $\mathbf{Q}_{it}^{(R)}(s_i, \mathbf{w})$ is equal to A, in condition (ii), is satisfied if the conditional choice probability function of player i is strictly monotonic in $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, the actual choice probabilities of the other players depend monotonically on the state variables in $\mathbf{S}_{-i}$. Note that for the identification of the payoff function we need that beliefs (or the choice probabilities of players other than i) depend monotonically on $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. And for the identification of beliefs we also need that the choice probability of the own player i depends on $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, to identify beliefs we need that player i is playing a game such that the values of the state variables of the other players affect his decision through the effect of these variables in their beliefs. If the other players' actions do not have any effect on the payoff of player i, then his beliefs do not have any effect on his actions and therefore his actions cannot reveal any information about his beliefs.

Remark 2. In games with only two players, we can get identification of payoffs and beliefs by imposing the restriction of unbiased beliefs at only $R = A$ values of the player-specific state variable. When the number of players increases, identification requires that we impose the restriction of unbiased beliefs in an increasing fraction of states. For instance, in a binary choice model ($A = 2$) with $|\mathcal{S}| = 10$ states, the minimum value of the ratio $R/|\mathcal{S}|$ to achieve identification is 20% in a model with two players, 72% with three players, 87% with four players, 93% with five players, and so on. In the limit, as the number of players goes to infinity, identification requires that the ratio $R/|\mathcal{S}|$ goes to one, i.e., in the limit we need to impose the restriction of unbiased beliefs at every possible state. This result is quite intuitive given that, as the number of players increases, the number of parameters in the payoff function increases exponentially according to the function A^{N-1}. Nevertheless, when the number of players is not too large, such as $N \leq 5$, beliefs and payoff differences are identified even when we allow beliefs to be biased in a non-negligible fraction of states.

Remark 3. Proposition 2 emphasizes how an exclusion restriction, that is common in applications of dynamic games, provides identification of contemporaneous beliefs and payoff differences under very weak restrictions on players' beliefs and on their evolution over time. However, this Proposition does not provide full identification of payoffs, only of payoff differences. A possible way to obtain full identification of payoffs is to impose one further restriction on payoffs. If, for every value of $(y_i, \mathbf{y}_{-i}, \mathbf{w})$, one of the $|S|$ payoff values $\pi_{it}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w})$ is known, either through a normalization or because the researcher has data on payoffs at these points, then payoffs are immediately identified everywhere.[15] For instance, in some applications of interest, particularly in empirical IO, only the number of competitors taking an action, not the identity of the competitors, enters in a player's payoff. The number of payoff parameters is substantially reduced in such applications and identification can be achieved even if we allow beliefs to be biased in a large fraction of states.

Also, if beliefs do not evolve over time so that $B^{(t)}_{ijt+s}$ is constant with respect to t and the game has a finite horizon, the model is identified. In such a case the researcher can use a backwards induction argument as discussed above in Section 3.2.2. This is a very strong assumption, and we do not consider it further here.

For the rest of the paper we focus our attention on dynamic games with two players. We use subindexes i and j to represent the two players.

3.2.6 Where to assume unbiased beliefs?

As we mentioned above, the choice of the subset $\mathcal{S}^{(R)}$ where we impose the restriction of unbiased beliefs is a potentially important modelling decision. Here we describe three different approaches that may help the researcher when making this modelling decision.

(a) Applying the test of equilibrium beliefs. Consider a two-player binary-choice version of the game. We can apply the test of unbiased beliefs to any possible triple of values of the excluded state variable, say $(s_j^{(a)}, s_j^{(b)}, s_j^{(c)})$. If a triple passes the test (i.e., its p-value if greater than say 10%), we can select two of the values in the triple as members of the set $\mathcal{S}^{(R)}$ where we impose the restriction of unbiased beliefs. If multiple triples pass the test, then we can select two values from the triple that has the largest p-value in the test. This approach can be simply generalized to games with any number of players and choice alternatives.

To implement this method, especially in a case where the set of possible triples has large cardinality, the researcher needs to account for the fact that this is a problem of multiple testing. If decisions about the individual hypotheses are based on the unadjusted marginal p-values, pure sampling error will eventually lead the researcher to find triples in the sample where the null hypothesis of equilibrium beliefs can not be rejected, even if this is not true in the population. Bonferroni's correction is a simple and well-known approach to adjust p-values for multiple testing. The survey paper by Romano, Shaikh, and Wolf (2010) describes recent developments based on resampling that result in an improved ability to reject false hypotheses.

[15] If s^* is the value of S where payoffs are known, the researcher can use the identified difference $\pi_{it}(y_i, \mathbf{y}_{-i}, s^*, \mathbf{w}) - \pi_{it}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w})$ to recover $\pi_{it}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w})$ for any s_i, and then continue substituting in payoff differences to recover all other payoffs.

The effective application of this approach to select the points in the set $\mathcal{S}^{(R)}$ requires an additional condition. For the two-player binary-choice game, the DGP should be such that beliefs are unbiased at no less than three points in the support set $\mathcal{S}$. If beliefs are unbiased at only two points, then, in general, the test of unbiased beliefs will reject the null hypothesis at any triple and we cannot detect the points with unbiased beliefs using this approach. More generally, in a two-player game with A choice alternatives, to detect A points with unbiased beliefs, we need that in the DGP there are at least $A+1$ points where beliefs are unbiased.

(b) Testing for the monotonicity of beliefs and using this restriction. Suppose that the state variable S_j is ordered and the CCP function $P_{jt}(y_j|s_i, s_j)$ is strictly monotonic in this state variable. In subsection 3.2.4 above, we showed that we can use our estimate of $\delta_q \equiv [q_{it}(1, s_i, S_j = s^{(3)}) - q_{it}(1, s_i, S_j = s^{(2)})] \ / \ [q_{it}(1, s_i, S_j = s^{(2)}) - q_{it}(1, s_i, S_j = s^{(1)})]$ to test for the monotonicity of the beliefs function $B_{ijt}^{(t)}$ with respect to S_j, i.e., strict monotonicity implies that $\delta_q > 0$. Suppose that we cannot reject the strict monotonicity of the beliefs function. Then, if the data generating process is such that the player-specific variable has a large support on the real line, the monotonicity of both CCPs and beliefs implies that the CCP function P_{jt} and the beliefs function $B_{ijt}^{(t)}$ converge to each other at extreme points of the support, and it is natural to assume unbiased beliefs at these extreme points.

(c) Minimization of the player's beliefs bias. Every choice of the set $\mathcal{S}^{(R)}$ implies a different estimate of payoffs and of the beliefs at states within and outside the set $\mathcal{S}^{(R)}$, and therefore a different distance between the vector of player contemporaneous beliefs $\mathbf{B}_{ijt}^{(t)}$ and the actual CCPs of player j, i.e., $\left\|\mathbf{B}_{ijt}^{(t)} - \mathbf{P}_{jt}\right\|$. The researcher may want to be conservative and minimize the departure of his model with respect to the paradigm of rational expectations or unbiased beliefs. If that is the case, the researcher can select the set $\mathcal{S}^{(R)}$ to minimize a bias criterion such as the distance $\left\|\mathbf{B}_{ijt}^{(t)} - \mathbf{P}_{jt}\right\|$.

In our empirical application, in section 6, we apply these arguments to justify our selection of the points in the state space where we impose the restriction of unbiased beliefs. We should note that the model selection methods proposed in this section can introduce a finite sample bias in our estimators of structural parameters and our inference using those estimators. This is the well-known problem of pre-testing (Leeb and Potscher, 2005) that is pervasive in many applications in econometrics. The sampling error at the model selection stage is not independent of the sampling error in the post-selection parameter estimates, and it can affect and distort the sampling distributions of these estimates. Different authors have advocated using bootstrap methods to construct correct post-selection inference methods. Recent work by Leeb and Potscher (2005, 2006) shows the limitations of some of these methods. In a recent paper, Berk et al. (2013) propose a new method to perform valid post model selection inference. Their method consists in doing simultaneous inference of the parameter estimates for all the possible models that can be selected. This method can be applied to our problem.

3.2.7 Unobserved market-specific heterogeneity

In Assumption ID-1 we require that a player has the same beliefs at any two markets with the same observable characteristics (to the econometrician). While this restriction[16] is strictly weaker than the assumption of equilibrium beliefs (which assumes common and *correct* beliefs across observationally equivalent markets), in some applications it can be quite restrictive and it is important to know whether it can be relaxed, and at what cost.

We can allow for non-common beliefs across observationally identical markets by including a common knowledge (to players) market level unobservable, say ω_m. Suppose that the unobservable variable ω_m is i.i.d. across markets with a distribution that has finite support. Payoff, beliefs, and CCP functions include variable ω as an argument, i.e., $\pi_{it}(\mathbf{Y}, \mathbf{X}, \omega)$, $B_{ijt}^{(t)}(Y_j|\mathbf{X}, \omega)$, and $P_{it}(Y_i|\mathbf{X}, \omega)$. Suppose, for the moment, that the CCP function $P_{it}(Y_i|\mathbf{X}, \omega)$ is identified for every value $(Y_i, \mathbf{X}, \omega)$. Under this condition, it is straightforward to show that our identification results in Propositions 1 and 2 extend to this model. Therefore, the only new identification problem associated to including unobserved market heterogeneity comes from the identification of the CCPs $P_{it}(Y_i|\mathbf{X}, \omega)$.

Kasahara and Shimotsu (2009) (hereafter KS) study the identification of CCPs and the distribution of unobserved types in nonparametric finite-mixture Markov decision models.[17] Proposition 4 in KS provides identification conditions when the problem is non-stationary, i.e., time-dependent CCPs and transition probabilities. Given the inherent non-stationarity of our model (since beliefs are not restricted over time), this is the relevant result in our context. The conditions for identification in KS Proposition 4 always hold in our model, except for one condition. They assume that the transition probability function $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$ has full support over the whole state space $\mathcal{X}$, i.e., $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t) > 0$ for all t and any $(\mathbf{X}_{t+1}, \mathbf{Y}_t, \mathbf{X}_t)$. This condition rules out dynamic models with a deterministic transition rule for some state variables, e.g., games of market entry.

4 Estimation and Inference

Our constructive proofs of the identification results in Propositions 1-3 suggest methods for estimation and testing of the nonparametric model. Section 4.1 presents our test for the null hypothesis of unbiased beliefs. Section 4.2 provides a description of a nonparametric estimation method. In most empirical applications, the payoff function is parametrically specified. For this reason, in section 4.3 we extend the estimation method to deal with parametric models. In the Appendix, we derive the asymptotic properties of the estimators and tests.

[16] See also Otsu, Pesendorfer, and Takahashi (2016) for a procedure to test for this restriciton.

[17] Hu and Shum (2013) extend identification results in Kasahara and Shimotsu (2009) to a richer model with time-variant serially correlated unobserved heterogeneity with a Markov chain structure.

4.1 Test of unbiased beliefs

Recall from Proposition 1 above that unbiased beliefs for player i implies the following $(A-1)^2$ *restrictions between CCPs of players i and j:*

$$\Delta\mathbf{Q}_{it}^{(a)}(s_i,\mathbf{w}) \ \left[\Delta\mathbf{Q}_{it}^{(b)}(s_i,\mathbf{w})\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(s_i,\mathbf{w}) \ \left[\Delta\mathbf{P}_{jt}^{(b)}(s_i,\mathbf{w})\right]^{-1} = \mathbf{0} \tag{20}$$

Remember that, by Hotz-Miller inversion Theorem, $q_{it}(y_i|\mathbf{x})$ is a known function of the vector of CCPs $\mathbf{P}_{it}(\mathbf{x})$, i.e., $q_{it}(y_i|\mathbf{x}) \equiv \Lambda^{-1}(y_i, \mathbf{P}_{it}(\mathbf{x}))$. In the context of the two player binary choice model,[18] the subsets $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ are single element sets: $\mathcal{S}^{(a)} = \{s^{(a)}\}$ and $\mathcal{S}^{(b)} = \{s^{(b)}\}$. For each $s^{(k)} \in \mathcal{S}_j \setminus \{s^{(a)}, s^{(b)}\}$ it is possible to show that the following restrictions hold if player i's beliefs are unbiased:

$$\frac{\Lambda^{-1}(P_{it}(1|s_i,s^{(k)},\mathbf{w})) - \Lambda^{-1}(P_{it}(1|s_i,s^{(a)},\mathbf{w}))}{\Lambda^{-1}(P_{it}(1|s_i,s^{(b)},\mathbf{w})) - \Lambda^{-1}(P_{it}(1|s_i,s^{(a)},\mathbf{w}))} = \frac{P_{jt}(1|s_i,s^{(k)},\mathbf{w}) - P_{jt}(1|s_i,s^{(a)},\mathbf{w})}{P_{jt}(1|s_i,s^{(b)},\mathbf{w}) - P_{jt}(1|s_i,s^{(a)},\mathbf{w})} \tag{21}$$

There are $|\mathcal{S}_j| - 2$ such restrictions for each value of $(S_i, \mathbf{W})$, for a total of $|\mathcal{S}_i||\mathcal{W}| \ (|\mathcal{S}_j| - 2)$ restrictions.[19]

Consider the nonparametric multinomial model where the probabilities are the CCPs of players i and j. The log-likelihood function of this multinomial model is:

$$\begin{aligned} \ell(\mathbf{P}_i, \mathbf{P}_j) \quad &= \quad \sum_{m,t} y_{imt} \ln P_{it}(\mathbf{x}_{mt}) + (1 - y_{imt}) \ln[1 - P_{it}(\mathbf{x}_{mt})] \\ &+ \quad \sum_{m,t} y_{jmt} \ln P_{jt}(\mathbf{x}_{mt}) + (1 - y_{jmt}) \ln[1 - P_{jt}(\mathbf{x}_{mt})] \end{aligned} \tag{22}$$

where $\mathbf{P}_i$ and $\mathbf{P}_j$ are the vectors of CCPs for player i and j, respectively, for every value period t and every value of $\mathbf{X}_{mt}$. These vectors of CCPs are the 'parameters' of this nonparametric model. The null hypothesis of unbiased beliefs imposes the set of restrictions (21) on the parameters $\mathbf{P}_i$ and $\mathbf{P}_j$.

A Likelihood Ratio (LR) Test seems a natural candidate for testing the set of restrictions (21) implied by the null hypothesis of unbiased beliefs. The Likelihood Ratio test statistic is given by:

$$LR = 2\left[\ell\left(\widehat{\mathbf{P}_i^u}, \widehat{\mathbf{P}_j^u}\right) - \ell\left(\widehat{\mathbf{P}_i^c}, \widehat{\mathbf{P}_j^c}\right)\right] \tag{23}$$

where $\left(\widehat{\mathbf{P}_i^u}, \widehat{\mathbf{P}_j^u}\right)$ is the Unconstrained Maximum Likelihood estimator of $(\mathbf{P}_i, \mathbf{P}_j)$ from the non-parametric multinomial likelihood in (22), i.e., the frequency estimator of the CCPs, and $\left(\widehat{\mathbf{P}_i^c}, \widehat{\mathbf{P}_j^c}\right)$ is the Constrained Maximum Likelihood estimator of $(\mathbf{P}_i, \mathbf{P}_j)$ given the set of restrictions in (21). Under mild regularity conditions, the asymptotic distribution of this likelihood ratio is Chi-squared with $T_{data} \ |\mathcal{S}_i| \ |\mathcal{W}| \ (|\mathcal{S}_j| - 2)$ degrees of freedom.[20]

[18] The extension to several players and actions is straightforward.

