Sufficient Statistics for Unobserved Heterogeneity in Dynamic Discrete Choice Models

Victor Aguirregabiria and Jiaying Gu

University of Toronto
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Overview

1) Conditional maximum likelihood estimator for structural parameters:

- One way to deal with incidental parameter problem in linear or non-linear panel data models is to use the conditional maximum likelihood method.
- Review of Sufficient Statistics with example on static binary choice panel data model.
- Conditional likelihood approach may be a lucky coincidence (Chamberlain (2010): only Logit model for discrete choice!).
- CMLE for dynamic (lagged choice dependence) discrete choice model without forward looking (Binary choice and multinomial choice).
- CMLE for discrete choice models accounting for duration dependence. (Chamberlain 1985, Frederiksen et al, 2007).

Overview

2) Partial identification results:

- Honoré and Tamer (2006): initial condition problem: $p(y_{i0}; \boldsymbol{x}_i, \omega_i)$ is not nonparametrically identified $\rightarrow$ characterize bounds.
- Chernozhukov et al (2003): Even in the logit model where we have point identification through CMLE for the structural parameters, marginal effects are not point identified unless stronger assumptions imposed.

3) Bias correction approach: look for estimators that have smaller biases as opposed to no bias at all.

- Modified conditional likelihood approach: Cox and Reid (1987).

4) Efficiency perspective: do we lose any information in using the conditional maximum likelihood approach?

$$P(\boldsymbol{y}_i; \beta, \omega_i) = \underbrace{P(\boldsymbol{y}_i | S(\boldsymbol{y}_i), \boldsymbol{\beta})}_{\text{conditional likelihood}} P(S(\boldsymbol{y}_i); \boldsymbol{\beta}, \omega_i).$$

$$P(\boldsymbol{y}_i; \beta) = \int P(\boldsymbol{y}_i; \beta, \omega) dF(\omega) = P(\boldsymbol{y}_i | S(\boldsymbol{y}_i), \boldsymbol{\beta}) \int P(S(\boldsymbol{y}_i); \boldsymbol{\beta}, \omega) dF(\omega)$$

- What do we think of the full MLE $(\hat{\beta}, \hat{F}(\omega))$ versus $\hat{\beta}_c = \text{argmax} \sum_i \log P(\boldsymbol{y}_i | S(\boldsymbol{y}_i), \boldsymbol{\beta})$.
- Is $\hat{\beta}$ asymptotically equivalent to $\hat{\beta}_c$? (i.e. is $\hat{\beta}_c$ is semiparametrically efficient?).

Sufficient Statistics

- Definition: Let $\{\mathcal{Y}, \mathcal{P} = \{P_\theta, \theta \in \Theta\}\}$ be a parametric model, the statistic S is sufficient for θ if the conditional distribution of Y given S does not depend on θ.
- Factorization Theorem: A necessary and sufficient condition for S to be a sufficient statistic is that the density factorizes

$$f(y;\theta) = \lambda(y)\psi(S(y);\theta) \; = f_1(y \mid S(Y) = S(y)) \; f_2(S(y), \theta)$$

- Sufficient statistic is not unique, consider iid sample $Y_1, Y_2, \ldots, Y_n \sim N(0, \sigma^2)$.
  - $S_1(Y) = (Y_1, Y_2, \ldots, Y_n)$
  - $S_2(Y) = (Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$
  - $S_3(Y) = (Y_1^2 + Y_2^2 + \cdots + Y_m^2, Y_{m+1}^2 + \cdots + Y_n^2)$
  - $S_4(Y) = \sum_i Y_i^2$ (minimum sufficient)
- Partial sufficiency: suppose $\theta = (\alpha, \beta) \in A \times B$, two cases of factorization for S being sufficient for α (nuisance):
  1. $f(y;\theta) = f_1(y \mid S(Y) = S(y), \beta) \; f_2(S(y); \alpha)$: α and β are likelihood orthogonal
  2. $f(y;\theta) = f_1(y \mid S(Y) = S(y), \beta) \; f_2(S(y); \alpha, \beta)$

  1. using f_1 for estimation of β leads to fully efficient estimator.
  2. using f_1 for estimation of β may or may not be fully efficient (in what sense?)

Example: static binary choice panel data model

- Model: $y_{it} = 1\{\boldsymbol{x}_{it}^{\top}\boldsymbol{\beta} + \omega_i + \epsilon_{it} > 0\}$, where $\epsilon_{it}|\omega_i, x_{it}$ are iid, $P(\epsilon_{it} \leq a) = \frac{\exp(a)}{1+\exp(a)}$
- Earliest example (Rasch 1961): Each individual i is given T items for test, $y_{it} = \{0, 1\}$ (incorrect, correct), x_{it} is dummy of item t, hence $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_T)$ measures item difficulty, we have
$$P(y_{it} = 1; \omega_i, \beta_t) = \frac{\exp(\omega_i - \beta_t)}{1 + \exp(\omega_i - \beta_t)}.$$
- Individual Likelihood for $\boldsymbol{y}_i = \{y_{i1}, \ldots, y_{iT}\}$:
$L_i(\boldsymbol{\beta}, \omega_i) = P(\boldsymbol{y}_i; \boldsymbol{\beta}, \omega_i) = \prod_{t=1}^{T} f(y_{it}; \boldsymbol{\beta}, \omega_i) = \frac{\exp(\omega_i \sum_t y_{it} - \sum_t y_{it}\beta_t)}{\prod_{t=1}^{T}(1+\exp(\omega_i - \beta_t))}$.
- Minimum Sufficient statistics for ω_i is: $S(\boldsymbol{y}_i) = \sum_t y_{it}$ (total score of individual).
- Factorization: $L_i(\boldsymbol{\beta}, \omega_i) = P(\boldsymbol{y}_i|S(\boldsymbol{y}_i), \boldsymbol{\beta}, \cancel{\omega_i})P(S_i; \boldsymbol{\beta}, \omega_i)$
- Conditional maximum likelihood estimator $\hat{\boldsymbol{\beta}}_c := \underset{\boldsymbol{\beta} \in B}{\operatorname{argmax}} \sum_i \log P(\boldsymbol{y}_i|S(\boldsymbol{y}_i), \boldsymbol{\beta})$.
- Remark: If S is not minimum sufficient, it could contain information about $\boldsymbol{\beta}$.