[19] The restrictions hold automatically for $k = a$ or $k = b$.

[20] In principle, we could also have used a standard Lagrange Multiplier (LM) test. This is asymptotically equivalent to the test we present here.

4.2 Estimation with nonparametric payoff function

Nonparametric estimation proceeds in two steps.

Step 1: Nonparametric estimation of CCPs and transition probabilities. For every player, time period, and state value $\mathbf{x}$ and $\mathbf{x}'$, we estimate CCPs $P_{it}(y_i|\mathbf{x})$, and (if necessary) the transition probabilities $f_t(\mathbf{x}'|\mathbf{y},\mathbf{x})$. We also construct estimates of $q_{it}(y_i,\mathbf{x})$ by inverting the mapping Λ, i.e., $q_{it}(y_i,\mathbf{x}) = \Lambda^{-1}(y_i, \mathbf{P}_{it}(\mathbf{x}))$.

Step 2: Estimation of preferences and beliefs. We select the subset $\mathcal{S}^{(R)}$ with the values of S_j for which we assume that player i's beliefs are unbiased. Given this set and the estimates in step 1, we construct, for any period t and any value of $(Y_i, S_i, \mathbf{W})$, the matrix $R \times A$ matrix $\mathbf{P}_{-it}^{(R)}(s_i,\mathbf{w})$ as defined in Assumption ID-4, and the $R \times 1$ vector $\mathbf{q}_{it}^{(R)}(y_i,s_i,\mathbf{w})$ with elements $\{q_{it}(y_i,s_i,\mathbf{s}_{-i},\mathbf{w}) : \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$. Remember that the function $g_{it}^{\mathbf{B}}(y_i,y_j,s_i,\mathbf{w})$ is the function the sum of current payoff and continuation values, i.e., $\pi_{it}(y_i,y_j,s_i,\mathbf{w}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i,y_j,\mathbf{w})$, and let $\mathbf{g}_{it}^{\mathbf{B}}(y_i,s_i,\mathbf{w})$ be the $A \times 1$ vector with $g_{it}^{\mathbf{B}}(y_i,y_j,s_i,\mathbf{w})$ for every value of y_j. Then, we apply the following formulas.

(i) In the proof of Proposition 2 we show that $\mathbf{g}_{it}^{\mathbf{B}}(y_i,s_i,\mathbf{w})$ is identified as:

$$\mathbf{g}_{it}^{\mathbf{B}}(y_i,s_i,\mathbf{w}) = \left[\mathbf{P}_{-it}^{(R)}(s_i,\mathbf{w})'\ \mathbf{P}_{-it}^{(R)}(s_i,\mathbf{w})\right]^{-1} \mathbf{P}_{-it}^{(R)}(s_i,\mathbf{w})'\ \mathbf{q}_{it}^{(R)}(y_i,s_i,\mathbf{w})\ ; \tag{24}$$

(ii) In that proof, we also show that contemporaneous beliefs are identified using the following expression,[21]

$$\mathbf{B}_{it}^{(t)}(\mathbf{x}) = \left[\widetilde{\mathbf{G}}_{it}(\mathbf{x})\right]^{-1} \mathbf{q}_{it}(\mathbf{x}) \tag{25}$$

where: $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ is an $A\times 1$ vector with $\{B_{it}^{(t)}(y_j|\mathbf{x}) : y_j \in \mathcal{Y}\}$; $\mathbf{q}_{it}(\mathbf{x})$ is an $A \times 1$ vector with elements $\{q_{it}(1,\mathbf{x}),\ ...,\ q_{it}(A-1,\mathbf{x})\}$ at rows 1 to $A-1$, and a 1 at the last row; and $\widetilde{\mathbf{G}}_{it}(\mathbf{x})$ is an $A \times A$ matrix where the element (y_i, y_j+1) is $g_{it}^{\mathbf{B}}(y_i,y_j,s_i,\mathbf{w})$ and the last row of the matrix is a row of ones. (iii) Finally, by definition of the function $g_{it}^{\mathbf{B}}(y_i,\mathbf{y}_{-i},s_i,\mathbf{w})$ we have that

$$\pi_{it}(y_i,y_j,s_i,\mathbf{w}) - \pi_{it}(y_i,y_j,s^*,\mathbf{w}) = g_{it}^{\mathbf{B}}(y_i,y_j,s_i,\mathbf{w}) - g_{it}^{\mathbf{B}}(y_i,y_j,s^*,\mathbf{w}) \tag{26}$$

for some value s^* of the player-specific state variable that we take as a benchmark. Then, we can apply the $|\mathcal{S}|$ restrictions in Proposition 3(i) to obtain the payoffs $\pi_{it}(y_i,y_j,s_i,\mathbf{w})$.

In the appendix, we prove the consistency and asymptotic normality of the estimators of payoffs and beliefs that apply this procedure.

[21] There is a somewhat subtle relationship between our idenfitication result here and the literature on nonparametric identification of finite mixture models (for example Hall and Zhou (2003) and Kasahara and Shimotsu (2009)). In particular, the result that payoffs are identified if beliefs are known and invertible at a sufficiently large subset of points in the state space has a parallel with the structure of finite mixture of distributions with an exclusion restriction. In that literature such an identification result is not particularly useful, as it requires knowledge of the mixture weights at different values of the excluded variable. In the present context it is motivated through elicited beliefs or theoretical assumptions.

4.3 Estimation with parametric payoff function

In most applications the researcher assumes a parametric specification of the payoff function. A class of parametric specifications that is common in empirical applications is the linear in parameters model:

$$\pi_{it}(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) = h(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t)\ \boldsymbol{\theta}_{it} \tag{27}$$

where $h(Y_i, Y_j, S_i, \mathbf{W})$ is a $1 \times K$ vector of known functions, and $\boldsymbol{\theta}_{it}$ is a $K \times 1$ vector of unknown structural parameters in player i's payoff function. Let $\boldsymbol{\theta}_i$ be the vector with all the parameters in the payoff of player i: $\boldsymbol{\theta}_i \equiv \{\boldsymbol{\theta}_{it} : t = 1, 2, ..., T_{data}\}$.

EXAMPLE 6: Consider the dynamic game in Example 1. The profit function in equation (2) can be written as $h(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t)\ \boldsymbol{\theta}_i$, where the vector of parameters $\boldsymbol{\theta}_i$ is $(\theta_i^M, \theta_i^D, \theta_{i0}^{FC}, \theta_{i1}^{FC}, \theta_i^{EC})'$ and

$$h(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) = Y_{it}\ \{\ H_t,\ - H_t Y_{jt},\ - 1,\ - Z_i, - 1\{Y_{it-1} = 0\}\ \} \qquad \blacksquare \tag{28}$$

To estimate $\boldsymbol{\theta}_i$ we propose a simple three steps method. The first two-steps are the same as for the nonparametric model.

Step 3: Given the estimates from step 2, we can apply a pseudo maximum likelihood method in the spirit of Aguirregabiria and Mira (2002) to estimate the structural parameters $\boldsymbol{\theta}_i$. Define the following pseudo likelihood function for the model with i.i.d. extreme value $\varepsilon' s$:

$$Q(\boldsymbol{\theta}_i, \mathbf{B}_i, \mathbf{P}_i) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T_{data}} \log \left(\frac{\exp\left\{ \widetilde{h}_{it}^{\mathbf{B},\mathbf{P}}(y_{imt}, \mathbf{x}_{mt})\ \boldsymbol{\theta}_i + \tilde{e}_{it}^{\mathbf{B},\mathbf{P}}(y_{imt}, \mathbf{x}_{mt}) \right\}}{\sum_{y_i=0}^{A-1} \exp\left\{ \widetilde{h}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{x}_{mt})\ \boldsymbol{\theta}_i + \tilde{e}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{x}_{mt}) \right\}} \right) \tag{29}$$

$\widetilde{h}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{x})$ is the discounted sum of the expected values of $\{h(y_{jt+s} y_{jt+s}, \mathbf{x}_{t+s}) : s = 0, 1, ..., T - t\}$ given that the state at period t is $\mathbf{x}$, that player i chooses alternative y_i at period t and then behaves according to the choice probabilities in $\mathbf{P}$, and believes that player j behaves according to the probabilities in $\mathbf{B}$. And $\tilde{e}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{x})$ is also a discounted sum, but of expected future values of $\sum_{y_i=0}^{A-1} P_{it+s}(y_i|\mathbf{x}_{mt+s})\ [\gamma - \ln P_{it+s}(y_i|\mathbf{x}_{mt+s})]$, that represents the expected value of $\varepsilon_{im,t+s}(Y_{imt+s})$ when $Y_{im,t+s}$ is optimally chosen, and γ is Euler's constant. From steps 1 and 2, we have consistent estimates of CCPs, $\hat{\mathbf{P}}_i$, and beliefs, $\hat{\mathbf{B}}_i$. Then, a consistent pseudo maximum likelihood estimator of $\boldsymbol{\theta}_i$ is defined as the value $\hat{\boldsymbol{\theta}}_i^{(1)}$ that maximizes $Q(\boldsymbol{\theta}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{P}}_i)$. Note that the sample criterion function $Q(\boldsymbol{\theta}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{P}}_i)$ is just the log likelihood function of a Conditional Logit model with the restriction that the parameter multiplying the discounted sum $\tilde{e}_{it}^{\mathbf{B},\mathbf{P}}$ is equal to 1. The estimator is root-M consistent and asymptotically normal.[22]

[22] Iterative procedures can often be used to improve on the finite sample properties of two step estimators in dynamic games which can suffer from imprecise nonparametric estimates in the first step (Aguirregabiria and Mira 2007, Kasahara and Shimotsu 2012). In principle, steps 1 to 3 here can also be applied recursively to try to improve the statistical properties of our estimators. Though the resulting K-step estimator is consistent and asymptotically

5 Monte Carlo Experiments

We use Monte Carlo experiments to illustrate the identification, estimation, and inference framework presented in previous sections. We study the ability of our test to reject the null hypothesis when it is false (the power of the test), and how frequently we reject the null when it is in fact true (the size of the test). We also study a key trade-off that a researcher faces when deciding to impose equilibrium restrictions on the data in the estimation of a dynamic game. By imposing the assumption of equilibrium beliefs the researcher is able to rely on the identification power afforded by the equilibrium restrictions, which results in more precise estimates. This is particularly relevant in small samples. However, the identification power associated with equilibrium restrictions comes with a price - the possibility of biased estimates if the restrictions are not true in the DGP (the model is mis-specified) We study the magnitude of this bias in the context of a simple application.

The model we consider in our experiments is a particular case of the *dynamic game of market entry and exit* in Example 1. We consider an infinite horizon game with two players. The per period profit functions of the two players are given by:

$$\begin{array}{rcl} \pi_{1mt}(1, Y_{2mt}, \mathbf{X}_{mt}) & = & (1 - Y_{2mt})\ \theta_1^M + Y_{2mt}\ \theta_1^D - \theta_{01}^{FC} - (1 - Y_{1mt-1})\ \theta_1^{EC} \\ \pi_{2mt}(1, Y_{1mt}, \mathbf{X}_{mt}) & = & (1 - Y_{1mt})\ \theta_2^M + Y_{1mt}\ \theta_2^D - \theta_{02}^{FC} - \theta_{12}^{FC}\ Z_{2mt} - (1 - Y_{2mt-1})\ \theta_2^{EC} \end{array} \tag{30}$$

We normalize the profits to not being active to be zero for both players: $\pi_{1mt}(0, Y_{2mt}, \mathbf{X}_{mt}) = \pi_{2mt}(0, Y_{1mt}, \mathbf{X}_{mt}) = 0$. The players' payoffs to being active are symmetric except for the variable Z_{2mt} which enters player 2's payoffs but not player 1's. Z_{2mt} is an exogenous time variant characteristic which affects the fixed cost of player 2, but does not have a (direct) effect on the payoff of player 1. Following the notation in previous sections, we can describe the vector of observable state variables by $\mathbf{X}_{mt} = (S_{1mt}, S_{2mt})$ with $S_{1mt} = Y_{1mt}$ and $S_{2mt} = (Z_{2mt}, Y_{2mt-1})$.

We focus on the estimation of the parameters in player 1's payoff and beliefs functions. Given the payoff structure in equation (30) above, only the payoffs and beliefs of player 1 are identified under our identification assumptions.[23] We can re-write the payoff function as:

$$\pi_{1t}(1, Y_{2mt}, \mathbf{X}_{mt}) \quad = \quad \alpha_1 - \delta_1\ Y_{2mt} + Y_{1m,t-1}\ \theta_1^{EC} \tag{31}$$

normal, stronger restrictions are needed to guarantee that these iterations improve the asymptotic or/and finite sample properties of the estimator relative to the two-step estimator. In the context of dynamic games with equilibrium beliefs, Kasahara and Shimotsu (2012) show that in the sequence of K-step estimators the finite sample bias declines monotonically only if the mapping associated to this iterative procedure satisfies a local contraction property. Pesendorfer and Schmidt-Dengler (2010) illustrate with an example that this iterative procedure can converge to an inconsistent estimator if the local contraction property does not hold. In our model with biased beliefs, the iterative mapping is different and the finite sample properties of these K-step estimators require further investigation.

[23] Specifically, there is no variable with at least three points of support (i.e., $A+1 = 3$) that enters player 1's payoffs directly and does not enter player 2's payoffs directly. In principle Y_{1mt-1} could play the role of the player-specific variable for identifying player 2's payoffs and beliefs, but since it can only take two values it is always at an "extreme point".

where the parameters α_1 and δ_1 are defined as $\alpha_1 \equiv \theta_1^M - \theta_{01}^{FC} - \theta_1^{EC}$ and $\delta_1 \equiv \theta_1^M - \theta_1^D$. The exogenous variable Z_{2mt} is independently and identically distributed over markets and time, with a discrete uniform distribution with support $\{-2, -1, 0, 1, 2\}$. This variable is key to identify the payoffs and beliefs of player 1. Essentially Z_{2mt} plays the role of an instrument in the sense that it satisfies an exclusion restriction. It affects player 1's payoffs only through its effect on the behavior of player 2.

We keep all the parameters in the payoff functions constant across all experiments. These values are: $\alpha_1 = \alpha_2 = 2.4$; $\delta_1 = \delta_2 = 3.0$; $\theta_2^{FC} = -1.0$; $\theta_1^{EC} = \theta_2^{EC} = 0.5$; $\beta_1 = \beta_2 = 0.95$; $Z_{2mt} \sim i.i.d.\ Uniform$ with support $\{-2, -1, 0, +1, +2\}$. To provide an economic interpretation for the magnitude of these parameters note that: the firm 1's entry cost represents 17.1% of its average profit as a monopolist; firm 1's reduction in profit from monopoly to duopoly is 103.4%; and firm's 2 profit as a monopolist increases by 81.6% when Z_2 goes from -2 to 2. For Experiment B with biased beliefs, $\lambda_1 = \lambda_2 = 1$ if $Z_2 \in \{-2, 2\}$ and $\lambda_1 = \lambda_2 = 0.5$ if $Z_2 \in \{-1, 0, 1\}$. In each experiment we consider, the sample is comprised by a total of $M = 500$ markets and $T_{data} = 5$ periods of data. This is a realistic sample size, and in fact is precisely the size of the sample we consider in our empirical application in section 6. We approximate the finite sample distribution of the estimators using $10,000$ Monte Carlo replications. The initial conditions for the endogenous state variables $\{Y_{1m0}, Y_{2m0}\}$ are drawn uniformly at random.

We implement two experiments. In experiment U players' beliefs are in equilibrium, the DGP is a Markov Perfect Equilibrium from the model. In experiment players' have biased beliefs according to the following model. For each player $i \in \{1, 2\}$ and any market m and period t, beliefs are $B_{imt}^{(t)}(\mathbf{x}_{mt}) = \lambda_{im}(\mathbf{x}_{mt})\ P_{jmt}(\mathbf{x}_{mt})$, where $\lambda_{im}(\mathbf{x}_{mt}) \in [0, 1]$ is an exogenous function that captures player i's bias in beliefs. Note that, given this specification of the DGP, beliefs are endogenous because they depend on the other player's choice probabilities that, of course, are endogenous. Therefore, to obtain these beliefs we need to solve for a particular equilibrium or fixed point problem. Given $\lambda_{1m}(.)$ and $\lambda_{2m}(.)$, players' choice probabilities $P_{1m}(\mathbf{x}_{mt})$ and $P_{2m}(\mathbf{x}_{mt})$ solve a fixed point problem that we could describe as a *biased beliefs Markov Perfect Equilibrium* such that $P_{im}(y|\mathbf{x}_{mt}) = \Lambda\left(y;\ \tilde{\mathbf{v}}_i^{\mathbf{B}}(\mathbf{x}_{mt})\right)$ and $\mathbf{B}_{im} = \lambda_{im}\ \mathbf{P}_{jm}$. We fix the following values for the bias functions λ_{im}:

$$\lambda_{im}(\mathbf{x}_{mt}) \quad = \quad \begin{cases} 1 & \text{if} \quad z_{2mt} \in \{-2, 2\} \\ \\ 0.5 & \text{if} \quad z_{2mt} \in \{-1, 0, 1\} \end{cases} \tag{32}$$

That is, if the exogenous characteristic z_{2mt} is at an 'extreme' value, i.e., $z_{2mt} \in \{-2, 2\}$, then there is not any strategic uncertainty or bias beliefs: players' beliefs are in equilibrium. However, if z_{2mt} lies in the interior of the support set, then beliefs are biased. More specifically, when beliefs are biased, both players are over-optimistic such that they underestimate (by 50%) the probability of the opponent will be active in the market. Note that given our choice of distribution of z_{2mt}, beliefs are (on average) out of equilibrium at 60% of the sample observations.

5.1 Test of Equilibrium Beliefs

Figures 1 and 2 present the results of applying our test to simulated data from the two experiments. To construct these figures: (i) we have calculated the LRT statistic for each of the 10,000 simulated sample; (ii) we have constructed a very fine grid of values in the interval $[0, 0.15]$ for the significance level (or size) of the test, α; and (iii) for each value α in this grid, we have obtained the corresponding critical value in the probability distribution of the Chi-square with 16 degrees of freedom, $\chi^2_{16}(\alpha)$, and then computed the frequency of simulated samples where the LRT statistic is greater than this critical value, i.e., the empirical frequency for rejecting the null hypothesis. Figures 1 and 2 present these empirical frequencies for every value of α. We also present the 45 degree line.

Figure 1 corresponds to the experiment with biased beliefs (B). The empirical frequency presented in figure 1 is *the power of the test*, i.e., the probability of rejecting the null hypothesis when it is false. This empirical probability of rejection is very close to one for any conventional value of α, showing that the test has strong power to reject the hypothesis of equilibrium beliefs when it is false.

Figure 2 deals with the experiment with unbiased beliefs (U). Here we see that the empirical frequency of rejecting the null tracks the true probability (i.e., significance level) very closely. That is, the test has the appropriate size, and we do not systematically over or under reject the null when it is true, at least given the DGP we are considering here. The asymptotic Chi-square distribution is a good approximation to the distribution of the test statistic under a realistic sample size.

5.2 Estimation of Preferences and Beliefs

Tables 3 and 4 summarize the remainder of the results of our experiments. Table 3 reports the Mean Absolute Bias (MAB) and the standard deviation from the Monte Carlo distribution of the estimators of payoffs, beliefs and choice probabilities. Table 4 presents MABs when we consider a large sample with one million observations. The motivation behind these experiments is to study the consequences of imposing the restriction of unbiased beliefs when it does not hold in the DGP, and to evaluate the loss of precision of our estimates when we do not impose the restriction of unbiased beliefs.

(a) Benchmark. Columns (5) and (6), in table 3 present MABs and standard deviations of estimates when beliefs are unbiased in the DGP and we impose this restriction in the estimation. This is the typical estimation framework in the dynamic games literature with a DGP that satisfies the assumptions that are typically maintained. In parentheses, we report the corresponding statistic as percentage of the true value of the parameter. The bias in the estimates of payoff parameters ranges from 6% for the parameter that captures competition effects (δ_1) to 18% for the entry

cost parameter (θ_1^{EC}). The estimates of beliefs and CCPs are quite precise: MABs and standard deviations for these estimates are always smaller than 9% of the true value.

(b) Loss of precision when relaxing the assumption of unbiased beliefs. The main purpose of experiments "U" is to evaluate the loss of identification power in a finite sample when we do not impose the restrictions of equilibrium beliefs. Columns (7) and (8) correspond to an estimation of the model that does not impose equilibrium restrictions but where the data comes from a population or DGP where beliefs are in equilibrium. We can see there is a price to pay for not exploiting the equilibrium restrictions: mean absolute bias and standard deviation increase substantially when we do not enforce the assumption of equilibrium beliefs. The standard deviation of the payoff estimates is 2-3 times larger when we do not impose the equilibrium restrictions. For instance, for the payoff parameter δ_1 that measures the competition effect, the bias increases from 5.9% to 19.4%, and the standard deviation increases from 4.5% to 17.0%. This highlights the value of being able to test for equilibrium beliefs before deciding on an estimation strategy, as relaxing the assumption of equilibrium beliefs does not come for free. Nevertheless, when we do not impose equilibrium restrictions, the estimates are still quite informative about the true value of the parameters. Note finally that the loss of precision in the estimation of beliefs and CCPs is substantially less severe than in the estimation of payoffs.

(c) Consequences of imposing the assumption of equilibrium beliefs when it is not true. In experiment "B" (columns 1-4 of table 3) the DGP is such that beliefs are not in equilibrium. Comparing columns 1 and 3, we can see the increase in bias induced by imposing the assumption of equilibrium beliefs incorrectly. The bias increases from 12.8% to 21.7% for the parameter α_1, and from 9.5% to 15.2% for parameter δ_1. Surprisingly, the bias of the estimate of the entry cost parameter actually *decreases* when we wrongly impose equilibrium restrictions. This is a finite sample property, and we have confirmed this point by implementing an experiment with 1,000,000 observations and a single Monte Carlo simulation, that we report in table 4. Columns 1 and 2 in table 4 show that the MABs of the entry cost parameter is only 0.6% when equilibrium restrictions are not imposed and it increases to 10.2% when these restrictions are wrongly imposed. Going back to columns 1 to 4 in table 3, we can see that though the precision of the estimates decreases significantly when we do not impose equilibrium restrictions, the combination of bias and variance shows very significant gains in the estimates of payoffs and beliefs when we allow for biased beliefs.

In sum, the experiments illustrate that it can be very important to choose the appropriate estimation framework given the DGP. Not imposing equilibrium beliefs in estimation when beliefs are in equilibrium in the DGP is costly in terms of precision and finite sample bias, while incorrectly imposing equilibrium restrictions in estimation can be very costly both in terms of finite sample *and* asymptotic bias. This underscores the importance of testing for equilibrium beliefs before deciding on an estimation strategy.

6 Empirical Application

We illustrate our model and methods with an application of a dynamic game of store location. Recently there has been significant interest in the estimation of game theoretic models of market entry and store location by retail firms. Most studies have assumed static games: see Mazzeo (2002), Seim (2006), Jia (2008), Zhu and Singh (2009), and Nishida (2014), among others. Holmes (2011) estimates a single-agent dynamic model of store location by Wal-Mart. Beresteanu and Ellickson (2005), Suzuki (2013), and Walrath (2016) propose and estimate dynamic games of store location.

We study store location of McDonalds (MD) and Burger King (BK) using data for the United Kingdom during the period 1991-1995. The dataset was collected by Otto Toivanen and Michael Waterson, who use it in their paper Toivanen and Waterson (2005). We divide the UK into local markets (districts) and study these companies' decision of how many stores, if any, to operate in each local market. The profits of a store in a market depends on local demand and cost conditions and on the degree of competition from other firms' stores and from stores of the same chain. There are sunk costs associated with opening a new store, and therefore this decision has implications for future profits. Firms are forward-looking and maximize the value of expected and discounted profits. Each firm has uncertainty about future demand and cost conditions in local markets. Firms also have uncertainty about the current and future behavior of the competitor.

In this context, the standard assumption is that firms have rational expectations about other firms' strategies, and that these strategies constitute a Markov Perfect Equilibrium. Here we relax this assumption. The main question that we want to analyze in this empirical application is whether the beliefs of each of these companies about the store location strategy of the competitor are consistent with the actual behavior of the competitor. The interest of this question is motivated by Toivanen and Waterson (2005) empirical finding that these firms' entry decisions do not appear to be sensitive to whether the competitor is an incumbent in the market or not. As we have illustrated in our Monte Carlo experiments, imposing the restriction of equilibrium beliefs can generate an attenuation bias in the estimation of competition effects when this restriction is not true in the DGP. We investigate here this possible explanation.[24]

[24] The nature of the econometric bias in the parameters that represent strategic interactions depends on the relationship between true and rational beliefs. It is useful to illustrate this issue using a simple model. Suppose that the relationship between $q_i(x)$ and the true beliefs $B_i(x)$ is $q_i(x) = \pi_0 + \pi_1 \ B_i(x)$, where π_0 and π_1 are structural parameters from the payoff function. Suppose that the relationship between actual beliefs $B_i(x)$ and the rational beliefs $P_j(x)$ is $B_i(x) = f(P_j(x))$, where $f(.)$ is a continuous and differentiable function in the space of probabilities. And suppose that the researcher imposes the restriction of rational beliefs, such that he estimates the model $q_i(x) = \pi_0 + \pi_1 \ P_j(x) + e$, where by construction the error term e is equal to $\pi_1[f(P_j(x)) - P_j(x)]$. It is straightforward to show that under mild regularity conditions the least square estimator of the parameter π_1 converges in probability to $\pi_1 \ f'(\mathbb{E}_x[P_j(x)])$, where $f'(.)$ is the derivative of the function $f(.)$. In this case, we have an attenuation bias in the estimator of the parameter π_1 iff $f'(\mathbb{E}_x[P_j(x)]) < 1$. Note that this condition is different to a condition on the level of the bias of beliefs. That is, the condition $f'(\mathbb{E}_x[P_j(x)]) < 1$ is compatible with $\mathbb{E}_x[B_i(x)]$ being either smaller or larger than $\mathbb{E}_x[P_j(x)]$.

6.1 Data and descriptive evidence

Our working sample is a five year panel that tracks 422 *local authority districts* (local markets), including the information on the stock and flow of MD and BK stores into each district. It also contains socioeconomic variables at the district level such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the UK headquarters of each of the firms. The *local authority district* is the smallest unit of local government in the UK, and generally consists of a city or a town sometimes with a surrounding rural area. There are almost 500 *local authority districts* in Great Britain. Our working sample of 422 districts does not include those that belong to Greater London.[25] The median district in our sample has an area of 300 square kilometers and a population of 95,000 people.[26] Table 5 presents descriptive statistics for socioeconomic and geographic characteristics of our sample of local authority districts.

Table 6 presents descriptive statistics on the evolution of the number of stores for the two firms.[27] In 1990, MD had more than three times the number of stores of BK, and it was active in more than twice the number of local markets than BK. Conditional on being active in a local market, MD had also significantly more stores per market than BK. These differences between MD and BK have not declined significantly over the period 1991-1995. While BK has entered in more new local markets than MD (69 new markets for BK and 48 new markets for MD), MD has opened more stores (143 new stores for BK and 166 new stores for MD).

Table 7 presents the annual transition probabilities of market structure in local markets as described by the number of stores of the two firms. According to this transition matrix, opening a new store was an irreversible decision during this sample period, i.e., no store closings are observed during this sample period. In Britain during our sample period, the fast food hamburger industry was still young and expanding, as shown by the large proportion of observations/local markets without stores (41.6%). Although there is significant persistence in every state, the less persistent market structures are those where BK is the leader. For instance, if the state is "$BK = 1$ & $MD = 0$", there is a 20% probability that the next year MD opens at least one store in the market. Similarly, when the state is "$BK = 2$ & $MD = 1$", the chances that MD opens one more store the next year are 31%.

Table 8 presents estimates of reduced form Probit models for the decision to open a new store. We obtain separate estimates for MD and BK. Our main interest is in the estimation of the effect

[25] The reason we exclude the districts in Greater London from our sample is that they do not satisfy the standard criteria of isolated geographic markets.

[26] As a definition of geographic market for the fast food retail industry, the district is perhaps a bit wide. However, an advantage of using district as definition of local market is that most of the markets in our sample are geographically isolated. Most districts contain a single urban area. And, in contrast to North America where many fast food restaurants are in transit locations, in UK these restaurants are mainly located in the centers of urban areas.

[27] Toivanen and Waterson present a detailed discussion of why the retail chain fast food hamburger industry in the UK during this period can be assumed as a duopoly of BK and MD.

of the previous year's number of stores (own stores and competitor's stores) on the probability of opening new stores. We include as control variables population, GDP per capita, population density, proportion of population 5-14, proportion population 15-29, average rent, and proportion of claimants of unemployment benefits. To control for unobserved local market heterogeneity we also present two fixed effects estimations, one with county fixed effects and the other with local district fixed effects.[28] We only report estimates of the marginal effects associated with the dummy variables that represent previous year number of stores. The main empirical result from table 8 is that, regardless of the set of control variables that we use, the own number of stores has a strong negative effect on the probability of opening a new store but the effect of the competitor's number of stores is either negligible or even positive. This finding is very robust to different specifications of the reduced form model and it is analogous to the result from the reduced form specifications in Toivanen and Waterson (2005). The estimate of the marginal effect of the number of own stores increases significantly when we control for unobserved heterogeneity using fixed effects. However, the estimated marginal effect of the number of competitor's stores barely changes. The estimates show also a certain asymmetry between the two firms: the absence of response to the competitor's number of stores is more clear for BK than for MD. In particular, when BK has three stores in the market there is a significant reduction in MD's probability of opening a new store. This negative effect does not appear in the reduced form probit for BK.