Example: static binary choice panel data model

- Suppose $T = 2$, and $\beta_2 = 0$, so $\beta_1 \equiv \beta$ is difficulty level of test item 1 relative to test item 2.
- $S(y_{i1}, y_{i2}) = \{0, 1, 2\}$.
  1. $P(y_{i1}, y_{i2} | y_{i1} + y_{i2} = 0; \beta, \omega_i) = 1$
  2. $P(y_{i1}, y_{i2} | y_{i1} + y_{i2} = 1; \beta, \omega_i) = 1\{y_{i1} = 1, y_{i2} = 0\} \frac{\exp(-\beta)}{1+\exp(-\beta)} + 1\{y_{i1} = 0, y_{i2} = 1\} \frac{1}{1+\exp(-\beta)}$
  3. $P(y_{i1}, y_{i2} | y_{i1} + y_{i2} = 2; \beta, \omega_i) = 1$
- Conditional MLE: $\hat{\boldsymbol{\beta}}_c = \underset{\boldsymbol{\beta} \in B}{\text{argmax}} \sum_{i \in I} 1\{y_{i1} = 1, y_{i2} = 0\} \ln \frac{1}{1+\exp(\beta)} + 1\{y_{i1} = 0, y_{i2} = 1\} \ln \frac{\exp(\beta)}{1+\exp(\beta)} = \log \frac{\#\{0,1\}}{\#\{1,0\}}$. where $I = \{i : y_{i1} + y_{i2} = 1\}$.
- If $\#\{0,1\} > \#\{1,0\}$, then $\hat{\boldsymbol{\beta}}_c > 0$ (test item 1 more difficult).
- CMLE $\hat{\boldsymbol{\beta}}_c$ is root-n consistent under suitable regularity conditions [Anderson (1970)].
  - Usual MLE regularity conditions.
  - ω_i could not take a sequence of too extreme values such that $P(y_{i1} + y_{i2} = 1)$ vanishes as $n \to \infty$.

Example: static binary choice panel data model

- Inference can be conducted in the usual way, asymptotic variance is inverse of Fisher information.
- $L = \sum_{i \in I} w_i \ln \frac{\exp(\beta)}{1+\exp(\beta)} + (1 - w_i) \ln \frac{1}{1+\exp(\beta)}$ with $w_i = 1\{y_{i1} = 0, y_{i2} = 1\}$.
- $\frac{\partial^2 L}{\partial \beta^2} = -\sum_{i \in I} \frac{\exp(\beta)}{(1+\exp(\beta))^2}$ and $\mathbb{E}[-\frac{\partial^2 L}{\partial \beta^2}] = \sum_i P_i \frac{\exp(\beta)}{(1+\exp(\beta))^2}$ with $P_i = \mathbb{E}[1\{y_{i1} + y_{i2} = 1\} | \omega_i, \beta]$.
- Since we do not have a consistent estimator for ω_i, it seems hard to construct consistent estimator for the asymptotic variance.
- However, $\frac{1}{N}\left[-\frac{\partial^2 L}{\partial \beta^2}\right] = \frac{\exp(\beta)}{(1+\exp(\beta))^2} \frac{1}{N} \sum_i 1\{y_{i1} + y_{i2} = 1\}$ converges to $\frac{\exp(\beta)}{(1+\exp(\beta))^2} P_i$ almost surely by LLN.

Identification

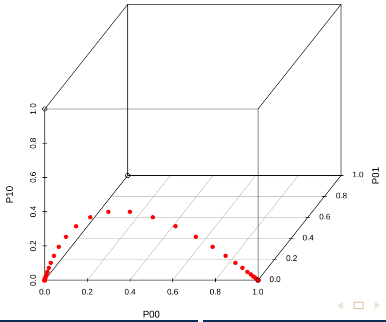
- Not all discrete choice model obtains the root-n consistent CMLE.
- For static binary choice model, Chamberlain (2010) shows that only the logistic model has identification.
- Consider model $P(y_{it} = 1|x_i, \omega_i) = F(\omega_i + x_{it}\beta_0)$, for given x, β, ω_i, the parametric model represents a vector in a $K-1$ ($K = 2^T$) dimensional space, if $T = 2$, it is a 3 dimensional unit simplex.

$$p(x, \beta, \omega) = \begin{pmatrix} P(0,0|x,\omega) \\ P(0,1|x,\omega) \\ P(1,0|x,\omega) \end{pmatrix}$$

- Identification: β_0 is identified if there doesn't exist $\beta' \neq \beta_0$ along with distribution $F'(\omega|x)$ and $F_0(\omega|x)$, such that $\int p(x, \beta_0, \omega) dF_0(\omega|x) = \int p(x, \beta', \omega) dF'(\omega|x)$.