Taken at face value, the empirical evidence suggests that Burger King is either indifferent to or prefers to enter in markets where McDonald's already has a presence. This behavior cannot be rationalized by standard static models of market entry where firms compete and sell substitute products. In such a model, Burger King's current profit is always higher (ceteris paribus) if it enters in a market where McDonalds is not present than if its entry is in a market where McDonalds already has a store. In the case of complementary goods, a firm may like to locate near another to capitalize positive spillover effects on business/traffic. However, it seems quite reasonable to consider that a MD store and a BK store are substitutes from the point of view of consumer demand at a given point in time. We discuss below other possible sources of positive spillover effects. We explore three, non-mutually exclusive, explanations for BK's observed behavior: (a) spillover effects; (b) forward looking behavior (dynamic game); and (c) biased beliefs about the behavior of the competitor.

(a) Spillover effects. The competitor's presence may have a positive spillover effect on the profit of a firm. There are several possible sources of this spillover effect. For example one firm may infer from another's decision to open a store in a particular market that market conditions are favorable (informational spillover effects). Alternatively, one firm may benefit from another firm's entry through cost reductions, or from product expansion through advertising. As such, we allow

[28] With only five observations (time periods) per district, the estimator with district fixed-effects may contain substantial biases. Despite its potential bias, the comparison of the district-FE estimator with the county-FE estimator and the estimator without fixed effects provide some interesting results.

for the possibility of spillover effects in our specification of demand, but since we do not have price and quantity data at the level of local markets, we do not try to identify the source of the spillover effect. While the natural interpretation of the spillover effect in the context of our model is a product expansion due to an advertising effect of retail stores, this should be interpreted as a reduced form' specification of different possible spillover effects.

(b) Forward looking behavior. Opening a store is a partly irreversible decision that involves significant sunk costs. Therefore, it is reasonable to assume that firms are forward looking when they make this decision. Moreover, dynamic strategic effects may help explain the apparent absence of competitive effects when we study behavior in the context of a static model of entry. Suppose that firms anticipate, with some uncertainty, the total number of hamburger stores that a local market can sustain in the long-run given the size and the socioeconomic characteristics of the market. For simplicity, suppose that this number of "available slots" does not depend on the ownership of the stores because the products sold by the two firms are very close substitutes. In this context, firms play a game where they 'race' to fill as many 'slots' as possible with their own stores. Diseconomies of scale and scope may generate a negative effect of the own number of stores on the decision of opening new stores. However, in this model, during most of the period of expansion the number of slots of the competitor has zero effect on the decision of opening a new store. Only when the market is filled or close to being filled do the competitor's stores have a significant effect on entry decisions.

(c) Biased beliefs. Competition in actual oligopoly industries is often characterized by strategic uncertainty. Firms face significant uncertainty about the strategies of their competitors. Although MD and BK should know a lot about each others strategies from a long history of play, the UK in the early 90s represented a relatively new market.[29] So while MD and BK likely know the possible strategies and thus the set of potential equilibria, the firms are competing for the first time in a new setting and may have not been sure, particularly during the initial stages of competition, which of the equilibria would be played by the opponent. While the possible equilibrium best responses are common knowledge, there is strategic uncertainty about which of these will be played. In the context of our application, it may be the case that MD's or/and BK's beliefs overestimate the negative effect of the competitor's stores on the competitor's entry decisions. For instance, if MD has one store in a local market, BK may believe that the probability that MD opens a second store is close to zero. These over-optimistic beliefs about the competitor's behavior may generate an apparent lack of response of BK's entry decisions to the number of MD's stores.

[29] See Toivanen and Waterson (2011) for an historical account of the early years of the hamburger fast food restaurant industry in UK. McDonalds opened its first restaurant in UK in 1974, but it was not until 1981 that it opened outlets outside the London area. Burger King started operating in UK in 1988 after acquiring Wimpy.

6.2 Model

Consider two retail chains competing in a local market. Each firm sells a differentiated product using its stores. Let $K_{imt} \in \{0, 1, ..., |\mathcal{K}|\}$ be the state variable that represents the number of stores of firm i in market m at period $t-1$. And let $Y_{imt} \in \{0, 1, ..., A-1\}$ be the number of *new* stores that firm i opens in the market during period t.[30] Following the empirical evidence during our sample period, we assume that opening a store is an irreversible decision. Also, for almost all the observations in the data we have that $Y_{imt} \in \{0, 1\}$, and therefore we consider a binary choice model for Y_{imt}, i.e., $A = 2$.[31]

The total number of stores of firm i in market m at period t is $N_{imt} \equiv K_{imt} + Y_{imt}$. Firm i is active in the market at period t if N_{imt} is strictly positive. Every period, the two firms know the stocks of stores in the market, K_{imt} and K_{jmt}, and simultaneously choose the new (additional) number of stores, Y_{imt} and Y_{jmt}. Firm i's total profit function is equal to variable profits minus entry costs and minus fixed operating costs: $\Pi_{imt} = VP_{imt} - EC_{imt} - FC_{imt}$.

The specification of the variable profit function is:

$$VP_{imt} = (\mathbf{W}_m \ \gamma) \ N_{imt} \ \left[\theta_{0i}^{VP} + \theta_{can,i}^{VP} N_{imt} + \theta_{com,i}^{VP} N_{jmt}\right] \tag{33}$$

$\mathbf{W}_m$ is a vector of exogenous market characteristics such as population, population density, percentage of population in age group 15-29, GDP per capita, and unemployment rate. γ is a vector of parameters where the coefficient associated to the Population variable in $\mathbf{W}_{mt}$ is normalized to one. Therefore, the index $\mathbf{W}_m \gamma$ is measured in number of people and we interpret it as "market size". According to this specification, the term $\theta_{0i}^{VP} + \theta_{can,i}^{VP} \ N_{imt} + \theta_{com,i}^{VP} \ N_{jmt}$ represents variable profits per-capita and per-store. $\theta_{0i}^{VP} + \theta_{can,i}^{VP}$ is the variable profit (per capita) when firm i has a single store in the market. The term $\theta_{can,i}^{VP} N_{imt}$ captures cannibalization effects between stores of the same chain as well as possible economies of scale and scope in variable costs. The term $\theta_{com,i}^{VP} N_{jmt}$ captures the effect of competition from the other chain.

Entry cost have the following form:

$$EC_{imt} \ = \ 1\{Y_{imt} > 0\} \ \left[\theta_{0i}^{EC} + \theta_{K,i}^{EC} \ 1\{K_{imt} > 0\} + \theta_{Z,i}^{EC} \ Z_{imt} + \varepsilon_{it}\right] \tag{34}$$

$1\{.\}$ is the indicator function, and θ_{0i}^{EC}, $\theta_{K,i}^{EC}$, and $\theta_{Z,i}^{EC}$ are parameters. θ_{0i}^{EC} is an entry cost that is paid the first time that the firm opens a store in the local market. $\theta_{0i}^{EC} + \theta_{K,i}^{EC}$ is the cost of opening a new store when the firm already has stores in the market. If there are economies of scope in the operation of multiple stores in a market, we expect the parameter $\theta_{K,i}^{EC}$ to be negative such that the

[30] We abstract from store location within a local market and assume that every store of the same firm has the same demand.

[31] The empirical distributions of store openings in our sample are the following. For McDonalds: zero stores, 1,954 observations (92.6%); one store, 146 (6.9%); two stores, 10 (0.5%). For Burger King: zero stores, 1,982 observations (93.9%); one store, 115 (5.5%); two stores, 11 (0.5%); three stores, 2 (0.1%).

entry cost of the first store is greater than the entry cost of additional stores. Z_{imt} represents the geographic distance between market m and the closest market where firm i has stores at period $t-1$ (i.e., Z_{imt} is zero if $K_{imt} > 0$). The term $\theta_{Z,i}^{EC}\ Z_{imt}$ tries to capture economies of density as in Holmes (2011). The random variable ε_{it} is a private information shock in the cost of opening a new store, and it is i.i.d. normally distributed.[32]

The specification of fixed costs is:

$$FC_{imt} \quad = \quad 1\{N_{imt} > 0\}\ \left[\theta_{0i}^{FC} + \theta_{lin,i}^{FC}\ N_{imt} + \theta_{qua,i}^{FC}(N_{imt})^2\right] \tag{35}$$

θ_{0i}^{FC} is a lump-sum cost associated with having any positive number of stores in the market. The term $\theta_{lin,i}^{FC}\ N_{imt} + \theta_{qua,i}^{FC}\ (N_{imt})^2$ takes into account that operating costs may increase (or decline) with the number of stores in a quadratic form.

Given this specification, the vector of state variables involved in the exclusion restriction of Assumption ID-3 is $S_{jmt} = (K_{jmt}, Z_{jmt})$. A firm's variable profit and fixed cost depend on both his own and his opponents current number of stores, N_{imt} and N_{jmt}. These two components of the profit function do not incorporate an exclusion restriction. Instead, our exclusion restrictions appear in the specification of the entry cost function. The entry cost of firm i depends on his own stock of stores at previous period, K_{imt}, and on the distance from market m to the closest store of the chain at year $t-1$, Z_{imt}. However, the competitors' number of stores in the previous year, K_{jmt}, and the distance from market m to the closest store of the competitor in the previous year, Z_{jmt}, do not directly affect the current profit of the firm. This satisfies the exclusion restriction in assumption ID-3. Of course a firm's beliefs about the probability distribution of the opponents' choice, Y_{jmt}, depend on $S_{jmt} = (K_{jmt}, Z_{jmt})$.

Note that the stock variable K_{jmt} does enter player i's payoffs through the current number of stores, i.e., $N_{jmt} = K_{jmt} + Y_{jmt}$. However, the number of stores of the competitor is in fact his decision at period t. The game can be described either using as decisions variables the incremental number of stores, Y_{jmt}, or the number of stores, N_{jmt}. We have preferred using the incremental number of stores as the decision variable to emphasize that it is a binary choice model. Therefore, once we condition on the competitor's current number of stores N_{jmt} (i.e., the competitor's current decision), the competitor's stock of stores, K_{jmt}, is inconsequential for player i's payoff. However, firm j cares about his own stock because the cost of adding new stores to his existing stock depends on how many he already has open.

The maximum value of K_{imt} in the sample is 13, but it is less than or equal to three for 99% of the observations in the sample. We assume that the set of possible values of K_{imt} is $\{0, 1, 2, 3\}$,

[32] Here we assume that the entry decision is made and the entry cost is paid at the same year that the store opens and starts operating in the market. In other words, we assume there is no "time-to-build", or at least that it is substantially shorter than one year. This timing assumption is quite realistic for franchise stores of large retail chains.

where $K_{imt} = 3$ represents a number of stores greater or equal than three. When $K_{imt} = 3$, we impose the restriction that firm i does not open additional stores in this market: $P_{imt}(1|\mathbf{x}_{mt}, K_{imt} = 3) = 0$. The variable Z_{imt}, that represents the distance to the closest chain store, is discretized into 8 cells of 30 miles intervals: $Z_{imt} = 1$ represents a distance of less than 30 miles, $Z_{imt} = 2$ for a distance of between 30 and 60 miles, ..., $Z_{imt} = 7$ for a distance of between 180 and 210 miles, and $Z_{imt} = 8$ for a distance greater than 210 miles. Market characteristics in the vector $\mathbf{W}_m$ have very little time variability in our sample and we treat them as time invariant state variables in order to reduce the dimensionality of the state space.[33] Therefore, the set $\mathcal{S}$ is equal to $\{0, 1, 2, 3\} \times \{1, 2, ..., 8\}$ and it has 32 grid points, and the whole state space $\mathcal{X}$ is equal to $\mathcal{S} \times \mathcal{S}$ and it has $1,024$ points.

Assumption ID-4, which restricts beliefs over a subset of the state space, takes the following form in this application. We assume that the two firms have unbiased beliefs about the entry behavior of the opponent in markets which are relatively close to the opponents network, i.e., for small values of the distance Z_{jmt}. However, beliefs may be biased for markets that are farther away to the opponent's network. More formally, we assume that:

$$B_{imt}(y_j|\mathbf{x}_{mt}) = P_{jmt}(y_j|\mathbf{x}_{mt}) \qquad \text{if } z_{jmt} \leq z^* \tag{36}$$

We have estimated the model for different values of z^*. The main intuition behind this assumption is that markets that are far away from a firm's network are unexplored markets for which there is more strategic uncertainty.

The selection of the points in the support of Z where we impose the restriction of unbiased beliefs is based on the criteria that we have proposed in section 3.2.6 above. *Criterion 'testing for the monotonicity of beliefs and using this restriction'*: the probabilities of market entry for BK and MD are strictly decreasing in their own distance variable Z. Furthermore, we cannot reject the monotonicity of beliefs with respect to this player-specific variable. According to this criterion, we could impose unbiased beliefs either at the smallest or at the largest values in the support of variable Z. *Criterion 'minimization of the player's beliefs bias'*: we have estimated the model under different selections for the points in the support of Z where we impose unbiased beliefs. In table 9, we present estimates under two different selections for unbiased beliefs: $Z \in \{0, 1\}$ and $Z \in \{0, 1, 2\}$. The estimation results are very similar under these two selections. We have also estimated the model imposing unbiased beliefs at the largest values of Z, i.e., $Z \in \{6, 7, 8\}$. The estimation results were quite different. In particular, we obtained substantially larger biases for beliefs. Therefore, a conservative criterion, based on minimizing the deviation with respect to the

[33] For those market characteristics with some time variation, we fix their values at their means over the sample period. We have also estimated the model using different values, such as the value at the first year in the sample, or at the last year (i.e., perfect forecast), and all the estimated parameters did not change up to the fourth significant digit.

paradigm of unbiased beliefs, recommends imposing the restriction of unbiased beliefs at small values of the player-specific state variable.

Our assumption on players' beliefs implies that the degree of bias in firms' beliefs declines over time with the geographic expansion of these retail chains. When the retail chains have sufficiently expanded geographically, we have that the distances z_{jmt} become smaller than the threshold value z^* such that firms' beliefs become unbiased for every market and state. The probability of this event increases over time. It is straightforward to check if this condition is satisfied for every market and firm in the data after some year in the sample. For our choices of the threshold value z^*, this condition is almost, but not exactly, satisfied in the last year of our sample, 1995.

6.3 Estimation of the structural model

Table 9 presents estimates of the dynamic game under three different assumptions on beliefs. Columns (1) and (2) present estimates under the assumption that beliefs are unbiased for every value of the state variables. In columns (3) and (4), we impose the restriction of unbiased beliefs only when the distance to the competitor's network is shorter than 60 miles, i.e., $z^* = 2$. In columns (5) and (6), beliefs are unbiased when that distance is shorter than 30 miles , i.e., $z^* = 1$. For each of these three scenarios, the proportion of observations at year 1995 for which we impose the restriction of unbiased beliefs is 100%, 38%, and 29%, respectively.

(a) Estimation with unbiased beliefs. The estimation shows substantial differences between estimated parameters in the variable profit function of the two firms. The parameter θ_{can}^{VP} is negative and significant for BK but positive and also statistically significant for MD. Cannibalization effects dominate in the case of BK. In contrast, economies of scope in variable profits seem important for MD. The estimates of the parameter that captures the competitive effect, θ_{com}^{VP}, are smaller in magnitude than the estimates of θ_{can}^{VP}, but they are statistically significant. According to these estimates the competitive effect of MD's market presence on BK's profits is smaller than the reverse effect.

The estimates of fixed cost parameters illustrates a similarity across firms in the structure of fixed costs of operation. The fixed operating cost increases linearly, not quadratically, with the number of stores, and the lump-sum component of the cost is relatively small. However, there a substantial economic differences between the firms in the magnitude of these costs. The fixed cost that BK pays per additional store is almost twice the fixed cost MD pays.

Entry costs are particularly important in this setting because they play a key role in the identification of the dynamic game, through the exclusion restrictions. The estimates of these costs are very significant, both statistically and economically. Entry costs depend significantly on the number of installed stores of the firm, K, and on the distance to the firm's network, Z. The signs of these effects, negative for θ_K^{EC} and positive for θ_Z^{EC}, are consistent with the existence of economies of scope and density between the stores of the same chain. McDonalds has smaller entry costs, and a larger absolute value of the parameter θ_K^{EC}, which indicates that there are stronger economies of scope in the network of McDonalds stores.

In summary, the estimated model with unbiased beliefs shows significant differences in the variable profits and entry costs of the firms. Cannibalization is stronger between BK stores, while MD exhibits substantial economies of scope both in variables profits and entry costs. Competition effects seem relatively weak but statistically significant.

(b) Tests of unbiased beliefs. Our test of unbiased beliefs clearly rejects the null hypothesis for BK, with a p-value of 0.00029, though we cannot reject the hypothesis of unbiased beliefs for MD.[34]

(c) Estimation with biased beliefs. As expected, (bootstrap) standard errors increase significantly when we estimate the model allowing for biased beliefs. Nevertheless, these standard errors are not large and the estimation provides informative and meaningful results. Comparing these parameter estimates with those in the model with equilibrium restrictions, the most important changes are in the parameters of variable profits of BK. In particular, the estimate of the parameter that measures the competitive effect of MD on BK is now more than twice the initial estimate with equilibrium beliefs. In contrast to the result with unbiased beliefs, we find that the competitive effect of MD on BK is stronger that the effect of BK on MD. This result is consistent with the findings in our Monte Carlo experiments: imposing the restriction of unbiased beliefs when it is incorrect introduces a "measurement error" in beliefs which in turn generates an attenuation bias in the estimate of the parameter associated with the strategic interactions. For the identification of this structural parameter the sample variation in beliefs plays an important role.

Interestingly, BK's estimated profit function has a lower level when we allow for biased beliefs than when we enforce unbiased beliefs: variable profits are lower, and fixed costs and entry costs are larger. This is fully consistent with our finding that the bias in BK's beliefs are mostly in the direction of underestimating the true probability that MD will enter in unexplored markets. If we impose the assumption of unbiased beliefs, BK's profit must be relatively high in order to rationalize entry into markets where MD is also likely to enter or to expand its number of stores. Once we take into account the over-optimistic beliefs of BK about the behavior of MD, revealed preference shows that BK profits are not as high as before. In fact, in the estimates that allow for biased beliefs we find that the differences in the profit function of MD and BK are even larger.

(d) Implications of biased beliefs on BK's profits. Finally, we have implemented a counterfactual experiment to obtain a measure of the effects of biased beliefs on BK's profits in the UK, or more specifically on its profits in the set of local markets that we include in our analysis, that excludes Greater London districts. We compare the value of BK's profits during years 1991 to 1994 given its actual entry decisions with this firm's profits if its entry decisions were based on unbiased beliefs

[34] To implement this test we use a vector $\widehat{\delta}_i = \{\widehat{\delta}_i(S_i) : S_i \in \mathcal{S}\}$ of $|\mathcal{S}| = 32$ statistics.

on MD's behavior. According to our estimates, having unbiased would increase BK's total profits in these markets by the following magnitudes: 2.78% in year 1991, 2.11% in 1992, 1.20% in 1993, and 0.87% in 1994. Remember that biased beliefs occur in markets which are relatively far away from the firm's network of stores, that these markets are relatively smaller, and that biased beliefs decline over time in the sample period as the result of geographic expansion. Though the magnitude of these gains from correct beliefs seem modest in this case, they also illustrate that they can be substantial for firms smaller than Burger King that operate only in a few local markets where beliefs are biased.

7 Conclusion

This paper studies a class of dynamic games of incomplete information where players' beliefs about the other players' actions may not be in equilibrium. We present new results on identification, estimation, and inference of structural parameters and beliefs in this class of games when the researcher does not have data on elicited beliefs, or these data are limited to players' beliefs at only some values of the state variables. Specifically, we propose a new test of the null hypothesis that beliefs are in equilibrium. This test is based on standard exclusion restrictions in dynamic games. We also derive sufficient conditions under which payoffs and beliefs are point identified. These conditions then lead naturally to a sequential estimator of payoffs and beliefs. We illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by McDonalds and Burger King. They key conditions for the identification of beliefs and payoffs in our application are the following. The first condition is an exclusion restriction in a firm's profit function that establishes that the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit through the existence of sunk entry costs and economies of density in these costs. The second condition restricts firms' beliefs to be unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores. However, beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs for Burger King. We find that Burger King underestimated the probability of entry of McDonalds in markets that were relatively far away from McDonalds' network of stores. Furthermore, we find that imposing the restriction of unbiased beliefs, when this restriction is rejected, generates a substantial attenuation bias in the estimation of the competition effects.

APPENDIX

[A.1] Aradillas-Lopez and Tamer's bounds approach in dynamic games

The purpose of this part of the appendix is to explain why Aradillas-Lopez and Tamer's bounds approach, while useful for identification and estimation of static binary choice games, has very limited applicability to dynamic games. Aradillas-Lopez and Tamer (2008) (we use the abbreviation ALT from now on) consider a static, two-player, binary-choice game of incomplete information. The model they consider can be seen as a specific case of our framework. To see this, consider the final period of the game T in our model. For the sake of notational simplicity, we omit here the vector of state variables $\mathbf{X}$ as an argument of payoff and belief functions. At the last period T, the decision problem facing the players is equivalent to that of a static game. At period T there is no future and the difference between the conditional choice value functions is simply the difference between the conditional choice current profits. For the binary choice game with two players, the Best Response Probability Function (BRP) function is:

$$P_{iT}(1) = \Lambda\left(B_{iT}^{(T)}(0)\ [\pi_{iT}(1,0) - \pi_{iT}(0,0)] + B_{iT}^{(T)}(1)\ [\pi_{iT}(1,1) - \pi_{iT}(0,1)] \right) \tag{A.1.1}$$

ALT assume that players' payoffs are submodular in players' decisions (Y_i,Y_j), i.e., for every value of the state variables $\mathbf{X}$, we have that $[\pi_{it}(1,0) - \pi_{it}(0,0)] > [\pi_{it}(1,1) - \pi_{it}(0,1)]$. Under this restriction, they derive informative bounds around players' conditional choice probabilities when players are *level-k rational*, and show that the bounds become tighter as k increases. For instance, without further restrictions on beliefs (i.e., rationality of level 1), player i's conditional choice probability $P_{iT}(1)$ takes its largest possible value when $B_{iT}^{(T)}(1) = 0$, and it takes its smallest possible value when beliefs are $B_{iT}^{(T)}(0) = 1$. This result yields informative bounds on the period T choice probabilities of player i:

$$\Lambda\left(\pi_{iT}(1,1) - \pi_{iT}(0,1)\right) \leq P_{iT}(1) \leq \Lambda\left(\pi_{iT}(1,0) - \pi_{iT}(0,0)\right) \tag{A.1.2}$$

These bounds on conditional choice probabilities can be used to "set-identify" the structural parameters in players' preferences.

In their setup, the monotonicity of players' payoffs in the decisions of other players implies monotonicity of players' BRP functions in the beliefs about other players actions. This type of monotonicity is very convenient in their approach, not only from the perspective of identification, but also because it yields a very simple approach to calculate upper and lower bounds on conditional choice probabilities. In particular, the maximum and minimum possible values of the CCPs are reached when the belief probability is equal to 0 or 1, respectively. Unfortunately, this property does not extend to dynamic games, even the simpler ones. We now discuss this issue.

Consider the two-players, binary-choice, dynamic game at some period t smaller than T. To obtain bounds on players' choice probabilities analogous to the ones obtained at the last period, we need to find, for every value of the state variables $\mathbf{X}$, the value of beliefs $\mathbf{B}^{(t)}$ that generate the smallest (and the largest) values of the best response probability $\Lambda(v_{it}^{\mathbf{B}(t)}(1,\mathbf{X}) - v_{it}^{\mathbf{B}(t)}(0,\mathbf{X}))$. That is, we need to minimize (or maximize) this best response probability with respect to the vector of beliefs $\{B_{it}^{(t)}, B_{it+1}^{(t)}, ..., B_{iT}^{(t)}\}$. Without making further assumptions, this best response function is not monotonic in beliefs at every possible state. In fact, this monotonicity is only achieved under very strong conditions not only on the payoff function but also on the transition probability of the state variables and on belief functions themselves.

Therefore, in a dynamic game, to find the largest and smallest value of a best response (and ultimately the bounds on choice probabilities) at periods $t < T$, one needs to explicitly solve a non-trivial optimization problem. In fact, the maximization (minimization) of the BRP function with respect to beliefs is a extremely complex task. The main reason is that the best response probability evaluated at a value of the state variables depends on beliefs at every period in the future and at every possible value of the state variables in the future. Therefore, to find bounds on best responses we must solve an optimization problem with a dimension equal to the number of values in the space of state variables times the number of future periods. This is because, in general, the maximization (or minimization) of a best response with respect to beliefs does not have a time-recursive structure except under very special assumptions (see Aguirregabiria, 2008). For instance, though $B_{iT}^{(T)}(1|\mathbf{x}) = 0$ maximizes the best response at the last period T, in general the maximization of a best response at period $T-1$ is not achieved setting $B_{iT}^{(T-1)}(1|\mathbf{x}) = 0$ for any value $\mathbf{x}$. More generally, the beliefs from period t to T that provide the maximum (minimum) value of the best response at period t are not equal to the beliefs from period t to T that provide the maximum (minimum) value of the best response at $t-1$. So at each point in time we need to re-optimize with respect to beliefs about strategies at every period in the future. That is, while the optimization of expected and discounted payoffs has the well-known time-recursive structure, the maximization (or minimization) of the value of BRP functions does not.

[A.2] Integrated Value Function and Continuation Values

Our proofs of Propositions 1 - 3 apply the concepts of the *integrated value function* and the *continuation value function*. The *integrated value function* is defined as $\bar{V}_{it}^{\mathbf{B}(t)}(\mathbf{X}_t) \equiv \int V_{it}^{\mathbf{B}(t)}(\mathbf{X}_t, \varepsilon_{it})$ $dG_{it}(\varepsilon_{it})$ (see Rust, 1994). Applying this definition to the Bellman equation, we obtained the *integrated Bellman equation:*

$$\begin{aligned}\bar{V}_{it}^{\mathbf{B}(t)}(\mathbf{x}_t) &= \int \max_{y_i \in \mathcal{Y}} \left\{ v_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}_t) + \varepsilon_{it}(y_i) \right\} dG_{it}(\varepsilon_{it}) \\ &= \int \max_{y_i \in \mathcal{Y}} \left\{ \pi_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1}) \; f_{it}^{\mathbf{B}}(\mathbf{x}_{t+1}|y_i, \mathbf{x}_t) + \varepsilon_{it}(y_i) \right\} dG_{it}(\varepsilon_{it})\end{aligned} \tag{A.2.1}$$

where it is understood that the function $\bar{V}_{it+1}^{\mathbf{B}(t)}$ depends on current beliefs about future events, that is, $B_{it+1}^{(t)}, B_{it+2}^{(t)}$... and so on. The expected payoff function $\pi_{it}^{\mathbf{B}}$ by contrast depends only on contemporaneous beliefs $B_{it}^{(t)}$. If $\{\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(A-1)\}$ are i.i.d. extreme value type 1, the integrated Bellman equation has the following closed-form expression:

$$\bar{V}_{it}^{\mathbf{B}(t)}(\mathbf{x}_t) = \gamma + \ln\left(\sum_{y_i \in \mathcal{Y}} \exp\left\{ \pi_{it}^{\mathbf{B}}(y_i, \mathbf{x}_t) + \beta \sum_{\mathbf{X}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1}) \; f_{it}^{\mathbf{B}}(\mathbf{x}_{t+1}|y_i, \mathbf{x}_t) \right\} \right) \tag{A.2.2}$$

where γ is Euler's constant. In the case of a finite horizon model with beliefs that do not vary over time (i.e., $B_{it+s}^{(t)} = B_{it+s}^{(t')}$ for every pair of periods t and t'), with knowledge of payoffs and beliefs, we could use this formula to obtain the integrated value function by backwards induction, starting at the last period T.

The *continuation value function* provides the expected and discounted value of *future* payoffs given current choices of all the players and beliefs of player i about future decisions. It is defined as:

$$c_{it}^{\mathbf{B}}(\mathbf{y}_t, \mathbf{x}_t) \equiv \beta \sum_{\mathbf{x}_{t+1}} \bar{V}_{it+1}^{\mathbf{B}(t)}(\mathbf{x}_{t+1}) \; f_t(\mathbf{x}_{t+1}|\mathbf{y}_t, \mathbf{x}_t) \tag{A.2.3}$$

Note that continuation values $c_{it}^{\mathbf{B}}$ depend on beliefs for decisions at periods $t+1$ and later, but not on beliefs for decisions at period t. By definition, the relationship between the conditional choice value function $v_{it}^{\mathbf{B}(t)}$ and the continuation value function $c_{it}^{\mathbf{B}(t)}$ is the following:

$$v_{it}^{\mathbf{B}(t)}(y_i, \mathbf{x}) \quad = \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} \left[\pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{x}) + c_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{x})\right] \; B_{it}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}) \tag{A.2.4}$$

Finally, we define two other objects that will be useful in what follows. First, the continuation value differences:

$$\widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i}, \mathbf{x}) \equiv c_{it}^{\mathbf{B}}(1, \mathbf{y}_{-i}, \mathbf{x}) - c_{it}^{\mathbf{B}}(0, \mathbf{y}_{-i}, \mathbf{x}) \tag{A.2.5}$$

and the sum of current payoffs and the continuation value differences:

$$g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{x}) \equiv \pi_{it}(y_i, \mathbf{y}_{-i}, \mathbf{x}) + \widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i}, \mathbf{x}) \tag{A.2.6}$$

[A.3] Proof of Proposition 1

The proof has two parts. First, we show that given CCPs of player i only, it is possible to identify a function that depends on beliefs of players but not on payoffs. Second, under the assumption of equilibrium beliefs, the identified function of beliefs can be also identified using only CCPs of player j. Therefore, we have identified the same object using two different sources of data. If the hypothesis of equilibrium beliefs is correct, the two approaches should give us the same result, but if beliefs are biased the two approaches provide different results. This can be used to construct a test statistic.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ is $(S_i, S_j, \mathbf{W})$, and players' actions are y_i and y_j. Under Assumption ID-3(iv), the transition of the state variables has the form $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{W}_t)$ and we have that continuation values $c_{it}^{\mathbf{B}}(\mathbf{Y}_t, \mathbf{X}_t)$ do not depend on $\mathbf{S}_t$. Therefore, the restrictions of the model can be written as:

$$q_{it}(y_i, \mathbf{x}) \quad = \mathbf{B}_{it}^{(t)}(\mathbf{x})' \; \mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i, \mathbf{w}) \tag{A.3.1}$$

where $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ and $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i, \mathbf{w})$ are $A \times 1$ vectors. For notational simplicity and without loss of generality, we omit $\mathbf{W}$ for the rest of this proof.