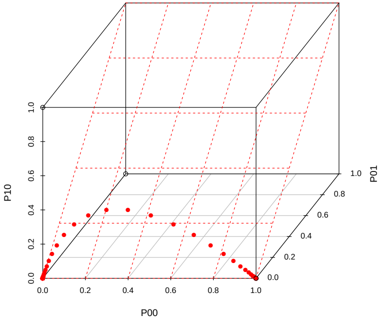
Identification: geometry

- Fix x and β, as ω changes, the model $p(x, \beta, \omega)$ defines a set of points in the 3-d unit simplex
$$C_{x\beta} = \{q \in \mathbb{S}^3 : \exists \omega \text{ such that } q = p(x, \beta, \omega)\}$$
- If F is continuous, then $C_{x\beta}$ traces out a continuous curve in the 3-d unit simplex.
- The observed data frequency $p(x, \beta) = \int p(x, \beta, \omega) dF(\omega|x)$ lives in the convex hull of $H_{x\beta} \equiv coC_{x\beta}$.
- Identification fails if the observed data frequency lies in two different convex hull $H_{x\beta_0}$ and $H_{x\beta'}$.
- Plot the vector $p(x, \beta, \omega)$ with $x_1 = 2, x_2 = 3, \beta = 0, \omega \in [-20, 20]$.



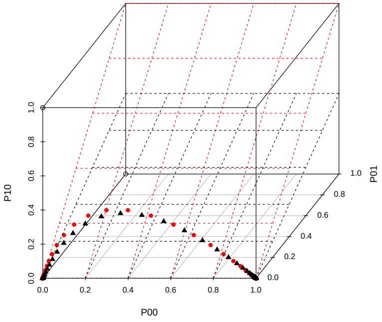
Identification: geometry

- Chamberlain shows that a necessary condition that we have identification of β is when $C_{x\beta}$ lies on a hyperplane of dimension $K-2$. (i.e. $\forall\omega$, $\sum_{j=1}^{K-1} c_j p_j(x, \beta_0, \omega) = c_0$).
- This can only be true if F is the logistic CDF.
- Below I am plotting the regression plane of regressing $P10$ against $P00$ and $P01$, residual is zero, perfect fit!



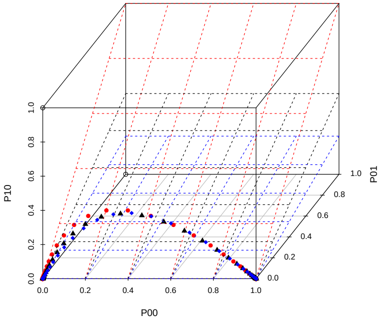
Identification: geometry

- Change β to 0.75 (black).



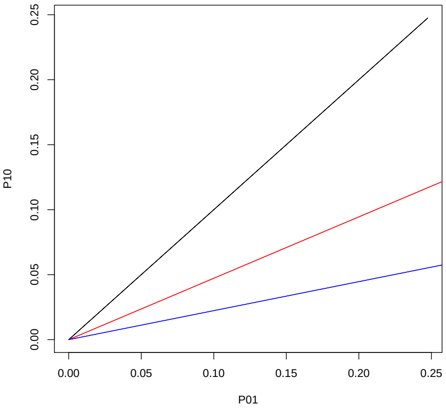
Identification: geometry

- Change β to 1.5 (blue).



Identification: geometry

- For Logit model, in fact $\frac{P10}{P01} = \exp((x_{i1} - x_{i2})\beta)$.

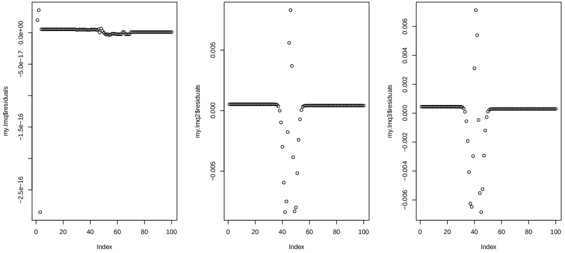


Identification failure: Probit model

- Consider $P(y_{it} = 1|x_i, \omega_i) = \Phi(\omega_i + x_{it}\beta)$: probit model.
- We also have

$$p(x, \beta, \omega) = \begin{pmatrix} P(0,0|x,\omega) \\ P(0,1|x,\omega) \\ P(1,0|x,\omega) \end{pmatrix}$$

- Plot the regression residual of $P10$ regress on $P00$ and $P01$, with $x_{i1} = 2, x_{i2} = 3$
- Left: $\beta = 0$, Middle: $\beta = 0.75$, Right: $\beta = 1.5$.