Let s_j^0 be an arbitrary value in the set $\mathcal{S}$. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s_j^0\}$ such that they satisfy two conditions: (1) each of these sets has $A-1$ elements;

and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A+1$, it is always possible to construct two subsets that satisfy these conditions. Given one of these subsets, say $\mathcal{S}^{(a)}$, we can construct the following system of $A-1$ equations:

$$\Delta \mathbf{q}_{it}^{(a)}(y_i, s_i) \quad = \Delta \mathbf{B}_{it}^{(a)}(S_i) \; \widetilde{\mathbf{g}}_{it}(y_i, s_i) \tag{A.3.2}$$

where: $\Delta \mathbf{q}_{it}^{(a)}(y_i, s_i)$ is an $(A-1)\times 1$ vector with elements $\{q_{it}(y_i, s_i, s_j) - q_{it}(y_i, s_i, s_j^0) : \text{for } s_j \in \mathcal{S}^{(a)}\}$; $\Delta \mathbf{B}_{it}^{(a)}(s_i)$ is a $(A-1)\times(A-1)$ matrix with elements $\{B_{it}^{(t)}(y_j, s_i, s_j) - B_{it}^{(t)}(y_j, s_i, s_j^0) : \text{for } y_j \in \mathcal{Y}-\{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$; and $\widetilde{\mathbf{g}}_{it}(y_i, s_i)$ is a $(A-1)\times 1$ vector with elements $\{g_{it}^{\mathbf{B}}(y_i, y_j, s_i) - g_{it}^{\mathbf{B}}(y_i, 0, s_i) : y_j \in \mathcal{Y}\}$. Using the other subset, $\mathcal{S}^{(b)}$, we can construct a similar system of $A-1$ equations. Given that matrices $\Delta \mathbf{B}_{it}^{(a)}(s_i)$ and $\Delta \mathbf{B}_{it}^{(b)}(s_i)$ are non-singular, we can use these systems to obtain two different solutions for $\widetilde{\mathbf{g}}_{it}(y_i, s_i)$:

$$\begin{aligned} \widetilde{\mathbf{g}}_{it}(y_i, s_i) \quad &= \quad \left[\Delta \mathbf{B}_{it}^{(a)}(s_i)\right]^{-1} \Delta \mathbf{q}_{it}^{(a)}(y_i, s_i) \\ &= \quad \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \Delta \mathbf{q}_{it}^{(b)}(y_i, s_i) \end{aligned} \tag{A.3.3}$$

For given s_i, we have these two solutions of $\widetilde{\mathbf{g}}_{it}(y_i, s_i)$ for every value of y_i in the set $\mathcal{Y}-\{0\}$. Putting these $A-1$ solutions in matrix form, we have:

$$\left[\Delta \mathbf{B}_{it}^{(a)}(s_i)\right]^{-1} \Delta \mathbf{Q}_{it}^{(a)}(s_i) \quad = \quad \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \Delta \mathbf{Q}_{it}^{(b)}(s_i) \tag{A.3.4}$$

where $\Delta \mathbf{Q}_{it}^{(a)}(s_i)$ and $\Delta \mathbf{Q}_{it}^{(b)}(s_i)$ are $(A-1)\times(A-1)$ matrices with columns $\Delta \mathbf{q}_{it}^{(a)}(y_i, s_i)$ and $\Delta \mathbf{q}_{it}^{(b)}(y_i, s_i)$, respectively. Given that $\Delta \mathbf{Q}_{it}^{(a)}(s_i)$ is an invertible matrix, we can rearrange the previous system in the following way:

$$\Delta \mathbf{B}_{it}^{(a)}(s_i) \; \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \quad = \quad \Delta \mathbf{Q}_{it}^{(a)}(s_i) \; \left[\Delta \mathbf{Q}_{it}^{(b)}(s_i)\right]^{-1} \tag{A.3.5}$$

This expression shows that we can identify the $(A-1)\times(A-1)$ matrix $\Delta \mathbf{B}_{it}^{(a)}(s_i) \; \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1}$ that depends only on beliefs, using only the CCPs of player i. That is, we can identify $(A-1)\times(A-1)$ objects or functions of beliefs.

Under the assumption of unbiased beliefs, we can use the CCPs of the other player, j, to identify matrix $\Delta \mathbf{B}_{it}^{(a)}(s_i) \; \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1}$:

$$\Delta \mathbf{B}_{it}^{(a)}(s_i) \; \left[\Delta \mathbf{B}_{it}^{(b)}(s_i)\right]^{-1} \quad = \quad \Delta \mathbf{P}_{jt}^{(a)}(s_i) \; \left[\Delta \mathbf{P}_{jt}^{(b)}(s_i)\right]^{-1} \tag{A.3.6}$$

where $\Delta \mathbf{P}_{jt}^{(a)}(s_i)$ is $(A-1)\times(A-1)$ matrix with elements $\{P_{jt}(y_j, s_i, s_j) - P_{jt}(y_j, s_i, s_j^0) : \text{for } y_j \in \mathcal{Y}-\{0\}$ and $s_j \in \mathcal{S}^{(a)}\}$, and $\Delta \mathbf{P}_{jt}^{(b)}(s_i)$ has a similar definition. Therefore, under the assumption of unbiased beliefs by player i the CCPs of player i and player j should satisfy the following $(A-1)^2$ restrictions:

$$\Delta \mathbf{Q}_{it}^{(a)}(s_i) \; \left[\Delta \mathbf{Q}_{it}^{(b)}(s_i)\right]^{-1} - \Delta \mathbf{P}_{jt}^{(a)}(s_i) \; \left[\Delta \mathbf{P}_{jt}^{(b)}(s_i)\right]^{-1} \quad = \quad \mathbf{0} \tag{A.3.7}$$

These restrictions are testable. ■

[A.4] Proof of Proposition 2

Proposition 2. Part (2.2). Identification of values. The restrictions of the model that come from best response behavior of player i can be represented using the following equation. For any $(y_i, \mathbf{x}) \in \mathcal{Y} \times \mathcal{X}$,

$$q_{it}(y_i, \mathbf{x}) = \mathbf{B}_{it}^{(t)}(\mathbf{x})' \; \left[\boldsymbol{\pi}_{it}(y_i, \mathbf{x}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})\right] \tag{A.4.1}$$

where $\mathbf{B}_{it}^{(t)}(\mathbf{x})$, $\boldsymbol{\pi}_{it}(y_i, \mathbf{x})$, and $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$ are vectors with dimension $A^{N-1} \times 1$ containing beliefs, payoffs, and continuation values, respectively, for every possible value of $\mathbf{y}_{-i}$ in the set $\mathcal{Y}^{N-1}$. Recalling our definition: $\mathbf{g}_{it}^{\mathbf{B}}(y_i, \mathbf{x}) \equiv \boldsymbol{\pi}_{it}(y_i, \mathbf{x}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$, we can re-write equation A.4.1 as $q_{it}(y_i, \mathbf{x}) = \mathbf{B}_{it}(\mathbf{x})' \; \mathbf{g}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$, and $\mathbf{g}_{it}^{\mathbf{B}}(y_i, \mathbf{x})$ is also a vector with dimension $A^{N-1} \times 1$.

Let $\mathcal{S}_{-i}^{(R)}$ be the set $[\mathcal{S}^{(R)}]^{N-1}$. By assumption ID-4, for any $\mathbf{x}$ such that $\mathbf{s}_{-i} \in \mathcal{S}_{-i}^{(R)}$ we have that $B_{it}^{(t)}(\mathbf{y}_{-i}|\mathbf{x}) = P_{-it}(\mathbf{y}_{-i}|\mathbf{x})$ and $P_{-it}(\mathbf{y}_{-i}|\mathbf{x})$ is known to the researcher. Consider the system of equations formed by equation (A.4.1) at a fixed value of $(y_i, s_i, \mathbf{w})$ and for every value of $\mathbf{s}_{-i}$ in $\mathcal{S}_{-i}^{(R)}$. This is a system of R^{N-1} equations, and we can represent this system in vector form using the following expression:

$$\mathbf{q}_{it}^{(R)}(y_i, s_i) = \mathbf{P}_{-it}^{(R)}(s_i) \; \mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i) \tag{A.4.2}$$

where $\mathbf{P}_{-it}^{(R)}(s_i)$ is the $R^{N-1} \times A^{N-1}$ matrix $\{P_{-it}(\mathbf{y}_{-i}|s_i, \mathbf{s}_{-i}) : \mathbf{y_{-i}} \in \mathcal{Y}^{\mathbf{N-1}}, \, \mathbf{s}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$. Under conditions (i)-(ii) in Proposition 2, matrix $\mathbf{P}_{-it}^{(R)}(s_i)'\mathbf{P}_{-it}^{(R)}(s_i)$ is non-singular and therefore we can solve for vector $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i)$ in the previous system of equations:

$$\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i) = \left[\mathbf{P}_{-it}^{(R)}(s_i)' \; \mathbf{P}_{-it}^{(R)}(s_i)\right]^{-1} \mathbf{P}_{-it}^{(R)}(s_i)' \; \mathbf{q}_{it}^{(R)}(y_i, s_i) \tag{A.4.3}$$

This expression shows that, given continuation values at period t, the vector of payoffs $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i)$ is identified, i.e., part (b) of Proposition 2.

Proposition 2. Part (2.1). Identification of beliefs. Now, we show the identification of the beliefs function for states outside the subset $\mathcal{S}_{-i}^{(R)}$. Again, we start with the system equations implied by the best response restrictions, but now we take into account that the vector $\mathbf{g}_{it}^{\mathbf{B}}(y_i, s_i)$ is identified and then look at the identification of beliefs at states $\mathbf{x}$ with $\mathbf{s}_{-i}$ outside the subset $\mathcal{S}_{-i}^{(R)}$. We stack equation (A.3.1) for every value of $y_i \in \mathcal{Y} - \{0\}$ to obtain a system of equations. Note that $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ is a vector of A^{N-1} probabilities, one element for each value of $\mathbf{y}_{-i}$ in $\mathcal{Y}^{N-1}$. The probabilities in this vector should sum to one, and therefore, $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ satisfies the restriction $\mathbf{1}'\mathbf{B}_{it}(\mathbf{x}) = 1$, where $\mathbf{1}$ is a vector of ones. Therefore, we have the following system of A equations:

$$\mathbf{q}_{it}(\mathbf{x}) = \widetilde{\mathbf{V}}_{it}(\mathbf{x}) \; \mathbf{B}_{it}^{(t)}(\mathbf{x}) \tag{A.4.4}$$

$\mathbf{q}_{it}(\mathbf{x})$ is an $A \times 1$ vector with elements $\{q_{it}(1, \mathbf{x}), ..., q_{it}(A-1, \mathbf{x})\}$ at rows 1 to $A-1$, and a 1 at the last row. And $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ is an $A \times A^{N-1}$ matrix where rows 1 to $A-1$ are $\mathbf{g}_{it}^{\mathbf{B}}(1, s_i, \mathbf{w})'$, ..., $\mathbf{g}_{it}^{\mathbf{B}}(A-1, s_i, \mathbf{w})'$, and the last row of the matrix is a row of ones. Since the values $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i, \mathbf{w})$ are identified from Proposition 2.2, we have that matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ is identified. We now prove that condition (ii) implies that matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ is non-singular. Our proof of part (b) implies that:

$$\widetilde{\mathbf{V}}_{it}(\mathbf{x}) = \mathbf{Q}_{it}^{(R)}(s_i) \; \mathbf{P}_{-it}^{(R)}(s_i) \; \left[\mathbf{P}_{-it}^{(R)}(s_i)' \; \mathbf{P}_{-it}^{(R)}(s_i)\right]^{-1} \tag{A.4.5}$$

and $\mathbf{Q}_{it}^{(R)}(s_i)$ is the $A \times R$ matrix with $\mathbf{q}_{it}^{(R)}(y_i, s_i)'$ at the first $A-1$ rows, and ones at the last row. By Assumption ID-4, $\mathbf{P}_{-it}^{(R)}(s_i)$ is full column rank, and then a sufficient condition for $\widetilde{\mathbf{V}}_{it}(\mathbf{x})$ to be non-singular matrix is that $\mathbf{Q}_{it}^{(R)}(s_i)$ has rank A, which is a condition in Proposition 2. Therefore, the vector of beliefs $\mathbf{B}_{it}^{(t)}(\mathbf{x})$ is identified as:

$$\mathbf{B}_{it}^{(t)}(\mathbf{x}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{x})\right]^{-1} \mathbf{q}_{it}(\mathbf{x}) \tag{A.4.6}$$

Proposition 2. Part (2.3): Identification of payoff differences. Consider the function $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i)$ at two different values of s_i, say $s^{(a)}$ and $s^{(b)}$, i.e., $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(a)}) = \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(a)}) + \widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i})$ and $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(b)}) = \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(b)}) + \widetilde{c}_{it}^{\mathbf{B}}(\mathbf{y}_{-i})$. By assumption ID-3(iv), variable s_i enters $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s_i)$ through the current payoff. Therefore, the difference: $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(a)}) - g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(b)})$ gives the payoff difference:

$$g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(a)}) - g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, s^{(b)}) = \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(a)}) - \pi_{it}(y_i, \mathbf{y}_{-i}, s^{(b)}) \qquad \blacksquare \tag{A.4.7}$$

[A.5] Asymptotic distribution of two-step estimators

The derivation of the asymptotic distribution of our two-step estimators of payoffs and beliefs is an application of properties of two-step semiparametric estimators as shown in Newey (1994), Andrews (1994), and McFadden and Newey (1994). In fact, given our maintained assumption that the space of state variables $\mathcal{X}$ is discrete and finite, all the structural functions in our model live in a finite dimensional Euclidean space. Therefore, we do not need to apply stochastic equicontinuity results, as in Newey (1994) and Andrews (1994), to show root-M consistency and asymptotic normality of these estimators. Here we apply results in Newey (1984) who provides a methods of moments interpretation of sequential estimators.