Generalize this idea: Bonhomme (2011)

- The idea is to find a moment condition for β that does not depend on ω_i.
- Let $[L_{\beta,x}F](y) = \int p(x,\beta,\omega)dF(\omega)$: maps distribution function F to the data frequency.
- Suppose we can find a function $\varphi(\cdot,x,\beta)$ such that for every $F(\omega)$:

$$\int_{\mathcal{Y}} \varphi(y,x,\beta)[L_{\beta,x}F](y)dy = 0$$

- Then we have the moment condition which doesnt depend on ω: $\mathbb{E}[\varphi(y_i,x_i,\beta_0)|x_i = x] = \int_{\mathcal{Y}} \varphi(y,x,\beta_0)[L_{\beta_0,x}F_0](y)dy = 0$.
- In the Chamberlain example, it boils down to find for all x and ω:

$$\sum_{y\in\{0,1\}^T} \varphi(y,x,\beta)P(\boldsymbol{y}_i = y|x,\omega,\beta) = 0.$$

- which leads to

$$\sum_{y\in\{0,1\}^T} 1\{\sum_t y_t = s\}\varphi(y,x,\beta)\exp(\sum_t y_t x_t \beta) = 0$$

  if T=2, the solution for φ is

$$\varphi(\boldsymbol{y} = (0,0),x,\beta) = \varphi(\boldsymbol{y} = (1,1),x,\beta) = 0$$
$$\varphi(\boldsymbol{y} = (1,0),x,\beta)\exp(x_{i1}\beta) + \varphi(\boldsymbol{y} = (0,1),x,\beta)\exp(x_{i2}\beta) = 0$$

Dynamic binary choice model

Econometrics Model: $y_{it} = 1\{\theta y_{i,t-1} + \omega_i + v_{it} \geq 0\}$

- Recall the example of a firm's entry-exit (1-0) decision without forward looking behavior (myopic).
- Endogenous state variable $x_{it} = y_{i,t-1}$.
- $U_{it}(y) = u(y, x_{it}, \omega_i) + \epsilon_{it}(y) = \omega_i(y) + \beta(y, y_{it-1}) + \epsilon_{it}(y)$, with unobservable $\epsilon_{it}(0), \epsilon_{it}(1)$ iid over (i,t),
- Dynamics is captured by $\beta(y, y_{it-1}) = 1\{y \neq y_{it-1}\}\beta(y, y_{it-1})$.
- Observed decision $y_{it} = 1\{\omega_i(1) - \omega_i(0) + \beta(1, x_{it}) - \beta(0, x_{it}) + \epsilon_{it}(1) - \epsilon_{it}(0) \geq 0\}$,
- Since $\beta(1, y_{it-1}) - \beta(0, y_{it-1}) = \beta(1,0)(1 - y_{it-1}) - \beta(0,1)y_{it-1} = \beta(1,0) + y_{it-1}(-\beta(1,0) - \beta(0,1))$.
- Let $\omega_i \equiv \omega_i(1) - \omega_i(0) + \beta(1,0)$ and $v_{it} \equiv \epsilon_{it}(1) - \epsilon_{it}(0)$ and $\theta = -\beta(1,0) - \beta(0,1)$ gives the econometrics model, and θ is the sunk cost of a firm (cost of entry and cost of exit).
- Chamberlain (1985) shows the CMLE of θ is root-n consistent.

Chamberlain (1985): CMLE for dynamic binary choice logit model

- Suppose $\epsilon_{it}(y)$ follows extreme value Type I distribution, and suppose $T = 3$, then
- Individual likelihood for $\boldsymbol{y}_i = \{y_{i0}, y_{i1}, y_{i2}, y_{i3}\}$:

$$
\begin{aligned}
& L_i(\theta, \omega_i) \\
&= P(y_{i0}|\theta, \omega_i) \frac{e^{(y_{i1}+y_{i2}+y_{i3})\omega_i} e^{\theta \sum_{t=1}^{3} y_{it} y_{it-1}}}{(1+e^{(\theta y_{i0}+\omega_i)})(1+e^{(\theta y_{i1}+\omega_i)})(1+e^{(\theta y_{i2}+\omega_i)})} \\
&= \frac{P(y_{i0}|\theta, \omega_i)}{(1+e^{\theta y_{i0}+\omega_i})} \\
&\times \frac{e^{(y_{i1}+y_{i2})\omega_i}}{((1+e^{\theta+\omega_i})^2)^{1\{y_{i1}+y_{i2}=2\}}((1+e^{\theta+\omega_i})(1+e^{\omega_i}))^{1\{y_{i1}+y_{i2}=1\}}((1+e^{\omega_i})^2)^{1\{y_{i1}+y_{i2}=0\}}} \\
&\times e^{y_{i3}\omega_i} \\
&\times e^{\theta \sum_{t=1}^{3} y_{it} y_{it-1}}
\end{aligned}
$$

- Minimum Sufficient Statistics for ω_i: $S(\boldsymbol{y}_i) = \{y_{i0}, y_{i1} + y_{i2}, y_{i3}\}$.
  1. $P(\boldsymbol{y}_i | y_{i0}, y_{i1} + y_{i2} = 0, y_{i3}; \theta, \omega_i) = 1.$
  2. $P(\boldsymbol{y}_i | y_{i0}, y_{i1} + y_{i2} = 2, y_{i3}; \theta, \omega_i) = 1.$
  3. $P(\boldsymbol{y}_i | y_{i0}, y_{i1} + y_{i2} = 1, y_{i3}; \theta, \omega_i) = 1\{y_{i1} = 1, y_{i2} = 0\} \frac{\exp(\theta(y_{i0}-y_{i3}))}{1+\exp(\theta(y_{i0}-y_{i3}))} + 1\{y_{i1} = 0, y_{i2} = 1\} \frac{1}{1+\exp(\theta(y_{i0}-y_{i3}))}$
- CMLE $\hat{\beta}_c = \log\left(\frac{\#\{1,1,0,0\}+\#\{0,0,1,1\}}{\#\{0,1,0,1\}+\#\{1,0,1,0\}}\right)$ (Minimum T =3).

General T: CMLE for dynamic binary choice logit model

- For general $T \geq 3$, minimum sufficient statistic $S(\boldsymbol{y}_i) = \{y_{i0}, \sum_{t=1}^{T-1} y_{it}, y_{iT}\}$.
- Conditional likelihood function: $P(\boldsymbol{y}_i \mid S_i, \omega_i, \theta) = \frac{\exp(\theta \sum_{t=1}^{T} y_{it} y_{it-1})}{\sum_{d \in B_i} \exp(\theta \sum_{t=1}^{T} d_t d_{t-1})}$ where

$$B_i = \left\{ (d_0, d_1, \ldots, d_T) \in \{0,1\}^{T+1} : d_0 = y_{i0}, d_T = y_{iT}, \sum_{t=1}^{T-1} d_t = \sum_{t=1}^{T-1} y_{it} y_{it-1} \right\}$$

- Disadvantage: no other observables other than y_{it-1}.
- Honoré and Kyriazidou (2000) considers $y_{it} = 1\{\theta y_{it-1} + x_{it}\beta + \omega_i + v_{it} \geq 0\}$ where x_{it} are exogenous random variables.

Honoré and Kyriazidou (2000), T = 3

- Individual likelihood for $\boldsymbol{y}_i = \{y_{i0}, y_{i1}, y_{i2}, y_{i3}\}$ given $\boldsymbol{x}_i = \{x_{i1}, \dots, x_{iT}\}$:

  $L_i(\theta, \omega_i)$

  $$= P(y_{i0}|\boldsymbol{x}_i, \theta, \omega_i) \frac{e^{(y_{i1}+y_{i2}+y_{i3})\omega_i} e^{\theta \sum_{t=1}^{3} y_{it} y_{it-1} + \beta \sum_t y_{it} x_{it}}}{(1 + e^{(\theta y_{i0} + x_{i1}\beta + \omega_i)})(1 + e^{(\theta y_{i1} + x_{i2}\beta + \omega_i)})(1 + e^{(\theta y_{i2} + x_{i3}\beta + \omega_i)})}$$

  $$\overset{x_{i2}=x_{i3}}{=} \frac{P(y_{i0}|\boldsymbol{x}_i, \theta, \omega_i)}{(1 + e^{\theta y_{i0} + x_{i1}\beta + \omega_i})}$$

  $$\times \frac{e^{(y_{i1}+y_{i2})\omega_i}}{((1 + e^{\theta + x_{i2}\beta + \omega_i})^2)^{1\{y_{i1}+y_{i2}=2\}} ((1 + e^{\theta + x_{i2}\beta + \omega_i})(1 + e^{\omega_i}))^{1\{y_{i1}+y_{i2}=1\}} ((1 + e^{\omega_i})^2)^{1\{y_{i1}+y_{i2}=}}$$

  $$\times e^{y_{i3}\omega_i}$$

  $$\times e^{\theta \sum_{t=1}^{3} y_{it} y_{it-1} + \beta \sum_t y_{it} x_{it}}$$

- Minimum Sufficient Statistics is still $S_i = \{y_{i0}, y_{i1} + y_{i2}, y_{iT}\}$.

Honoré and Kyriazidou (2000), T = 3

- Conditional log Likelihood:
$\sum_i 1\{y_{i1} + y_{i2} = 1\} 1\{x_{i2} - x_{i3} = 0\} \log\left(\frac{\exp((x_{i1}-x_{i2})\beta+\theta(y_{i0}-y_{i3}))^{y_{i1}}}{1+\exp((x_{i1}-x_{i2})\beta+\theta(y_{i0}-y_{i3}))}\right)$
- If all x_{it} are discrete random variable and $P(x_{i2} = x_{i3}) > 0$, then the CMLE $(\hat{\beta}_c, \hat{\theta}_c)$ is root-n consistent.
- If x_{it} are continuous random variables, modify the conditional log likelihood with a kernel weights.

$$\sum_i 1\{y_{i1} + y_{i2} = 1\} K\left(\frac{x_{i2} - x_{i3}}{h_n}\right) \log\left(\frac{\exp((x_{i1} - x_{i2})\beta + \theta(y_{i0} - y_{i3}))^{y_{i1}}}{1 + \exp((x_{i1} - x_{i2})\beta + \theta(y_{i0} - y_{i3}))}\right)$$

- But apparently, age is not allowed as a regressor using this strategy.
- Kernel bandwidth $h_n \to 0$ as $n \to \infty$ and $K(v) \to 0$ as $\|v\| \to \infty$.
- The resulting kernel type CMLE converges to Normal at the rate $\sqrt{nh_n^k}$ where k is column dimension of $\boldsymbol{x}$.