We begin by establishing the consistency and asymptotic normality of our estimator of CCPs. The estimator of the CCP $P_{it}(y|\mathbf{x})$ is based on the moment condition:

$$\mathbb{E}\left[1\{\mathbf{X}_{mt} = \mathbf{x}\} \ \left(1\{Y_{imt} = y\} - P_{it}(y|\mathbf{x})\right)\right] = 0 \tag{A.5.1}$$

In vector form, we have the system, $\mathbb{E}\left[f_{\mathbf{x}}(\mathbf{X}_{mt}, Y_{imt}, \mathbf{P}_{it,\mathbf{x}})\right] \equiv \mathbb{E}\left[1\{\mathbf{X}_{mt} = \mathbf{x}\} \ (\mathbf{1}_{Y_{imt}} - \mathbf{P}_{it,\mathbf{x}})\right] = \mathbf{0}$, where $\mathbf{1}_{Y_{imt}}$ and $\mathbf{P}_{it,\mathbf{x}}$ are the $(A-1) \times 1$ vectors $\mathbf{1}_{Y_{imt}} \equiv \{1\{Y_{imt} = y\} : y = 1, 2, ..., A-1\}$ and $\mathbf{P}_{it,\mathbf{x}} \equiv \{P_{it}(y|\mathbf{x}) : y = 1, 2, ..., A-1\}$, respectively. The corresponding sample moment condition that defines the estimator $\widehat{\mathbf{P}}_{it,\mathbf{x}}$ is: $\sum_{m=1}^{M} f_{\mathbf{x}}(\mathbf{X}_{mt}, Y_{imt}, \widehat{\mathbf{P}}_{it,\mathbf{x}}) = \mathbf{0}$. For notational simplicity, for the rest of this Appendix we omit the player and time subindexes (i, t) from variables, parameters, and functions. As the observations are *i.i.d.* across markets, this estimator satisfies the standard regularity conditions for consistency and asymptotic normality of the Method of Moments estimator, such that as M goes to infinity, we have that $\widehat{\mathbf{P}}_{\mathbf{x}} \to_p \mathbf{P}_{\mathbf{x}}$, and

$$\sqrt{M}\left(\widehat{\mathbf{P}}_{\mathbf{x}} - \mathbf{P}_{\mathbf{x}}\right) \to_d N\left(0 \ , \ \mathbf{F}_{\mathbf{x}}^{-1} \ \mathbf{\Omega}_{ff} \ \mathbf{F}_{\mathbf{x}}^{-1\prime}\right) \tag{A.5.2}$$

where $\mathbf{F}_{\mathbf{x}} \equiv \mathbb{E}\left[\partial f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}}) / \partial \mathbf{P}_{\mathbf{x}}'\right]$ and $\mathbf{\Omega}_{ff} \equiv \mathbb{E}\left[f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}}) \ f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})'\right]$.

We now establish the consistency and asymptotic normality of the estimator of the function $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{X})$. This implies the distribution of the estimator of payoffs $\pi_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{X})$ is also consistent and asymptotically normal as, by Proposition 3, payoffs are a deterministic, linear combination of $g_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{X})$. The population restrictions that our estimator of payoff must satisfy at a given value of $(y_i, S_i, \mathbf{W})$ are given by:

$$\mathbf{q}_i^{(R)}(y_i, S_i, \mathbf{W}) - \mathbf{P}^{(R)}(S_i, \mathbf{W})\ \mathbf{g}_i^{\mathbf{B}}(y_i, S_i, \mathbf{W}) = 0 \tag{A.5.3}$$

Or in vector form, for any value of y_i (and omitting the player subindex i), $h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \boldsymbol{\pi}_{S,\mathbf{W}}) \equiv \mathbf{q}^{(R)}(., S, \mathbf{W}) - \mathbf{P}^{(R)}(., S, \mathbf{W})\ \boldsymbol{\pi}_{S,\mathbf{W}}(., S, \mathbf{W}) = \mathbf{0}$. In the just-identified nonparametric model, the estimator $\widehat{\mathbf{g}}_{S,\mathbf{W}}$ of the vector of payoffs $\mathbf{g}_{S,\mathbf{W}}$ is the value that solves the system of equations $h_{S,\mathbf{W}}(\widehat{\mathbf{P}}_{\mathbf{x}}, \widehat{\boldsymbol{\pi}}_{S,\mathbf{W}}) = \mathbf{0}$. Under the conditions of Proposition 2, the mapping $h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \boldsymbol{\pi}_{S,\mathbf{W}})$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem (or Mann-Wald Theorem) such that as M goes to infinity, we have that $\widehat{\mathbf{g}}_{S,\mathbf{W}} \rightarrow_p \mathbf{g}_{S,\mathbf{W}}$, and $\sqrt{M}\,(\widehat{\mathbf{g}}_{S,\mathbf{W}} - \mathbf{g}_{S,\mathbf{W}}) \rightarrow_d N\left(0\ ,\ \mathbf{V}_{\mathbf{g}_{S,\mathbf{W}}}\right)$, where applying Newey (1984)

$$\mathbf{V}_{\mathbf{g}_{S,\mathbf{W}}} = \mathbf{H}_{\mathbf{g}}^{-1}\left(\ \boldsymbol{\Omega}_{hh} + \mathbf{H}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{ff}\ \mathbf{F}_{\mathbf{x}}^{-1\prime}]\ \mathbf{H}_{\mathbf{P}}' - \mathbf{H}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{f,h}\ +\ \boldsymbol{\Omega}_{h,f}\mathbf{F}_{\mathbf{x}}^{-1\prime}]\mathbf{H}_{\mathbf{P}}'\ \right)\ \mathbf{H}_{\mathbf{f}}^{-1\prime} \tag{A.5.4}$$

with $\mathbf{H}_{\mathbf{g}} \equiv \partial h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})/\partial \mathbf{g}_{S,\mathbf{W}}'$, $\mathbf{H}_{\mathbf{P}} \equiv \partial h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})/\partial \mathbf{P}_{\mathbf{x}}'$, $\boldsymbol{\Omega}_{hh} \equiv \mathbb{E}[h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})\ h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})']$, $\boldsymbol{\Omega}_{fh} \equiv \mathbb{E}[f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})\ h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})']$, and $\boldsymbol{\Omega}_{hf} \equiv \mathbb{E}[h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \mathbf{g}_{S,\mathbf{W}})\ f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})']$.

Our estimator of beliefs takes as given the estimates of CCPs and payoffs. Specifically, for a given vector of the state variables $\mathbf{x}$, beliefs are given by the system

$$\ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}) \equiv \widetilde{V}_i(\mathbf{X})\ \mathbf{B}_i(\mathbf{X}) - \mathbf{q}_i(\mathbf{X}) = \mathbf{0} \tag{A.5.5}$$

The estimator $\widehat{\mathbf{B}}_{\mathbf{x}}$ of the vector of beliefs $\mathbf{B}_{\mathbf{x}}$ is the value that solves the system of equations $\ell_{\mathbf{x}}\left(\widehat{\mathbf{P}}_{\mathbf{x}}, \widehat{\mathbf{B}}_{\mathbf{x}}\right) = 0$. Under the conditions of Proposition 2, the mapping $\ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem such that as M goes to infinity, we have that $\widehat{\mathbf{B}}_{\mathbf{x}} \rightarrow_p \mathbf{B}_{\mathbf{x}}$, and $\sqrt{M}\left(\widehat{\mathbf{B}}_{\mathbf{x}} - \mathbf{B}_{\mathbf{x}}\right) \rightarrow_d N(0\ ,\ \mathbf{V}_{\mathbf{B}_{\mathbf{x}}})$, where applying Newey (1984),

$$\mathbf{V}_{\mathbf{B}_{\mathbf{x}}} = \mathbf{L}_{\mathbf{B}}^{-1}\left(\ \boldsymbol{\Omega}_{\ell\ell} + \mathbf{L}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{ff}\ \mathbf{F}_{\mathbf{x}}^{-1\prime}]\ \mathbf{L}_{\mathbf{P}}' - \mathbf{L}_{\mathbf{P}}[\mathbf{F}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{f,\ell}\ +\ \boldsymbol{\Omega}_{\ell,f}\mathbf{F}_{\mathbf{x}}^{-1\prime}]\mathbf{L}_{\mathbf{P}}'\ \right)\ \mathbf{L}_{\mathbf{B}}^{-1\prime} \tag{A.5.6}$$

with $\mathbf{L}_{\mathbf{B}} \equiv \partial \ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})/\partial \mathbf{B}_{\mathbf{x}}'$, $\mathbf{L}_{\mathbf{P}} \equiv \partial \ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})/\partial \mathbf{P}_{\mathbf{x}}'$, $\boldsymbol{\Omega}_{\ell\ell} \equiv \mathbb{E}[\ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})\ \ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})']$, $\boldsymbol{\Omega}_{f,\ell} \equiv \mathbb{E}[f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})\ \ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})']$, and $\boldsymbol{\Omega}_{\ell,f} \equiv \mathbb{E}[\ell_{\mathbf{x}}(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}})\ f_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})']$. ■

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Table 1. Order Condition for Identification
Models without Exclusion Restrictions in Payoffs

Number of parameters & restrictions for each player-period

Number of parameters & restrictions	**Models without exclusion restrictions (A) Unrestricted Beliefs**	**(B) Unbiased (Equil) Beliefs**
(1) Restrictions from observed behavior	$(A-1)\ \lvert\mathcal{X}\rvert$	$(A-1)\ \lvert\mathcal{X}\rvert$
(2) Restrictions from unbiased beliefs	0	$(N-1)\ (A-1)\ \lvert\mathcal{X}\rvert$
(3) Free parameters in payoffs	$(A-1)\ \lvert\mathcal{X}\rvert\ A^{N-1}$	$(A-1)\ \lvert\mathcal{X}\rvert\ A^{N-1}$
(4) Free parameters in beliefs	$(N-1)\ (A-1)\ \lvert\mathcal{X}\rvert$	$(N-1)\ (A-1)\ \lvert\mathcal{X}\rvert$
(5) Free parameters in $\widetilde{c}$	$(A-1)\ \lvert\mathcal{X}\rvert$	$(A-1)\ \lvert\mathcal{X}\rvert$
(1)+(2)-(3)-(4)-(5) Over-under identifying rest.	$-(A-1)\lvert\mathcal{X}\rvert\left[A^{N-1}+(N-1)\right]$	$-(A-1)\ \lvert\mathcal{X}\rvert\ \left[A^{N-1}\right]$
Is the Model identified?	**NO**	**NO**

Table 2. Order Condition for Identification
Models WITH Exclusion Restrictions in Payoffs

Number of parameters & restrictions for each player-period

Number of parameters & restrictions	Models with exclusion restrictions (A) Unrestricted Beliefs	(B) Unbiased (Equil) Beliefs
(1) Restrictions from observed behavior	$(A-1)\ \vert\mathcal{S}\vert^N$	$(A-1)\ \vert\mathcal{S}\vert^N$
(2) Restrictions from unbiased beliefs	0	$(N-1)\ (A-1)\ \vert\mathcal{S}\vert^N$
(3) Free parameters in payoffs	$(A-1)\ \vert\mathcal{S}\vert\ A^{N-1}$	$(A-1)\ \vert\mathcal{S}\vert\ A^{N-1}$
(4) Free parameters in beliefs	$(N-1)\ (A-1)\ \vert\mathcal{S}\vert^N$	$(N-1)\ (A-1)\ \vert\mathcal{S}\vert^N$
(5) Free parameters in $\widetilde{c}$	$(A-1)$	0
(1)+(2)-(3)-(4)-(5) Over-under identifying rest.	$(A-1)\vert\mathcal{S}\vert^N \left[1 - \frac{A^{N-1}}{\vert\mathcal{S}\vert^{N-1}} - (N-1) - \frac{1}{\vert\mathcal{S}\vert^N}\right]$	$(A-1)\vert\mathcal{S}\vert^N \left[1 - \frac{A^{N-1}}{\vert\mathcal{S}\vert^{N-1}}\right]$
Is the Model identified?	**NO (For any N$\geq$ 2)**	**YES (For any $\vert\mathcal{S}\vert \geq A$)**

Table 3
Monte Carlo Experiments[1]

Parameter (True value)	Biased Beliefs in the DGP: Estimation with equilibrium restrictions (1) MAB[2] (%)[3]	(2) Std (%)	Estimation no equilibrium restrictions (3) MAB (%)	(4) Std (%)		Unbiased Beliefs in the DGP: Estimation with equilibrium restrictions (5) MAB (%)	(6) Std (%)	Estimation no equilibrium restrictions (7) MAB (%)	(8) Std (%)
Payoffs									
α_1 (2.4)	0.521	0.332	0.308	0.245	α_1 (2.4)	0.180	0.135	0.525	0.441
	(21.7%)	(13.8%)	(12.8%)	(10.2%)		(7.5%)	(5.6%)	(21.9%)	(18.4%)
δ_1 (3.0)	0.456	0.277	0.284	0.224	δ_1 (3.0)	0.178	0.134	0.581	0.509
	(15.2%)	(9.2%)	(9.5%)	(7.5%)		(5.9%)	(4.5%)	(19.4%)	(17.0%)
θ_1^{EC} (0.5)	0.108	0.081	0.205	0.160	θ_1^{EC} (0.5)	0.091	0.069	0.172	0.128
	(21.6%)	(16.3%)	(40.9%)	(32.0%)		(18.3%)	(13.8%)	(34.5%)	(25.7%)
Beliefs at $Z_2 = 0 : B_1(Y_{1t-1}, Y_{2t-1})$									
$B_1(0,0)$ (0.410)	0.409	0.066	0.148	0.110	$B_1(0,0)$ (0.658)	0.058	0.044	0.101	0.082
	(99.8%)	(16.0%)	(36.2%)	(26.8%)		(8.8%)	(6.6%)	(15.4%)	(12.4%)
$B_1(0,1)$ (0.442)	0.442	0.035	0.111	0.086	$B_1(0,1)$ (0.814)	0.031	0.023	0.081	0.114
	(100%)	(7.9%)	(25.1%)	(19.5%)		(3.8%)	(2.8%)	(9.9%)	(14.0%)
$B_1(1,0)$ (0.403)	0.403	0.035	0.091	0.072	$B_1(1,0)$ (0.559)	0.031	0.024	0.068	0.053
	(100%)	(8.7%)	(22.7%)	(17.9%)		(5.6%)	(4.3%)	(12.2%)	(9.5%)
$B_1(1,1)$ (0.437)	0.437	0.021	0.070	0.053	$B_1(1,1)$ (0.727)	0.026	0.020	0.054	0.041
	(100%)	(4.8%)	(16.0%)	(12.2%)		(3.6%)	(2.7%)	(7.4%)	(5.7%)
CCPs at $Z_2 = 0 : P_1(Y_{1t-1}, Y_{2t-1})$									
$P_1(0,0)$ (0.829)	0.073	0.034	0.051	0.038	$P_1(0,0)$ (0.704)	0.037	0.028	0.063	0.047
	(8.8%)	(9.7%)	(4.1%)	(4.6%)		(5.2%)	(4.0%)	(8.9%)	(6.6%)
$P_1(0,1)$ (0.814)	0.090	0.026	0.035	0.026	$P_1(0,1)$ (0.598)	0.025	0.019	0.046	0.040
	(11.1%)	(3.1%)	(4.3%)	(3.2%)		(4.2%)	(3.2%)	(7.6%)	(6.7%)
$P_1(1,0)$ (0.891)	0.056	0.015	0.022	0.027	$P_1(1,0)$ (0.841)	0.014	0.011	0.028	0.021
	(6.3%)	(1.7%)	(2.5%)	(3.1%)		(1.9%)	(1.3%)	(3.3%)	(2.5%)
$P_1(1,1)$ (0.880)	0.072	0.013	0.017	0.012	$P_1(1,1)$ (0.761)	0.016	0.012	0.023	0.018
	(8.1%)	(1.4%)	(1.9%)	(1.4%)		(2.1%)	(1.6%)	(3.1%)	(2.3%)

Note (1): Summary of DGPs in Monte Carlo Experiments. In all the experiments the values of the parameters are $\alpha_1 = \alpha_2 = 2.4$, $\delta_1 = \delta_2 = 3.0$, $\theta_2^{FC} = -1.0$, $\theta_1^{EC} = \theta_2^{EC} = 0.5$, and $\beta_1 = \beta_2 = 0.95$.
$Z_{2mt} \sim Uniform$ over $\{-2, -1, 0, +1, +2\}$ $i.i.d.$
In Experiment B with Biased beliefs, $\lambda_1 = \lambda_2 = 1$ if $Z_2 \in \{-2, 2\}$ and $\lambda_1 = \lambda_2 = 0.5$ if $Z_2 \in \{-1, 0, 1\}$.
Number of Monte Carlo Replications is 10,000. And number of observations in each replication, M=500 and T=5.
Note (2): MAB = Mean Absolute Bias. Std = Standard deviation.
Note (3): (%) = Percentage of the true value of the parameter.