Honoré and Kyriazidou (2000), General $T \geq 3$

- For general T, in order for $\{y_{i0}, \sum_{t=1}^{T-1} y_{it}, y_{iT}\}$ to be minimum sufficient statistics, we would have to require $x_{i2} = x_{i3} = \ldots x_{iT}$, this implies the rate of convergence be $\sqrt{nh_n^{(T-2)k}}$ which is too slow!
- Honoré and Kyriazidou (2000) suggests a pair-wise approach to maintain $\sqrt{nh_n^k}$ rate: identification of (θ, β) is based on sequences of histories such that $y_{is} + y_{it} = 1$ for $1 \leq t < s \leq T-1$.
- Consider
  - $A = \{y_{i0} = d_0, \ldots, y_{it-1} = d_{t-1}, y_{it} = 1, y_{it+1} = d_{t+1}, \ldots, y_{is-1} = d_{s-1}, y_{is} = 0, y_{is+1} = d_{s+1}, \ldots, y_{iT} = d_T\}$.
  - $B = \{y_{i0} = d_0, \ldots, y_{it-1} = d_{t-1}, y_{it} = 0, y_{it+1} = d_{t+1}, \ldots, y_{is-1} = d_{s-1}, y_{is} = 1, y_{is+1} = d_{s+1}, \ldots, y_{iT} = d_T\}$
- $P(A|\mathbf{x}_i, \omega_i, A \cup B, x_{it+1} = x_{is+1}) = \frac{\exp\left((x_{it}-x_{is})\beta+\theta(d_{t-1}-d_{s+1})+\theta(d_{t+1}-d_{s-1})1\{s-t>1\}\right)}{1+\exp\left((x_{it}-x_{is})\beta+\theta(d_{t-1}-d_{s+1})+\theta(d_{t+1}-d_{s-1})1\{s-t>1\}\right)}$

  independent of ω_i
- Conditional likelihood:

$$\sum_i \sum_{1 \leq t < s \leq T-1} 1\{y_{is} + y_{it} = 1\} K\left(\frac{x_{it+1} - x_{is+1}}{h_n}\right)$$

$$\times \log\left(\frac{\exp\left((x_{it} - x_{is})\beta + \theta(d_{t-1} - d_{s+1}) + \theta(d_{t+1} - d_{s-1})1\{s - t > 1\}\right)^{y_{it}}}{1 + \exp\left((x_{it} - x_{is})\beta + \theta(d_{t-1} - d_{s+1}) + \theta(d_{t+1} - d_{s-1})1\{s - t > 1\}\right)}\right)$$

Extension to multinomial choice with dynamics

Only covariate: y_{it-1}

- $P(y_{it}=k|y_{it-1}=j,\omega_i,\theta)=\frac{\exp(\omega_{ik}+\theta_{jk})}{\sum_{h=0}^{J}\exp(\omega_{ih}+\theta_{jh})}$.
- Magnac (2000) shows that minimum sufficient statistics for $\boldsymbol{\omega}_i$ is $S_i=\{y_{i0},y_{iT},\sum_{t=1}^{T-1}1\{y_{it}=k\},\forall k\}$ (initial and termination state, and numbers of occurances of all states during period 1 to $T-1$.)
- Conditional likelihood function: $P(\boldsymbol{y}_i|S_i,\boldsymbol{\omega}_i,\boldsymbol{\theta})=\frac{\exp\left(\sum_k\sum_j\left(\sum_{t=1}^{T}1\{y_{it}=k\}1\{y_{it-1}=j\}\theta_{jk}\right)\right)}{\sum_{d\in B_i}\exp\left(\sum_k\sum_j\left(\sum_{t=1}^{T}1\{d_t=k\}1\{d_{t-1}=j\}\theta_{jk}\right)\right)}$ independent of $\boldsymbol{\omega}_i$ where $B_i=\{d=(d_0,\ldots,d_T)\in\{0,\ldots,J\}^T,d_0=y_{i0},d_T=y_{iT},\sum_{t=1}^{T-1}1\{d_t=k\}=\sum_{t=1}^{T-1}1\{y_{it}=k\},\forall k\}$.

Extension to include exogenous covariates x_{jit} (Honoré and Kyriazidou (2000))

- $P(y_{it}=k|y_{it-1}=j,\omega_i,\theta)=\frac{\exp(x_{kit}\beta_k+\omega_{ik}+\theta_{jk})}{\sum_{h=0}^{J}\exp(x_{hit}\beta_h+\omega_{ih}+\theta_{jh})}$.
- Same pair-wise approach as binary choice case: identification via CMLE requires $x_{kit+1}=x_{kis+1}$ for $k=0,\ldots,J$.

Duration dependence (Frederiksen et al 2007)

- Single spell data with Grouped fixed effect: groups indexed by i and individuals within groups are indexed by $j = 1, \ldots, J_i$. Number of groups goes to ∞ relative to group size and time periods $t = 1, \ldots, T$.
- $y_{jit} = \{0, 1\}$ (unemployed-employed). Let T_{ji} be the time period in which unemployment spell ends.

$$y_{jit} = 1\{\delta_{S_{jit}} + \omega_i + v_{jit} \geq 0\} \text{for } t = 1, \ldots, T_{ji}$$

  where $S_{jit} = S_{ji1} + t$.
- We observe $\{y_{jit}, T_{ji}\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, J_i$.
- Suppose $J_i = 2$ for all i, the likelihood function for $\{y_{1i1}, \ldots, y_{1iT_{1i}}, y_{2i1}, \ldots, y_{2iT_{2i}}\}$ is

$$\left(\prod_{s=1}^{T_{1i}-1} \frac{1}{1+\exp(\delta_{S_{i1s}} + \omega_i)}\right) \frac{\exp(\delta_{S_{1iT_{1i}}} + \omega_i)}{1+\exp(\delta_{S_{1iT_{1i}}} + \omega_i)}$$

$$\times \left(\prod_{s=1}^{T_{2i}-1} \frac{1}{1+\exp(\delta_{S_{2is}} + \omega_i)}\right) \frac{\exp(\delta_{S_{2iT_{2i}}} + \omega_i)}{1+\exp(\delta_{S_{2iT_{2i}}} + \omega_i)}$$

$$= \frac{\exp(2\omega_i)\exp(\delta_{S_{1iT_{1i}}} + \delta_{S_{2iT_{2i}}})}{\prod_{s=1}^{T_{1i}}(1+\exp(\delta_{S_{i1s}} + \omega_i))\prod_{s=1}^{T_{2i}}(1+\exp(\delta_{S_{i1s}} + \omega_i))}$$

  Sufficient statistics is (T_{1i}, T_{2i}) with no further reduction, hence no CMLE for δ_t.
- No SS $\neq$ No identification or root-n consistent estimator.

Duration dependence (Frederiksen et al 2007)

- Identification comes from comparing two events:
  - $A = \{T_{1i} = t_1, T_{2i} > t_2\}$ and $B = \{T_{1i} > t_1, T_{2i} = t_2\}$; WLOG $t_1 < t_2$.
  - $P(A|A \cup B) = \frac{a_1}{a_1 + a_2}$ with

$$
\begin{aligned}
a_1 &= P_{t_1}(y_{1it_1} = 1, y_{2it_1} = 0 \mid \{y_{1is} = 0, y_{2is} = 0\}_{s<t_1}) \\
&\quad \times P_{t_2}(y_{2it_2} = 0 \mid \{y_{1is} = 0\}_{s<t_1}, y_{1it_1} = 1, \{y_{2is} = 0\}_{s<t_2}) \\
&= F(\delta_{t_1+S_{1i1}} + \omega_i)(1 - F(\delta_{t_1+S_{2i1}} + \omega_i))(1 - F(\delta_{t_2+S_{2i1}} + \omega_i)) \\
a_2 &= P_{t_1}(y_{1it_1} = 0, y_{2it_1} = 0 \mid \{y_{1is} = 0, y_{2is} = 0\}_{s<t_1}) \\
&\quad \times P_{t_2}(y_{2it_2} = 1 \mid \{y_{1is} = 0\}_{s\leq t_1}, \{y_{2is} = 0\}_{s<t_2}) \\
&= (1 - F(\delta_{t_1+S_{1i1}} + \omega_i))(1 - F(\delta_{t_1+S_{2i1}} + \omega_i))F(\delta_{t_2+S_{2i1}} + \omega_i)
\end{aligned}
$$

  - If F is logistic distribution function, $P(A|A \cup B) = \frac{\exp(\delta_{t_1+S_{1i1}} - \delta_{t_2+S_{2i1}})}{1+\exp(\delta_{t_1+S_{1i1}} - \delta_{t_2+S_{2i1}})}$.

- Extremum estimator (root-n consistent):

$$
\begin{aligned}
\delta_t = \text{argmax} \sum_i \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} & 1\{T_{1i} = t_1, T_{2i} > t_2\} \log \frac{\exp(\delta_{t_1+S_{1i1}} - \delta_{t_2+S_{2i1}})}{1 + \exp(\delta_{t_1+S_{1i1}} - \delta_{t_2+S_{2i1}})} \\
& + 1\{T_{1i} > t_2, T_{2i} = t_2\} \log \frac{1}{1 + \exp(\delta_{t_1+S_{1i1}} - \delta_{t_2+S_{2i1}})}
\end{aligned}
$$

Duration dependence (Chamberlain 1985)

- If there is no duration dependence, then the choice prior to y_{it-1} should have no effect on the probability of the choice y_{it}.
- Chamberlain (1985) proposes to test $H_0 : \gamma_2 = 0$ from the following model

$$P(y_{it} = 1 | y_{it-1}, y_{it-2}) = \frac{\exp(\omega_i + \gamma_{1i} y_{it-1} + \gamma_2 y_{it-2})}{1 + \exp(\omega_i + \gamma_{1i} y_{it-1} + \gamma_2 y_{it-2})}$$

- Sufficient statistics for (ω_i, γ_{1i}): $S_i = \{y_{i0}, y_{i1}, \sum_{t=2}^{T-2} y_{it}, \sum_{t=2}^{T-1} y_{it} y_{it-1}, y_{iT-1}, y_{iT}\}$.
- Conditional likelihood function:

$$P(\mathbf{y}_i | S_i, \omega_i, \gamma_{1i}, \gamma_2) = \frac{\exp(\gamma_2 \sum_{t=2}^{T} y_{it} y_{it-2})}{\sum_{d \in B_i} \exp(\gamma_2 \sum_{t=2}^{T} d_t d_{t-2})}$$

  with $B_i = \{d = \{d_0, \ldots, d_T\} : d_0 = y_{i0}, d_1 = y_{i1}, \sum_{t=2}^{T-2} d_t = \sum_{t=2}^{T-2} y_{it}, \sum_{t=2}^{T-1} d_t d_{t-1} = \sum_{t=2}^{T-1} y_{it} y_{it-1}, d_{T-1} = y_{iT-1}, d_T = y_{iT}\}$
- Requires at least $T = 5$.