Table 4
Monte Carlo Experiment(1): One replication with 1 million observations

Parameter (True value)	Biased Beliefs in the DGP: Estimation with equilibrium restrictions (1) MAB(2) (%)(3)	Biased Beliefs in the DGP: Estimation no equilibrium restrictions (2) MAB (%)		Unbiased Beliefs in the DGP: Estimation with equilibrium restrictions (3) MAB (%)	Unbiased Beliefs in the DGP: Estimation no equilibrium restrictions (4) MAB (%)
Payoffs					
α_1 (2.4)	0.523	0.005	α_1 (2.4)	0.002	0.017
	(21.7%)	(0.2%)		(0.1%)	(0.7%)
δ_1 (3.0)	0.365	0.009	δ_1 (3.0)	0.001	0.013
	(12.2%)	(0.3%)		(0.0%)	(0.4%)
θ_1^{EC} (0.5)	0.051	0.003	θ_1^{EC} (0.5)	0.002	0.004
	(10.2%)	(0.6%)		(0.3%)	(0.8%)
Beliefs at $Z_2 = 0 : B_1(Y_{1t-1}, Y_{2t-1})$					
$B_1(0,0)$ (0.410)	0.408	0.007	$B_1(0,0)$ (0.658)	0.003	0.007
	(99.6%)	(1.7%)		(0.4%)	(1.1%)
$B_1(0,1)$ (0.442)	0.444	0.007	$B_1(0,1)$ (0.814)	0.001	0.003
	(100%)	(1.6%)		(0.1%)	(0.4%)
$B_1(1,0)$ (0.403)	0.399	0.004	$B_1(1,0)$ (0.559)	0.004	0.001
	(99.0%)	(1.0%)		(0.6%)	(0.1%)
$B_1(1,1)$ (0.437)	0.437	0.006	$B_1(1,1)$ (0.727)	0.001	0.000
	(100%)	(1.5%)		(0.1%)	(0.0%)
CCPs at $Z_2 = 0 : P_1(Y_{1t-1}, Y_{2t-1})$					
$P_1(0,0)$ (0.829)	0.072	0.003	$P_1(0,0)$ (0.704)	0.001	0.005
	(8.7%)	(0.4%)		(0.2%)	(0.7%)
$P_1(0,1)$ (0.814)	0.092	0.003	$P_1(0,1)$ (0.598)	0.001	0.003
	(11.3%)	(0.4%)		(0.2%)	(0.5%)
$P_1(1,0)$ (0.891)	0.055	0.002	$P_1(1,0)$ (0.841)	0.001	0.000
	(6.3%)	(0.2%)		(0.1%)	(0.0%)
$P_1(1,1)$ (0.880)	0.077	0.001	$P_1(1,1)$ (0.761)	0.001	0.000
	(8.2%)	(0.2%)		(0.2%)	(0.0%)

Note (1): Summary of DGPs in Monte Carlo Experiments. In all the experiments the values of the parameters are $\alpha_1 = \alpha_2 = 2.4$, $\delta_1 = \delta_2 = 3.0$, $\theta_2^{FC} = -1.0$, $\theta_1^{EC} = \theta_2^{EC} = 0.5$, and $\beta_1 = \beta_2 = 0.95$.
$Z_{2mt} \sim Uniform$ over $\{-2, -1, 0, +1, +2\}$ $i.i.d.$
In Experiment B with Biased beliefs, $\lambda_1 = \lambda_2 = 1$ if $Z_2 \in \{-2, 2\}$ and $\lambda_1 = \lambda_2 = 0.5$ if $Z_2 \in \{-1, 0, 1\}$.
Number of Monte Carlo Replications is 1, and the number of observations is 1,000,000.
Note (2): MAB = Mean Absolute Bias.
Note (3): (%) = Percentage of the true value of the parameter.

Table 5
Descriptive Statistics on Local Markets (Year 1991)
422 local authority districts (excluding Greater London districts)

Variable	Median	Std. Dev.	Pctile 5%	Pctile 95%
Area (thousand square km)	0.30	0.73	0.03	1.67
Population (thousands)	94.85	93.04	37.10	280.50
Share of children: Age 5-14 (%)	12.43	1.00	10.74	14.07
Share of Young: 15-29 (%)	21.24	2.46	17.80	25.17
Share of Pensioners: 65-74 (%)	9.01	1.50	6.89	11.82
GDP per capita (thousand £)	92.00	12.14	74.40	112.70
Claimants of UB / Population ratio (%)	2.75	1.27	1.24	5.11
Avg. Weekly Rent per dwelling (£)	25.31	10.61	19.11	35.07
Council tax (thousand £)	0.24	0.05	0.11	0.31
Number of BK stores	0.00	0.62	0.00	1.00
Number of MD stores	1.00	1.16	0.00	3.00

Table 6
Evolution of the Number of Stores
422 local authority districts (excluding Greater London districts)

	Burger King					
	1990	**1991**	**1992**	**1993**	**1994**	**1995**
#Markets with Stores	71	98	104	118	131	150
Change in #Markets with Stores	-	17	6	14	13	19
# of Stores	79	115	128	153	181	222
Change in # of Stores	-	36	13	25	28	41
Mean #Stores per Market (Conditional on #Stores>0)	1.11	1.17	1.23	1.30	1.38	1.48
	McDonalds					
	1990	**1991**	**1992**	**1993**	**1994**	**1995**
#Markets with Stores	206	213	220	237	248	254
Change in #Markets with Stores		7	7	17	11	6
# of Stores	281	316	344	382	421	447
Change in # of Stores		35	28	38	39	26
Mean #Stores per Market (Conditional on #Stores>0)	1.36	1.49	1.56	1.61	1.70	1.76

Table 7
Transition Probability Matrix for Market Structure

Annual Transitions. Market structure: BK=x & MD=y, where x and y are number of stores

Market Structure at t	% **Market Structure at t+1** BK=0 MD=0	BK=0 MD=1	BK=0 MD$\geq$ 2	BK=1 MD=0	BK=1 MD=1	BK=1 MD$\geq$ 2	BK$\geq$ 2 MD=0	BK$\geq$ 2 MD=1	BK$\geq$ 2 MD$\geq$ 2
BK=0 &MD=0	**95.1**	3.6	0.2	1.0	-	-	-	0.1	-
BK=0 &MD=1	-	**87.2**	4.2	-	7.4	1.0	-	-	1.4
BK=0 &MD$\geq$ 2	-	-	**82.7**	-	-	15.8	-	-	1.4
BK=1 &MD=0	-	-	-	**76.0**	18.0	2.0	4.0	-	-
BK=1 &MD=1	-	-	-	-	**87.1**	8.1	-	3.3	1.4
BK=1 &MD$\geq$ 2	-	-	-	-	-	**86.5**	-	-	13.5
BK$\geq$ 2 &MD=0	-	-	-	-	-	-	**84.6**	15.4	-
BK$\geq$ 2 &MD=1	-	-	-	-	-	-	-	**69.0**	31.0
BK$\geq$ 2 &MD$\geq$ 2	-	-	-	-	-	-	-	-	**100.0**
Frequency	41.6	23.3	6.6	2.2	10.9	8.8	0.6	1.4	4.5

Table 8
Reduced Form Probits for the Decision to Open a Store

	Estimated Marginal Effects[1] ($\Delta P(x)$ when dummy from 0 to 1)					
	Burger King			**McDonalds**		
Explanatory Variable	**No FE**	**County FE**	**District FE**	**No FE**	**County FE**	**District FE**
Own number of stores at t-1						
Dummy: Own #stores = 1	-0.021**	-0.036**	-0.885**	-0.035**	-0.045**	-0.550**
	(0.005)	(0.007)	(0.063)	(0.010)	(0.012)	(0.056)
Dummy: Own #stores = 2	-0.023**	-0.030**	-0.210*	-0.047**	-0.060*	-0.757**
	(0.004)	(0.005)	(0.085)	(0.006)	(0.008)	(0.041)
Dummy: Own #stores $\geq$ 3	-0.019**	-0.027**	-0.056	-0.043**	-0.053**	-0.816**
	(0.005)	(0.005)	(0.036)	(0.006)	(0.008)	(0.038)
Competitor's number of stores at t-1						
Dummy: Comp.'s #stores = 1	0.032**	0.037*	-0.025	0.020	0.032*	0.052**
	(0.011)	(0.014)	(0.055)	(0.013)	(0.018)	(0.073)
Dummy: Comp.'s #stores = 2	0.045*	0.052*	-0.017	0.041	0.076	-0.007**
	(0.023)	(0.029)	(0.031)	(0.029)	(0.046)	(0.093)
Dummy: Comp.'s #stores $\geq$ 3	0.089*	0.101*	0.011	-0.041**	-0.050**	-0.104**
	(0.048)	(0.059)	(0.084)	(0.007)	(0.009)	(0.020)
Pred. Prob. Y=1 at mean X	0.024	0.027	0.014	0.045	0.054	0.085
Time dummies	YES	YES	YES	YES	YES	YES
Control variables[2]	YES	YES	YES	YES	YES	YES
County Fixed Effects	NO	YES	NO	NO	YES	NO
District Fixed Effects	NO	NO	YES	NO	NO	YES
Number of Observations[3]	2110	1715	535	2110	1855	640
Number of Local Districts[3]	422	343	107	422	371	128
log likelihood	-371.89	-340.26	-110.54	-467.46	-449.02	-198.50
Pseudo R-square	0.229	0.252	0.624	0.159	0.161	0.441

Note 1: Estimated Marginal Effects are evaluated at the mean value of the rest of the explanatory variables.

Note 2: Every estimation includes as control variables log-population, log-GDP per capita, log-population density, share population 5-14, share population 15-29, average rent, and proportion of claimants of unemployment benefits.

Note 3: FE estimations do not include districts where the dependent variable does not have enough time variation.

Table 9
Estimation of Dynamic Game for McDonalds and Burger King
Models with Unbiased and Biased Beliefs[1]

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)					
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
	Unbiased Beliefs		Biased Beliefs: $Z^* = 2$		Biased Beliefs: $Z^* = 1$	
	Burger King	McDonalds	Burger King	McDonalds	Burger King	McDonalds
Variable Profits:						
θ_0^{VP}	0.5413	0.8632	0.4017	0.8271	0.4342	0.8582
	(0.1265)*	(0.2284)*	(0.2515)*	(0.4278)*	(0.2820)	(0.4375)
θ_{can}^{VP} cannibalization	-0.2246	0.0705	-0.2062	0.0646	-0.1926	0.0640
	(0.0576)*	(0.0304)*	(0.1014)*	(0.0710)	(0.1140)*	(0.0972)
θ_{com}^{VP} competition	-0.0541	-0.0876	-0.1133	-0.0856	-0.1381	-0.0887
	(0.0226)*	(0.0272)	(0.0540)*	(0.0570)	(0.0689)*	(0.0622)
Fixed Costs:						
θ_0^{FC} fixed	0.0350	0.0374	0.0423	0.0307	0.0490	0.0339
	(0.0220)	(0.0265)	(0.0478)	(0.0489)	(0.0585)	(0.0658)
θ_{lin}^{FC} linear	0.0687	0.0377	0.0829	0.0467	0.0878	0.0473
	(0.0259)*	(0.0181)*	(0.0526)*	(0.0291)	(0.0665)	(0.0344)
θ_{qua}^{FC} quadratic	-0.0057	0.0001	-0.0007	0.0002	-0.0004	0.0004
	(0.0061)	(0.0163)	(0.0186)	(0.0198)	(0.0253)	(0.0246)
Entry Cost:						
θ_0^{EC} fixed	0.2378	0.1887	0.2586	0.1739	0.2422	0.1764
	(0.0709)*	(0.0679)*	(0.1282)*	(0.0989)*	(0.1504)	(0.1031)
θ_K^{EC} (K)	-0.0609	-0.107	-0.0415	-0.1190	-0.0419	-0.1271
	(0.043)	(0.0395)*	(0.096)	(0.0628)*	(0.109)*	(0.0762)*
θ_Z^{EC} (Z)	0.0881	0.0952	0.1030	0.1180	0.0902	0.1212
	(0.0368)*	(0.0340)*	(0.0541)*	(0.0654)*	(0.0628)	(0.0759)*
Log-Likelihood	-848.4		-840.4		-838.7	
Test of unbiased beliefs:						
For BK: $\widehat{D}$ (d.o.f) (p-value)			66.841 (32) (0.00029)		66.841 (32) (0.00029)	
For MD: $\widehat{D}$ (d.o.f) (p-value)			42.838 (32) (0.09549)		42.838 (32) (0.09549)	

Note 1: Bootstrap standard errors in parentheses.

Note 2: * and ** denote significance at the 5% and 1% level respectively

Figure 1: Rejection Probability when the Null is False (Experiment “B")

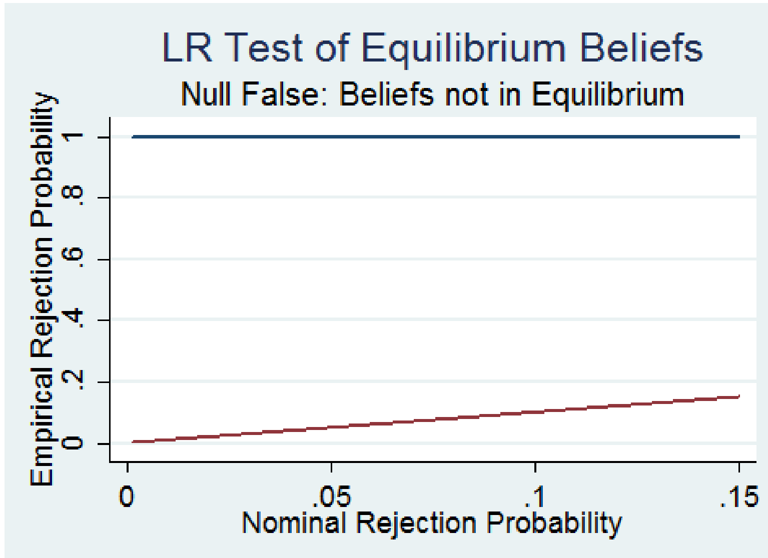

Figure 2: Rejection Probability when the Null is True (Experiment “U")