Duration dependence (Chamberlain 1985)

- $A_1 = \{1,0,1,0,0,0\}, A_2 = \{1,0,0,1,0,0\}$: $P(A_1|A_1 \cup A_2) = \frac{\exp(\gamma_2)}{1+\exp(\gamma_2)}$
- $B_1 = \{0,1,0,1,1,1\}, B_2 = \{0,1,1,0,1,1\}$: $P(B_1|B_1 \cup B_2) = \frac{\exp(\gamma_2)}{1+\exp(\gamma_2)}$
- $C_1 = \{1,1,0,1,1,0\}, C_2 = \{1,1,1,0,1,0\}$: $P(C_1|C_1 \cup C_2) = \frac{1}{1+\exp(\gamma_2)}$.
- $D_1 = \{0,0,1,0,0,1\}, D_2 = \{0,0,0,1,0,1\}$: $P(D_1|D_1 \cup D_2) = \frac{1}{1+\exp(\gamma_2)}$.

Duration dependence (AGL, 2018wp)

- Recall the example of firm's entry-exit (1,0) decision without forward looking behavior.
- Endogenous state variable $x_{it} = \{y_{it-1}, d_{it}\}$ where $d_{it+1} = 1\{y_{it} = y_{it-1}\}d_{it} + 1$.
- $U_{it}(y) = \omega_i(y) + \beta(y, y_{it-1}, d_{it}) + \epsilon_{it}(y)$
- Structural parameter $\beta(y, y_{it-1}, d_{it}) = 1\{y = y_{it-1}\}\beta_d(y, d_{it}) + 1\{y \neq y_{it-1}\}\beta_y(y, y_{it-1})$.
- Suppose $\beta_d(y, d) = \beta_d(y, d^*)$ for $d \geq d^* = 2$ for both $y = 0, 1$.

Duration dependence (AGL, 2018wp)

- Optimal decision rule

$$y_{it} = 1\Big\{ \ \omega_i(1) - \omega_i(0) + \beta(1, y_{it-1}, d_{it}) - \beta(0, y_{it-1}, d_{it}) + \epsilon_{it}(1) - \epsilon_{it}(0) \geq 0 \ \Big\}$$

$$\begin{aligned} \beta(1, y_{it-1}, d_{it}) &= y_{it-1}\beta_d(1, d_{it}) + (1 - y_{it-1})\beta_y(1, 0) \\ &= y_{it-1}y_{it-2}\beta_d(1, 2) + y_{it-1}(1 - y_{it-2})\beta_d(1, 1) + (1 - y_{it-1})\beta_y(1, 0) \\ \beta(0, y_{it-1}, d_{it}) &= (1 - y_{it-1})(1 - y_{it-2})\beta_d(0, 2) + (1 - y_{it-1})y_{it-2}\beta_d(0, 1) + y_{it-1}\beta_y(0, 1) \end{aligned}$$

- Combine terms, we have

$$\begin{aligned} y_{it} = 1\Big\{ \quad & \omega_i + y_{it-1}y_{it-2}(\beta_d(1, 2) - \beta_d(1, 1)) + y_{it-2}(1 - y_{it-1})(\beta_d(0, 2) - \beta_d(0, 1)) \\ & + y_{it-1}(\beta_d(1, 1) + \beta_d(0, 2) - \beta_y(1, 0) - \beta_y(0, 1)) + \epsilon_{it} \geq 0 \Big\} \end{aligned}$$

with $\omega_i = \omega_i(1) - \omega_i(0) + \beta_y(1, 0) - \beta_d(0, 2)$ and $\epsilon_{it} = \epsilon_{it}(1) - \epsilon_{it}(0)$.

Duration dependence (AGL, 2018wp)

$$y_{it} = 1\Big\{ \quad \omega_i + y_{it-1}y_{it-2}(\beta_d(1,2) - \beta_d(1,1)) + y_{it-2}(1 - y_{it-1})(\beta_d(0,2) - \beta_d(0,1)) \\ + y_{it-1}(\beta_d(1,1) + \beta_d(0,2) - \beta_y(1,0) - \beta_y(0,1)) + \epsilon_{it} \geq 0 \Big\}$$

- If $\delta_d \equiv \beta_d(0,2) - \beta_d(0,1) = \beta_d(1,2) - \beta_d(1,1)$, then

$$y_{it} = 1\Big\{\omega_i + y_{it-2}\delta_d + y_{it-1}\underbrace{(\beta_d(1,1) + \beta_d(0,2) - \beta_y(1,0) - \beta_y(0,1))}_{not-identified} + \epsilon_{it} \geq 0\Big\}$$

  Two lags dynamic binary choice model: δ_d is identified.

- If no duration dependence for zero choice, $\beta_d(0,2) = \beta_d(0,1) = 0$, and $\delta_d \equiv \beta_d(1,2) - \beta_d(1,1)$, then

$$y_{it} = 1\Big\{\omega_i + y_{it-1}y_{it-2}\delta_d + y_{it-1}(\beta_d(1,1) - \beta_y(1,0) - \beta_y(0,1)) + \epsilon_{it} \geq 0\Big\}$$

  Cumulative lag: both δ_d and $(\beta_d(1,1) - \beta_y(1,0) - \beta_y(0,1))$ are identified.

- AGL provides a generalization of the Chamberlain results which allows more general identification of duration dependence.